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ON THE APPROXIMATION OF THE EXTERIOR STOKES PROBLEM IN THREE DIMENSIONS (*)

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Abstract. — *We approximate the Stokes operator on an exterior domain in three dimensions by a truncated problem on a finite subdomain. Boundary conditions at the artificially introduced boundary are presented. Approximation results are discussed, both concerning the error in the solution of the problem posed over the truncated domain and the error due to the discretization by finite elements techniques.*

Résumé. — *L'opérateur de Stokes dans un domaine extérieur de* $\mathbb{R}^3$ *est approximé par un problème tronqué dans un sous-domaine fini. Les conditions aux limites à la frontière artificielle sont présentées. Des résultats d'approximation sont étudiés, résultats concernant l'erreur pour la solution du problème dans le domaine tronqué, et ceux concernant l'erreur due aux techniques de discrétisation par éléments finis.*

1. INTRODUCTION

So far, the study of numerical approximations to incompressible viscous flows has been largely restricted to the case of bounded domains. The rigorous mathematical study of the governing equations, known as the Navier-Stokes equations, is not an easy task to achieve even in bounded domains. Our goal in this paper is to discuss methods of approximating the linear model of viscous incompressible flow which is known as the Stokes problem. Let Λ_1 be a bounded star shaped set with respect to the origin in $\mathbb{R}^3$. Let Ω be the complement of its closure in $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3)$ denote a generic point in $\mathbb{R}^3$ and let $|x|$ denote the distance from the origin, given by :

$$|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2} .$$

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The problem, we consider, will be denoted in the sequel as the *continuous problem* and is given by :

$$-\Delta \underline{u} + \nabla p = \underline{f} \quad \text{in } \Omega \tag{1.1}$$

$$\operatorname{div} \underline{u} = 0 \quad \text{in } \Omega \tag{1.2}$$

$$\underline{u}\,|_{\delta\Omega} = 0 \tag{1.3}$$

$$\lim_{|x| \to \infty} \underline{u}(x) = 0 \tag{1.4}$$

or symbolically, $S(\underline{u}, p) = F$ where S denotes the Stokes operator.

First, we will need a suitable function space in which to pose the problem (1.1)-(1.4) and establish the existence, uniqueness and regularity of the solution. A variational formulation of the problem is more suitable if the problem is to be approximated later by a finite element method. This has been accomplished in [8]. Also, the continuous problem is not immediately suitable for discretization due to the fact that a finite sized grid would yield an infinite number of unknowns. Thus, we approximate the continuous problem by another problem defined on a finite subdomain Ω_R of the original domain Ω given by :

$$\Omega_R = \Omega \cap B(0\,;R)$$

where $B(0\,;R)$ denotes the sphere of radius R centered at the origin. We will denote this problem as the *truncated problem.* Let $\delta\Omega_R$ denote the boundary introduced by the construction of the truncated domain. Then the truncated problem is given by :

$$\begin{aligned} -\Delta \underline{u}_R + \nabla p_R &= \underline{f}_R && \text{in } \Omega_R \\ \operatorname{div} \underline{u}_R &= 0 && \text{in } \Omega_R \\ \underline{u}_R &= 0 && \text{on } \delta\Omega \\ \beta(\underline{u}_R, p_R) &= 0 && \text{on } \delta\Omega_R \end{aligned}$$

or symbolically, $S_R(\underline{u}_R, p_R) = \underline{F}_R$, where $\beta(\,.\,,\,.\,)$ denotes an artificial boundary condition to be imposed at the artificial boundary $\delta\Omega_R$.

Finally, the truncated problem can be approximated by any method using finite elements or finite difference. It is required to choose the boundary condition at the large boundary such that :

(i) The approximating operator S_R is invertible.

(ii) S_R^{-1} is a good approximation to S^{-1} i.e., there exists a constant $C > 0$ independent of the *truncation parameter* R and a positive constant δ such that :

$$\|(\underline{u}_R, p_R) - (\underline{u}, p)\|_{;\Omega_R} \leq \frac{C}{R^{\delta}}$$

where $\|\cdot\|_{;\Omega_R}$ is a suitable norm [8].

The various methods of approximating exterior problems will differ in their accuracy upon the particular choice of boundary condition at the boundary $\delta\Omega_R$. Ultimately, if $\delta = \infty$, we have what is known as capacitance matrix methods or exact boundary conditions. We would like to point out that this has been applied to the exterior Helmholtz equation [15, 16] and the Laplace equation [17] and [18]. Also this has been done for the exterior Stokes problem in two dimensions [19] and three dimensions [9]. Also for $\delta < \infty$, we mention the work in [10], [3] and [7]. We shall consider in this paper two types of artificial boundary conditions for the Exterior Stokes problem. The first, presented in Section 3, could be physically understood in the sense of the conservation of mass and is given by :

$$\beta_1(\underline{u}_R, p_R) = \underline{u}_R .$$

The second, discussed in Section 4, is given by :

$$\beta_2(\underline{u}_R, p_R) = -p_R \underline{n} + \underline{n} \cdot \operatorname{grad} \underline{u}_R + \frac{1}{R} \underline{u}_R .$$

It will be pointed in Section 4 that the term $1/R\underline{u}_R$ could be dropped from the artificial boundary condition. At that time the boundary condition could be understood in the sense of conservation of momentum. Then, in Section 5, we approximate the truncated problem by finite elements. Finally, combined error estimates will allow the balance of the discretization error with the « truncation » error [7].

2. PRELIMINARIES

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with λ_i, $i = 1, 2, 3$, non-negative integers be a multiindex. Let $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$. We use the weighted Sobolev spaces of Hanouzet [11]. For m a non-negative integer and $\alpha \in \mathbb{R}$ we define the weighted Sobolev space :

$$W^{m,\alpha}(\Omega) = \left\{ u \in D'(\Omega) : \int_\Omega (1 + r^2)^{\alpha - m + |\lambda|} |D^\lambda u|^2 \, dx < \infty, \ |\lambda| \leq m \right\}.$$

We briefly mention some of the properties that are needed in the analysis. The details can be found in [11] and [9].

1. $W^{m,\alpha}(\Omega)$ is a Hilbert space equipped with the inner product

$$(u, v)_{m,\alpha;\Omega} = \sum_{|\lambda| \leq m} \left\{ \int_\Omega (1 + r^2)^{\alpha - m + |\lambda|} D^\lambda u D^\lambda v \, dx \right\}$$

and norm,

$$\|u\|_{m,\alpha;\Omega} = [(u, u)_{m,\alpha;\Omega}]^{1/2} .$$

2. The following imbeddings are continuous :

$$W^{m,\alpha}(\Omega) \to W^{m-1,\alpha-1}(\Omega) \cdots \to W^{0,\alpha-m}(\Omega) .$$

3. There exists a linear boundary operator $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$ with :

$$\gamma : W^{m,\alpha}(\Omega) \to \prod_{j=0}^{j=m-1} W^{m-j-1/2,\alpha}(\delta\Omega)$$

such that

$$\gamma u = (u(\delta\Omega), \partial u/\partial n(\delta\Omega), \ldots, \partial u^{m-1}/\partial n(\delta\Omega))$$

where $\delta\Omega$ denotes the boundary of the domain Ω, and $\partial/\partial n$ denotes the distributional normal derivative and $W^{m-j-1/2,\alpha}(\delta\Omega)$ denote the trace spaces associated with the weighted Sobolev spaces $W^{m-j-1/2,\alpha}(\Omega)$ previously defined. The operator γ is onto.

4. $\gamma^{-1}(0) = \mathring{W}^{m,\alpha}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the $W^{m,\alpha}(\Omega)$ norm and we will denote its dual space by $W^{-m,-\alpha}(\Omega)$ with the norm

$$\|u\|_{-m,-\alpha;\Omega} = \sup_{v \in \mathring{W}^{m,\alpha}(\Omega)} \frac{\langle u, v\rangle}{\|v\|_{m,\alpha;\Omega}} .$$

5. For an exterior domain $W^{m-j-1/2,\alpha}(\delta\Omega) = H^{m-j-1/2}(\delta\Omega)$.

6. $C_b^\infty(\Omega)$ is dense in $W^{m,\alpha}(\Omega)$, where $C_b^\infty(\Omega)$ is the set of infinitely differentiable functions with bounded support in Ω.

7. Let $W^{0,0}(\Omega) = L^2(\Omega)$ be the space of square integrable functions over the domain Ω. We define the space $L_0^2(\Omega)$ to be :

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega) \mid \int_\Omega u\, dx = 0 \right\} .$$

We will also need the following approximation result.

PROPOSITION 2.1 : *Let $u \in C^\infty(\Omega) \cap W^{1,0}(\Omega)$ be such that $u = 0(r^{-\alpha})$ for $\alpha > 1/2$ and $r \geqslant R$. Let $\Omega_\infty = \Omega \backslash \bar{\Omega}_R$ denote the open domain exterior to Ω_R. Then we have the following* :

$$\|u\|_{s,s-1;\Omega_\infty} = 0(R^{-\alpha+1/2}) \quad s \geqslant 0 \tag{2.1}$$

$$\|u\|_{-s,-s+1;\Omega_\infty} = 0(R^{-\alpha+1/2}) \quad s \geqslant 0 \tag{2.2}$$

$$\|u\|_{s-1/2,s-1;\delta\Omega_R} = 0(R^{-\alpha+1/2}) \quad s \geqslant 1/2 \tag{2.3}$$

$$\|u\|_{-s+1/2,-s+1;\delta\Omega_R} = 0(R^{-\alpha+1/2}) \quad s \geqslant 1/2. \tag{2.4}$$

■

Remarks :

1. It is easy to establish the result (2.1) of the proposition by direct integration for s an integer, then proceed by interpolation for s fractional. In a similar way, (2.3) can be established. Finally, (2.2) and (2.4) can be obtained using duality arguments.

2. The proposition shows that all the seminorms converge to their limit on the unbounded domain Ω_∞ at the same rate.

We state the existence theorem for the continuous problem (1.1)-(1.4). The proof can be found in [8, 9]. As in the case of bounded domains, the variational formulation of the problem reduces to a Brezzi type saddle point problem [2].

THEOREM 2.2 : *The variational form of* (1.1)-(1.4) *given by* : *seek* $(\underline{u}, p) \in [\mathring{W}^{1,0}(\Omega)]^3 \times L^2(\Omega)$ *such that* :

$$a(\underline{u}, \underline{v}) + b(p, \underline{v}) = \underline{f}(\underline{v}) \quad \forall \underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3 \tag{2.5}$$

$$b(q, \underline{u}) = 0 \quad \forall q \in L^2(\Omega) \tag{2.6}$$

where,

$$a(\underline{u}, \underline{v}) = \int_\Omega \operatorname{grad} \underline{u} : \operatorname{grad} \underline{v} \, dx$$

and

$$b(p, \underline{v}) = -\int_\Omega p \operatorname{div} \underline{v} \, dx$$

has a unique solution pair $(\underline{u}, p)$ *for* $\underline{f} \in [W^{-1,0}(\Omega)]^3$. *Furthermore, there exists a constant* $C > 0$ *such that* :

$$\|\underline{u}\|_{1,0;\Omega} + \|p\|_{0,0;\Omega} \leq C \, \|\underline{f}\|_{-1,0;\Omega} . \qquad \blacksquare$$

Remarks :

1. If the support of $\underline{f}$ is not compact in Ω, then, the truncated problem that will be considered will correspond to an external force $\underline{f}_R$ with the support of $\underline{f}_R$ in Ω_R and $\underline{f}_R$ being a good approximation of $\underline{f}$. In that case we have the following initial approximation : let $(\underline{u}_1, p_1)$ represent the solution pair of the variational problem (2.5)-(2.6) with $\underline{f}_R$ replacing $\underline{f}$. In that case we will need the intermediate estimate :

$$\|\underline{u} - \underline{u}_1\|_{1,0;\Omega} + \|p - p_1\|_{0,0;\Omega} \leq C \, \|\underline{f} - \underline{f}_R\|_{-1,0;\Omega}$$

and therefore, it is not a loss of generality to consider only the case of the support of $\underline{f}$ compact in $\mathbb{R}^3$.

2. For the purpose of estimating errors between the solution of the truncated problem and the solution of the continuous problem, we need to be aware of the dependence of every constant in every estimate on the truncation parameter R. Thus, in spite of the equivalence of the weighted Sobolev space $[\mathring{W}^{1,0}(\Omega_R)]^3$ and the usual space $[H_0^1(\Omega_R)]^3$, we will still pose the truncated problem on the weighted Sobolev space rather than the regular one due to the fact that the constant existing in the Hardy inequality does not depend on the truncation parameter R while the Poincaré inequality holds with a constant dependent on R.

3. THE FIRST ARTIFICIAL BOUNDARY CONDITION

In this section, we study the approximation of the problem (1.1)-(1.4) by a truncated problem with zero velocity at the artificial interface $\delta\Omega_R$: this approximation will be denoted by the first truncated problem. We pose the problem with the boundary condition :

$$\beta_1(\underline{u}_R, p_R) = \underline{u}_R \,.$$

In this case the variational formulation of the first truncated problem is : seek $(\underline{u}_R, p_R) \in [\mathring{W}^{1,0}(\Omega_R)]^3 \times L_0^2(\Omega_R)$ such that :

$$a(\underline{u}_R, \underline{v}_R) + b(p_R, \underline{v}_R) = \underline{f}_R(\underline{v}_R) \quad \forall \underline{v}_R \in [\mathring{W}^{1,0}(\Omega)]^3 \tag{3.1}$$

$$b(q_R, \underline{u}_R) = 0 \quad \forall q_R \in L^2(\Omega)\,. \tag{3.2}$$

3.1. Existence of the solution

First, we will need to establish the existence of the solution of the first truncated problem (3.1)-(3.2). We intend to omit the details similar to the work in [8, 9], and we refer the reader to these references. The only exception to the work in [9] will be the handling of the tangential component of the velocity at the large boundary $\delta\Omega_R$ where it will be essential to establish estimates independent of the truncation parameter R. The following lemmas are essential for the stability condition [2] of the form $b(.,.)$ on $L_0^2(\Omega_R) \times [\mathring{W}^{1,0}(\Omega_R)]^3$.

LEMMA 3.1 : *Given* $\underline{a} \in [W^{1/2,0}(\delta\Omega_R)]^3$, $\underline{a}\,.\,\underline{n} = 0$, *there exists at least one* $\underline{v}_2 \in [W^{1,0}(\Omega_R)]^3$ *such that :*

$$\begin{aligned} \operatorname{div} \underline{v}_2 &= 0 \; in \; \Omega_R \\ \underline{v}_2 &= 0 \; on \; \delta\Omega \\ \underline{v}_2 &= \underline{a} \; on \; \delta\Omega_R \end{aligned}$$

Furthermore, there exists a constant $C > 0$, independent of R such that :

$$\|\underline{v}_2\|_{1,0;\Omega_R} \leq C \|\underline{a}\|_{1/2,0;\delta\Omega_R}.$$

Proof: This is a constructive proof. First, we would like to point out that the similar result for bounded domains has been established in [20] by using compactness arguments which are not applicable in our case. In addition, we would like to study whether the estimates are independent of the truncation parameter R. Without loss of generality, we show the result for $\delta\Omega_R$ the surface of a sphere. Also we will assume that $\Omega_1 \cap \Omega$ is not empty.

Step 1 : Let $\underline{g}(x) = \underline{a}(Rx/2)$ for $x \in \delta\Omega_2$. We can immediately show that :

$$\|\underline{g}\|_{s,s-1/2;(\delta\Omega_2)} \leq CR^{-1/2} \|\underline{a}\|_{s,s-1/2;\delta\Omega_R}, \quad s \geq 0 \tag{3.3}$$

Step 2 : Given $\underline{g}(x) \in [W^{1/2,0}(\delta\Omega_2)]^3$, by using the similar result for bounded domains, there exists at least one vector $\hat{\underline{v}}_2 \in [W^{1,0}(\Omega_2)]^3$ with

$$\operatorname{div} \hat{\underline{v}}_2 = 0$$

$$\hat{\underline{v}}_2 = \underline{g} \quad \text{on} \quad \delta\Omega_2$$

$\hat{\underline{v}}_2$ can be chosen to have support outside Ω_1. We now have the following estimate :

$$\|\hat{\underline{v}}_2\|_{1,0;\Omega_2} \leq C \|\underline{g}\|_{1/2,0;\delta\Omega_2}. \tag{3.4}$$

Step 3 : Now define $\underline{v}_2(x) = \hat{\underline{v}}_2(x/2R)$ for $1 \leq |x| \leq 2$. We can easily see that

$$\operatorname{div} \underline{v}_2 = 0$$

and we have the following estimate :

$$CR^{-1/2} \|\underline{v}_2\|_{1,0;(\Omega_R)} \leq \|\hat{\underline{v}}_2\|_{1,0;\Omega_2} \tag{3.5}$$

for some constant C independent of R. Finally, combining the estimates (3.3), (3.4) and (3.5), we get :

$$CR^{-1/2} \|\underline{v}_2\|_{1,0;\Omega_R} \leq \|\hat{\underline{v}}_2\|_{1,0;\Omega_2} \leq C \|\underline{g}\|_{1/2,0;\delta\Omega_2} \leq CR^{-1/2} \|\underline{a}\|_{1/2,0;\delta\Omega_R}$$

which completes the proof. ■

COROLLARY 3.2 : *Given* $\underline{a} \in W^{1/2,0}(\delta\Omega \cup \delta\Omega_R)$ *with* $\underline{a} \cdot \underline{n}|_{\delta\Omega \cup \delta\Omega_R}$, *there exists* $\underline{v}_2 \in [W^{1,0}(\Omega_R)]^3$ *with :*

$$\operatorname{div} \underline{v}_2 = 0 \quad in \quad \Omega_R$$

$$\underline{v}_2 = \underline{a} \quad on \quad \delta\Omega \cup \delta\Omega_R$$

with the estimate,

$$\|\underline{v}_2\|_{1,0;\Omega_R} \leqslant C \; \|\underline{a}\|_{1/2,0;\delta\Omega \cup \delta\Omega_R}$$

with C *independent of* R.

Proof: See [8, 9]. ■

LEMMA 3.3 : *Given* $p \in L_0^2(\Omega_R)$, *there exists at least one* $\underline{v} \in [\mathring{W}^{1,0}(\Omega_R)]^3$ *and a constant* $C > 0$, *independent of* R, *such that :*

$$\operatorname{div} \underline{v} = -p \quad in \quad \Omega_R$$

$$\|\underline{v}\|_{1,0;\Omega_R} \leqslant C \; \|p\|_{0,0;\Omega_R}.$$

Proof: See [8, 9]. ■

We are now ready to study the existence of the solution of the first truncated problem. This is clarified in the following result.

COROLLARY 3.4 :

$$\sup_{\underline{v} \in [\mathring{W}^{1,0}(\Omega_R)]^3} \frac{b(p_R, \underline{v})}{\|\underline{v}\|_{1,0;\Omega_R}} \geqslant C \; \|p_R\|_{0,0;\Omega_R} \quad \forall p_R \in L_0^2(\Omega_R). \tag{3.6}$$

■

Remark : The coercivity of the form $a(.,.)$

$$a(\underline{v}, \underline{v}) \geqslant C \; \|\underline{v}\|_{1,0;\Omega_R}^2 \quad \forall \underline{v} \in [\mathring{W}^{1,0}(\Omega_R)]^3$$

also follows from the Hardy inequality [8, 9, 14] with a constant C independent of R.

THEOREM 3.5 : *The variational form of* (3.1)-(3.2) *has a unique solution pair* $(\underline{u}_R, p_R) \in [\mathring{W}^{1,0}(\Omega_R)]^3 \times L_0^2(\Omega_R)$ *for* $\underline{f} \in [W^{-1,0}(\Omega_R)]^3$. *Furthermore, there exists a constant* C *independent of* R *such that :*

$$\|\underline{u}_R\|_{1,0;\Omega_R} + \|p_R\|_{0,0;\Omega_R} \leqslant C \; \|\underline{f}_R\|_{-1,0;\Omega_R}. \qquad ■$$

3.2. Error of the truncated problem

We are now ready to consider the approximation properties of the first truncated problem. Let $\underline{e}_R = \underline{u} - \underline{u}_R$ and $\mu_R = p - p_R$ denote the error in the velocity and the pressure respectively. Then the pair $(\underline{e}_R, \mu_R)$ satisfies :

$$\Delta \underline{e}_R + \nabla \mu_R = \underline{0} \quad \text{in } \Omega_R \tag{3.7}$$
$$\operatorname{div} \underline{e}_R = 0 \quad \text{in } \Omega_R \tag{3.8}$$
$$\underline{e}_R = 0 \quad \text{on } \delta\Omega \tag{3.9}$$
$$\underline{e}_R = \underline{u} \quad \text{on } \delta\Omega_R \,. \tag{3.10}$$

Let $\underline{u}_1$ be any velocity vector satisfying :

$$\underline{u}_1 = 0 \quad \text{on } \delta\Omega$$
$$\underline{u}_1 = \underline{u} \quad \text{on } \delta\Omega_R \,.$$

It follows immediately that the pair $(\underline{e}_R, \mu_R)$ satisfies the variational formulation :

Seek $(\underline{e}_R - \underline{u}_1, \mu_R) \in [\mathring{W}^{1,0}(\Omega_R)]^3 \times L_0^2(\Omega_R)$ such that :

$$a(\underline{e}_R - \underline{u}_1, \underline{v}_R) + b(\mu_R, \underline{v}_R) = -a(\underline{u}_1, \underline{v}_R) \quad \forall \underline{v}_R \in [\mathring{W}^{1,0}(\Omega_R)]^3 \tag{3.11}$$
$$b(q_R, \underline{e}_R - \underline{u}_1) = -b(q_R, \underline{u}_1) \quad \forall q_R \in L_0^2(\Omega_R) \tag{3.12}$$

and therefore we have the following estimate :

$$\|\underline{e}_R\|_{1,0;\Omega_R} + \|\mu_R\|_{0,0;\Omega_R} \leqslant C \;\|\underline{u}_1\|_{1,0;\Omega_R}$$

since $\underline{u}_1$ is arbitrary we obtain :

$$\|\underline{e}_R\|_{1,0;\Omega_R} + \|\mu_R\|_{0,0;\Omega_R} \leqslant$$
$$\leqslant C \inf_{\underline{u}_1 = \underline{u}|\delta\Omega_R} \|\underline{u}_1\|_{1,0;\Omega_R} = C\;\|\underline{u}\|_{1/2,0;\delta\Omega_R} \tag{3.13}$$

where again the constants are independent of R. It is known [14] that, for $\underline{f}$ with compact support, the solution pair $(\underline{u}, p)$ is $0(R^{-1}, R^{-2})$. Therefore, we have the following approximation result.

THEOREM 3.6 :

$$\|\underline{u}_R - \underline{u}\|_{1,0;\Omega_R} + \|p_R - p\|_{0,0;\Omega_R} \leqslant \frac{C}{R^{1/2}} \,. \tag{3.14}$$

■

Remarks :

1. In the estimate (3.14) it is important to note that the difference $p_R - p$ is not measured in $L^2(\Omega_R)$ but rather it is measured in $L_0^2(\Omega_R)$. Similar to bounded domains, the first truncated problem is posed on $L_0^2(\Omega_R)$: the pressure p_R is uniquely determined up to a constant while the pressure p of the continuous problem is determined in $L^2(\Omega) = L^2(\Omega)/\mathbb{R}$ (constants are not in $L^2(\Omega)$). The following argument explains how the sequence of spaces $L_0^2(\Omega_R)$, as a function of the truncation parameter R, does not converge to $L^2(\Omega)$ as $R \to \infty$. Let meas (Ω_R) denote the volume of the domain Ω_R. The constant C that takes any $p \in L^2(\Omega)$ into $p_R \in L_0^2(\Omega_R)$ can be written as :

$$C = -\frac{\int_\Omega p\kappa_R \, dx}{\text{meas }(\Omega_R)}$$

where κ_R denotes the characteristic function of the domain Ω_R. Define the map,

$$\beta_R : L^2(\Omega) \to L_0^2(\Omega_R)$$

by,

$$\beta_R p = p\kappa_R + C$$

we obtain,

$$\|\beta_R p\|_{0,0;\Omega_R}^2 = \|p\|_{0,0;\Omega_R}^2 - C^2 \text{meas }(\Omega_R) \leq 2 \; \|p\|_{0,0;\Omega}^2 \quad (3.15)$$

and therefore,

$$\|\beta_R p - p\|_{0,0;\Omega_R} \leq \|p\|_{0,0;\Omega_R} \leq \|p\|_{0,0;\Omega} \quad (3.16)$$

which does not necessarily converge to zero as $R \to \infty$ for a general $p \in L^2(\Omega)$. When $p \in L^2(\Omega) \cap L^{2-\varepsilon}(\Omega)$ with $0 < \varepsilon < 1$, estimates of the type (3.16) will yield

$$\|\beta_R p - p\|_{0,0;\Omega_R} \leq \|p\|_{L^{2-\varepsilon}(\Omega)} [\text{meas }(\Omega_R)]^{\frac{-\varepsilon}{2(2-\varepsilon)}} \quad (3.17)$$

thus showing convergence. When the support of f is compact in Ω, the pressure $p = 0(R^{-2})$ (i.e., p could be in $L^{2-\varepsilon}(\Omega)$ for $\varepsilon = 1/2$). Hence the constant C will be $0(R^{-2})$ and

$$\|\beta_R p - p\|_{0,0;\Omega_R} = 0(R^{-1/2}) .$$

2. Theorem 3.6 shows that for supp $(\underline{f})$ compact in Ω the convergence rate is only of order of $R^{-1/2}$, which is not satisfactory. Improving the approximating behavior of the truncated problem can be achieved only through the use of higher order boundary conditions. This is considered in the next section.

4. THE SECOND ARTIFICIAL BOUNDARY CONDITION

Again, we consider only the case where $\underline{f}$ with compact support in Ω_R. As in the previous case we discuss first the existence of the solution of the truncated problem which we denote by the second truncated problem. We pose the problem with the boundary condition :

$$\beta_2(\underline{u}_R, p_R) = -p_R\,\underline{n} + \underline{n}\,.\,\mathrm{grad}\,\underline{u}_R + \frac{1}{R}\underline{u}_R \quad \text{on } \delta\Omega_R\,.$$

In this case the variational formulation of the problem is sought in the following space for the velocity :

$$W^R(\Omega_R) = \left\{\kappa_R\,\underline{u}\,|\,\underline{u} \in [\mathring{W}^{1,0}(\Omega)]^3\right\}$$

where κ_R denotes the characteristic function of the domain Ω_R. For the pressure we will use $L^2(\Omega_R)$. In this case the variational formulation of the problem is given by :

Seek $(\underline{u}_R, p_R) \in W^R(\Omega_R) \times L^2(\Omega_R)$ such that :

$$a_1(\underline{u}_R, \underline{v}_R) + b(p_R, \underline{v}_R) = \underline{f}_R(\underline{v}_R) \quad \forall \underline{v}_R \in W^R(\Omega_R) \tag{4.1}$$

$$b(q_R, \underline{u}_R) = 0 \quad \forall q_R \in L^2(\Omega_R) \tag{4.2}$$

where $a_1(\,.\,,\,.\,)$ is given by :

$$a_1(\underline{u}_R, \underline{v}_R) = \int_{\Omega_R} \mathrm{grad}\,\underline{u}_R : \mathrm{grad}\,\underline{v}_R\,dx + \int_{\delta\Omega_R} \frac{1}{R}\underline{u}_R\,.\,\underline{v}_R\,dS\,.$$

Remark : The second term in the form $a_1(\,.\,,\,.\,)$ is added to preserve the coercivity of the form $a_1(.\,,\,.)$ if the bounded star shaped set Λ_1 becomes empty. We note the importance of this term if the constants in the coercivity estimates on $a_1(\,.\,,\,.\,)$ are to be independent of R. This term may be left out in when $\Lambda_1 \neq \Phi$ and zero Dirichlet boundary conditions are applied on $\delta\Omega$.

4.1. Existence of the solution

PROPOSITION 4.1 : *Equipped with the norm* $\| \cdot \|_{1,0;\Omega_R}$, $W^R(\Omega_R)$ *is a Hilbert space.* ■

PROPOSITION 4.2 :

$$W^R(\Omega_R) = \left\{ \underline{u} \in [H^1(\Omega_R)]^3, \underline{u}|_{\delta\Omega} = 0 \right\}. \qquad ■$$

Remark : The proofs of the Propositions 4.1 and 4.2 use the equivalence of the weighted and the unweighted norms on the bounded domain Ω_R [11].

When the bounded set $\Lambda_1 \neq \Phi$ and Dirichlet boundary conditions are considered on $\delta\Omega$ we have the following :

PROPOSITION 4.3 : *The seminorm* $| \cdot |_{1,0;\Omega_R}$, *defined by :*

$$|\underline{u}|_{1,0;\Omega_R} = \left\{ \int_{\Omega_R} |\mathrm{grad}\ \underline{u}|^2\, dx \right\}^{1/2}$$

is a norm on $W^R(\Omega_R)$ *equivalent to the usual unweighted norm.*

Proof: We use the inequality [4] (equation (4.1) chapter 1, section 3)

$$\int_\Theta \frac{\varphi^2}{(1 + |x|)^2}\, dx \leq C \int_\Theta |\underline{v}|^2\, dx$$

where Θ is an unbounded domain, and where :

$$\varphi = \int_L \underline{v} \cdot d\underline{l}$$

with L a line segment $[x_0, x]$, $x \in \Theta$. Now we can easily establish the inequality on the domain Ω_R for the set :

$$\underline{u} \in K = \{ \underline{u} = \kappa_R\, \mathrm{grad}\, \psi,\ \psi \in C_0^\infty(\Omega) \}$$

where now x_0 is any point outside the support of ψ. Then the result can be extended to $W^R(\Omega_R)$ by using the density of K in $W^R(\Omega_R)$. ■

When $\Lambda_1 = \Phi$, we need instead the following proposition :

PROPOSITION 4.4 : *For* $u \in [W^{1,0}(\Omega_R)]^3$ *we have :*

$$\int_{\Omega_R} \frac{u^2}{(1 + r^2)}\, dx \leq C \left[\int_{\delta\Omega_R} \frac{u^2}{(1 + r^2)^{1/2}}\, dS + \int_{\Omega_R} \left[\frac{\partial u}{\partial r} \right]^2 dx \right].$$

Proof :

$$\int_{\delta\Omega_R} \frac{u^2}{(1+r^2)^{1/2}} \underline{e}_r \cdot \underline{n}\, dS = \int_{\Omega_R} \operatorname{div}\left[\frac{u^2}{(1+r^2)} \underline{e}_r\right] dx$$

$$= \int_{\Omega_R} \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2 u^2}{(1+r^2)^{1/2}}\right] dx$$

$$= \int_{\Omega_R} \frac{(2+r^2)\, u^2}{r(1+r^2)^{3/2}}\, dx + 2\int_{\Omega_R} \frac{1}{(1+r^2)^{1/2}}\left[u\frac{\partial u}{\partial r}\right] dx$$

$$\int_{\Omega_R} \frac{u^2}{(1+r^2)}\, dx \leqslant C \int_{\delta\Omega_R} \frac{u^2}{(1+r^2)^{1/2}}\, dS + 2\left[\int_{\Omega_R} \frac{u^2}{(1+r^2)}\, dx\right]^{1/2}\left[\int_{\Omega_R}\left|\frac{\partial u}{\partial r}\right|^2 dx\right]^{1/2}$$

$$\leqslant C \int_{\delta\Omega_R} \frac{u^2}{(1+r^2)^{1/2}}\, dS + \varepsilon\left[\int_{\Omega_R} \frac{u^2}{(1+r^2)}\, dx\right] + \frac{1}{\varepsilon}\left[\int_{\Omega_R}\left[\frac{\partial u}{\partial r}\right]^2 dx\right]$$

for $0 \leqslant \varepsilon < 1$. Finally, by collecting similar terms, we complete the proof. ■

COROLLARY 4.5 : *The symmetric bilinear form $a_1(\,.\,,\,.\,)$ defined on $W^R(\Omega_R) \times W^R(\Omega_R)$ is strongly coercive i.e., there exists a constant C independent of R such that :*

$$a_1(\underline{u}_R, \underline{u}_R) \geqslant C\,\|\underline{u}_R\|^2_{1,0\,;\,\Omega_R} \quad \forall \underline{u}_R \in W^R(\Omega_R)\,. \qquad ■$$

The bilinear form $b(\,.\,,\,.\,)$ is now defined on $L^2(\Omega_R) \times W^R(\Omega_R)$. The continuity of the form on the above spaces is clear, so we proceed to show the stability condition.

LEMMA 4.6 : *Given $p \in L^2(\Omega_R)$, there exists at least one $\underline{v}_R \in W^R(\Omega_R)$ and a constant $C > 0$, independent of R, such that :*

$$\operatorname{div} \underline{v}_R = -p \quad in \quad \Omega_R$$

$$\|\underline{v}_R\|_{1,0\,;\,\Omega_R} \leqslant C\,\|p\|_{0,0\,;\,\Omega_R}\,.$$

Proof: We extend p by zero outside Ω_R and denote the result by ν, then

$$\nu \in L^2(\Omega)$$

and

$$\|\nu\|_{0,0;\Omega} = \|p\|_{0,0;\Omega_R}.$$

Now using the techniques of [9] we can find at least one vector $\underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3$ such that :

$$\operatorname{div} \underline{v} = -\nu \quad \text{in} \quad \Omega$$

with the estimate

$$\|\underline{v}\|_{1,0;\Omega} \leqslant C\,\|\nu\|_{0,0;\Omega}.$$

Now choosing $\underline{v}_R$ to be the restriction of $\underline{v}$ to Ω_R, i.e.,

$$\underline{v}_R = \kappa_R\,\underline{v}$$

we get the required vector and estimate. This completes the proof. ■

We now have the following result :

LEMMA 4.7 :

$$\sup_{\underline{v} \in W^R(\Omega_R)} \frac{b(p_R, \underline{v})}{\|\underline{v}\|_{1,0;\Omega_R}} \geqslant \|p_R\|_{0,0;\Omega_R} \quad \forall p \in L^2(\Omega_R). \qquad ■$$

Thus, we have established the coercivity of the form $a(.,.)$ *on* $W^R(\Omega_R) \times W^R(\Omega_R)$ *as well as the stability of the form* $b(.,.)$ *on* $L^2(\Omega_R) \times W^R(\Omega_R)$. *Now it is possible to establish the result similar to Theorem* 3.5.

THEOREM 4.8 : *The variational form of* (4.1)-(4.2) *has a unique solution pair* $(\underline{u}_R, p_R) \in W^R(\Omega_R) \times L^2(\Omega_R)$ *for* $\underline{f}_R \in [W^{-1,0}(\Omega_R)]^3$. *Furthermore, there exists a constant C independent of R such that :*

$$\|\underline{u}_R\|_{1,0;\Omega_R} + \|p_R\|_{0,0;\Omega_R} \leqslant C\,\|\underline{f}_R\|_{-1,0;\Omega_R}. \tag{4.3}$$

■

4.2. The error in the truncated problem

LEMMA 4.9 [14] : *For R large enough, the solution pair* $(\underline{u}, p)$ *of the exterior Stokes problem, with support of* $\underline{f}$ *compact in* Ω, *satisfies :*

$$|\beta_2(\underline{u}_R, p_R)| = 0(R^{-2}). \qquad ■$$

As introduced earlier, we let $\underline{e}_R = \underline{u} - \underline{u}_R$ *and* $\mu_R = p - p_R$ *denote the error in the velocity and the pressure respectively. Then the pair* $(\underline{e}_R, \mu_R)$ *satisfies :*

$$- \Delta \underline{e}_R + \nabla \mu_R = \underline{0}_R \quad in \quad \Omega_R \tag{4.4}$$

$$\operatorname{div} \underline{e}_R = 0 \quad in \quad \Omega_R \tag{4.5}$$

$$\underline{e}_R = 0 \quad on \quad \delta\Omega \tag{4.6}$$

$$\beta_2(\underline{e}_R, \mu_R) = \beta_2(\underline{u}, p) \quad on \quad \delta\Omega_R . \tag{4.7}$$

The problem of estimating the pair $(\underline{e}_R, \mu_R)$ *can be formulated variationally as follows :*

Seek $(\underline{e}_R, \mu_R) \in W^R(\Omega_R) \times L^2(\Omega_R)$ *such that :*

$$a_1(\underline{e}_R, \underline{v}_R) + b(\mu_R, \underline{v}_R) = \int_{\delta\Omega_R} \beta_2(\underline{u}, p) \cdot \underline{v}_R \, dS_{\delta\Omega_R} \quad \forall \underline{v}_R \in W^R(\Omega_R) \tag{4.8}$$

$$b(q_R, \underline{u}_R) = 0 \quad \forall q_R \in L^2(\Omega_R) . \tag{4.9}$$

Remark : It is important to note that equation (4.7) holds in the sense of $[W^{-1/2,0}(\delta\Omega_R)]^3$.

THEOREM 4.10 : *For* supp $(\underline{f})$ *compact in* Ω *we have* :

$$\|\underline{e}_R\|_{1,0;\Omega_R} + \|\mu_R\|_{0,0;\Omega_R} \leqslant C \; \|\beta_2(\underline{u}, p)\|_{-1/2,0;\delta\Omega_R} \leqslant \frac{C}{R^{1.5}} \tag{4.10}$$

where again the constants are independent of R.

Proof: Using the estimate (4.3) and Proposition 2.1, (4.10) can be established. ■

5. DISCRETIZATION OF THE TRUNCATED PROBLEM

In this section we consider the approximation of the truncated problem by finite elements. We assume that Ω_R is a polygonal domain in R^3. Let $\{\tau_h\}$ be a family of regular triangulizations [9] of Ω_R such that :

$$\bar{\Omega}_R = \bigcup_{K \in \tau_h} K$$

where K denotes a simplex in R^3. We use a notation similar to the notation used in [6]. Let $h(K)$ be defined to be the maximum length of an edge belonging to the simplex K and let h be defined to be :

$$h = \max_{K \in \tau_h} h(K) .$$

Let V and S denote the velocity and pressure spaces associated with any of the truncated problems in Section 3 or Section 4. Let V_h and S_h denote finite dimensional subspaces of V and S, respectively. Define Z_h to be the null space associated with the form $b(.,.)$ and is given by :

$$Z_h = \{\underline{v}_{rh} \in V_h | b(q_{rh}, \underline{v}_{rh}) = 0 \quad \forall q_{rh} \in S_h\}.$$

The variational form of the discrete problem is now given by :

Seek $(\underline{u}_{rh}, p_{rh}) \in V_h \times S_h$ such that :

$$a(\underline{u}_{rh}, \underline{v}_{rh}) + b(p_{rh}, \underline{v}_{rh}) = \underline{f}_{rh}(\underline{v}_{rh}) \quad \forall \underline{v}_{rh} \in V_h \tag{5.1}$$

$$b(q_{rh}, \underline{u}_{rh}) = 0 \quad \forall q_{rh} \in S_h . \tag{5.2}$$

Again, in this section, we still use the weighted norms since our final goal is to derive estimates where the constants depend neither on the truncation parameter R nor the discretization parameter h. The most important question to address is the approximation properties in finite dimensional subspaces of the weighted spaces. Do we still obtain the same qualitative results as in [1, 4, 5, 6, 13, 20] ? The answer to this question is given in the next lemma.

LEMMA 5.1 : *There exists a map denoted by* Π_h

$$\Pi_h : W^{m, m-1}(\Omega_R) \to S_h$$

such that

$$\| v - \Pi_h v \|_{\nu, \nu - 1 ; \Omega_R} \leq C h^{\mu - \nu} | v |_{\mu, \mu - 1 ; \Omega_R}$$

for $\nu = 0, 1$ *and* $\nu + 1 \leq \mu \leq m$.

Proof: Such a map exists for the unweighted Sobolev spaces [4] with the well-known estimate

$$\| v - \Pi_h v \|_{\nu ; \Omega_R} \leq C h^{\mu - \nu} | v |_{\mu, \mu - 1 ; \Omega_R} .$$

We denote the unweighted norms and seminorms by $\| \cdot \|_{m ; \Omega_R}$ and $| \cdot |_{m ; \Omega_R}$ respectively. Now we can immediately write :

$$\| \cdot \|_{\nu . \nu - 1 ; \Omega_R} \leq \| \cdot \|_{\nu ; \Omega_R} \quad \text{for} \quad \nu = 0, 1$$

$$| \cdot |_{\mu . \mu - 1 ; \Omega_R} \geq | \cdot |_{\nu ; \Omega_R} \quad \text{for} \quad \nu \geq 2$$

which completes the proof. ■

5.1. Finite element approximation

Similar to the case of bounded domains, the following assumptions can be easily verified in the weighted norms for a variety of choices of finite element spaces [1, 4, 5, 6, 13, 20].

(A1) There exists a map denoted by

$$r_h \in \zeta([W^{2,1}(\Omega_R)]^3 \cap v \, ; z_h)$$

and a positive integer ν sucht that :

$$b(q_{Rh}, \operatorname{div} r_h \underline{v} - \underline{v}) = 0 \quad \forall q_{Rh} \in S_h$$

with

$$\|\underline{v} - r_h \underline{v}\|_{1,0\,;\Omega_R} \leqslant Ch^m \|v\|_{m+1,m\,;\Omega_R}$$

with $1 \leqslant m \leqslant \nu$.

(A2) The orthogonal projection denoted by

$$\rho_h \in \zeta(W^{m,m}(\Omega_R) \cap S)$$

satisfies :

$$\|q - \rho_h q\|_{0,0\,;\Omega_R} \leqslant Ch^m \|v\|_{m,m\,;\Omega_R} \quad \forall q \in W^{m,m}(\Omega_R) \cap S\,.$$

(A3) For every $q_{Rh} \in S_h$, there exists a $\underline{w}_{Rh} \in V_h$ such that :

$$(q_{Rh} - \operatorname{div}_h \underline{w}_{Rh}, s_h) = 0 \quad \forall s_h \in S_h$$

with

$$\|\underline{w}_{Rh}\|_{1,0\,;\Omega_R} \leqslant C\, \|q_{Rh}\|_{0,0\,;\Omega_R}.$$

We can now state the following theorem.

THEOREM 5.2 : *Under the hypotheses* (A1), (A2) *and* (A3) *problem* (5.1)-(5.2) *has exactly one pair* $(\underline{u}_{Rh}, p_{Rh}) \in V_h \times S_h$ *and* :

$$\lim_{h \to 0} \left\{ \|\underline{u}_{Rh} - \underline{u}_R\|_{1,0\,;\Omega_R} + \|p_{Rh} - p_R\|_{0,0\,;\Omega_R} \right\} = 0\,.$$

Moreover, if the pair

$$(\underline{u}_R, p_R) \in \left\{W^{s+1,s}(\Omega_R) \cap V\right\} \times \left\{W^{s,s}(\Omega_R)\right\}$$

we have the usual bound :

$$\left\{ \|\underline{u}_{Rh} - \underline{u}_R\|_{1,0\,;\,\Omega_R} + \|p_{Rh} - p_R\|_{0,0\,;\,\Omega_R} \right\} \leqslant$$

$$\leqslant Ch^m \left\{ \|\underline{u}_R\|_{m+1,m\,;\,\Omega_R} + \|p_R\|_{m,m\,;\,\Omega_R} \right\}$$

for $1 \leqslant m \leqslant s$, *and* C *independent of* h *and* R. ■

As mentioned earlier typical examples used in the case of bounded domains can be used. Verifying assumptions (A1)-(A3) is no different from the standard treatment with the exception that the results are obtained in the weighted norms. Therefore, we omit the details and refer the reader to the different references on particular choices of finite element spaces (see [1] for example).

5.2. A two parameter approximation

Combining the results concerning the errors due to truncation and the discretization leads to :

THEOREM 5.3 : *There exists a constant* C *independent of* h *and* R *such that :*

$$\left\{ \|\underline{u}_{Rh} - \underline{u}\|_{1,0\,;\,\Omega_R} + \|p_{Rh} - p\|_{0,0\,;\,\Omega_R} \right\} \leqslant C\,\{h^m + R^{-\delta}\} \tag{5.3}$$

where the index $\delta = 0.5,\ 1.5$ *depends on the artificial boundary condition imposed on* $\delta\Omega_R$ *and the index* m *depends on the choice of the finite element spaces used in the discretization of the truncated problem.* ■

As a concluding remark it is essential in this two parameter approximation to maintain the balance between the truncation error and the discretization error for optimality of the approximation. Also, we refer the reader to reference [7] for the discussion of how estimates of the type (5.3) can be used to generate a mesh grading technique where larger simplices can be used away from the support of the forcing term $\underline{f}$ i.e., in the far field. This is of course recommended instead of using a quasiuniform mesh since a smaller number of unknowns can achieve the same accuracy.

Remark : With the application of artificial boundary conditions, it has been observed through numerical experiments that the dependence of the error estimates on the truncation parameter R (i.e., the values of $\delta = 0.5$ and 1.5) does not change by replacing Ω_R by Q where Q is any fixed subdomain of the domain Ω.

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