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APPROXIMATE INERTIAL MANIFOLDS FOR THE PATTERN FORMATION CAHN-HILLIARD EQUATION

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Abstract — *An approximate inertial manifold for an evolution equation is a finite dimensional smooth manifold such that the orbits enter, after a transient time, a very thin neighbourhood of the manifold In this paper, we consider the Cahn-Hilliard equation and we present a method which allows to construct several approximate inertial manifolds providing better and better order approximations to the orbits. These approximate inertial manifolds exist, whether an exact inertial manifold is known to exist or not*

INTRODUCTION

The purpose of this article is to study some questions related to the large time behavior of the solutions of the Cahn-Hilliard equation. This equation is a model for pattern formation in phase transition and describes the so-called spinodal decomposition in binary alloys [1, 2, 6, 12]. The equation, which contains a fourth order dissipative term and a second order anti-dissipative term, reads

$$\frac{\partial u}{\partial t} + \Delta^2 u + a\,\Delta u - b\,\Delta(u^3) = 0\,, \quad a, b > 0\,. \tag{0.1}$$

Problem (0.1) has been studied by several authors [12, 10, 11, 3]. In particular, in space dimension $n \leqslant 3$, (0.1) possesses a global attractor [11]. Also, in the case where the spatial domain is a cube of $\mathscr{R}^n$, $n = 1, 2$, the existence of inertial manifolds has been derived [3, 11] ; we recall that an inertial manifold [5] is a finite dimensional Lipschitz invariant manifold which attracts exponentially all the orbits as time goes to infinity. We will

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consider here Problem (0.1) posed on arbitrary bounded subsets of $\mathcal{R}^n$, $n \leq 3$, and our aim is to show the existence of approximate inertial manifolds.

The concept of approximate inertial manifolds [4] constitutes a substitute to that of inertial manifold when either an inertial manifold does not exist or its existence is not know. These manifolds are finite dimensional smooth manifolds such that all the orbits enter after a transient time a very thin neighbourhood of the manifold. The existence of approximate inertial manifolds has been proved for the two-dimensional Navier-Stokes equations [4, 13] and also for reaction-diffusion equations in high space dimension [8] (for the latest equations non existence results of inertial manifolds are known when $n = 4$ [7]). Let us also mention that the concept of approximate inertial manifolds leads to new numerical schemes, well adapted to the long term integration of evolution equations [9].

We will construct several manifolds providing better and better order approximations to the orbits. We investigate a slightly more general equation than (0.1) which can be rewritten in the abstract form

$$\frac{du}{dt} + A^2 u + Af(u) = 0 ,$$

where $A = -\Delta$ associated to the appropriate boundary condition (Neumann or periodic) and f is a polynomial of odd degree with positive leading coefficient. The equation and its functional setting are described in Section 1. We consider, in Section 2, the orthonormal basis of $L^2(\Omega)$ consisting of the eigenvectors of A

$$Aw_j = \lambda_j w_j , \quad j = 1, 2, \ldots$$
$$0 \leq \lambda_1 \leq \lambda_2, \ldots, \lambda_j \to +\infty \quad \text{as} \quad j \to +\infty .$$

For fixed m, we consider the orthogonal projector P_m in $L^2(\Omega)$ onto the space spanned by $w_1, \ldots, w_m$ and we introduce the corresponding projections of u

$$p_m = P_m u , \quad q_m = (I - P_m) u .$$

We show that, after a transient period, p_m is comparable to u in norm, and q_m is small in comparison with p_m and u. This is the first step in the construction of approximate inertial manifolds, since these manifolds are closely related to convenient approximations of the different terms in the equation for q_m, taking into account the « smallness » of q_m :

$$\frac{dq_m}{dt} + A^2 q_m + (I - P_m) Af(p_m + q_m) = 0 . \tag{0.2}$$

For example, the simplest approximation will be given by

$$A^2 q_m + (I - P_m) A f(p_m) = 0 \tag{0.3}$$

and the corresponding manifold $\mathcal{M}_1$ has the equation

$$q_m = \Phi_1(p_m) ,$$

where, for any given p_m, $\Phi_1(p_m)$ denotes the solution of (0.3). The Sections 3 to 5 contain the main results. We prove the existence of six manifolds $\mathcal{M}_i$, $1 \leq i \leq 6$, of dimension m such that the orbits enter, after a transient time, a neighbourhood of $\mathcal{M}_i$ of thickness $(\lambda_2/\lambda_{m+1})^{i+2}$. These manifolds are analytic and explicitly defined. Of course, in each case, by choosing m sufficiently large, we can make the neighbourhood of $\mathcal{M}_i$ arbitrarily thin (i.e. λ_{m+1}/λ_2 sufficiently large). The manifolds $\mathcal{M}_i$ are defined one after another thanks to improved approximations of (0.2). We believe that the method we present here leads to the construction of a whole family $\mathcal{M}_i$ providing better and better order approximations of the orbits (of the order of $(\lambda_2/\lambda_{m+1})^{i+2}$) and we intend in a separate paper to give the construction of the whole family.

CONTENTS

1. THE EQUATION

Let Ω denote an open bounded set of $\mathcal{R}^n$, $n = 1, 2$ or 3, with a smooth boundary Γ. We consider the following equation involving a scalar function $u = u(x, t)$, $x \in \Omega$, $t \geq 0$

$$\frac{\partial u}{\partial t} - \Delta K(u) = 0 , \qquad \text{in} \quad \Omega \times \mathcal{R}_+ , \tag{1.1}$$

$$K(u) = - \Delta u + f(u) .$$

Here, f is a polynomial of degree $(2p - 1)$ with positive leading coefficient

$$f(u) = \sum_{j=1}^{2p-1} a_j u^j , \quad a_{2p-1} > 0 , \tag{1.2}$$

and we assume that

$$p \geq 2 \quad \text{if} \quad n = 1, 2 \qquad \text{and} \qquad p = 2 \quad \text{if} \quad n = 3 . \tag{1.3}$$

For $p = 2$, one recovers the usual Cahn-Hilliard equation ($f(u) = -au + bu^3$, $a, b > 0$, see [1, 2]).

This equation is supplemented with the initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \Omega , \tag{1.4}$$

and with boundary conditions which can be of either Neumann or periodic type

$$\frac{\partial u}{\partial \nu} = \frac{\partial K(u)}{\partial \nu} = 0 \quad \text{on } \Gamma , \tag{1.5$_1$}$$

$$\Omega = \prod_{i=1}^{n}]0, L_i[\, , \quad L_i > 0 , \qquad \text{and} \quad u \text{ is } \Omega\text{-periodic} . \tag{1.5$_2$}$$

Note that $(1.5)_1$ is also equivalent to

$$\frac{\partial u}{\partial \nu} = \frac{\partial \, \Delta u}{\partial u} = 0 \quad \text{on } \Gamma .$$

For the mathematical setting of the problem, we introduce $H = L^2(\Omega)$ (equipped with its usual scalar product $(. \, , .)$ and norm $| . |$). Let A denote the linear unbounded positive self-adjoint operator on H given by

$$Au = -\Delta u ,$$
$$D(A) = \{u \in H^2(\Omega), \text{the considered boundary condition holds} \} .$$

Then, (1.1) (1.4) (1.5) is equivalent to the abstract evolution equation

$$\frac{du}{dt} + A^2 u + Af(u) = 0 , \tag{1.6}$$

$$u(0) = u_0 . \tag{1.7}$$

As shown in [10], for u_0 given in H, the initial value problem (1.6) (1.7) possesses a unique solution u defined for all $t \geq 0$, such that

$$u \in \mathscr{C}(\mathscr{R}^+ ; H) \cap L^2(0, T ; D(A)) \cap L^{2p}(0, T ; L^{2p}(\Omega)) , \quad \forall T > 0 .$$

Furthermore, if $u_0 \in D(A) \cap L^{2p}(\Omega)$,

$$u \in \mathscr{C}(\mathscr{R}^+ ; D(A) \cap L^{2p}(\Omega)) \cap L^2(0, T ; D(A^2)) , \quad \forall T > 0 .$$

Now, let us recall briefly some of the results in [10, 11] concerning the long time behaviour of the solutions of (1.6) (1.7). It is easy to see that the average of u is conserved

$$\overline{u(t)} = \frac{1}{|\Omega|} \int_{\Omega} u(x, t)\, dx = \bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0(x)\, dx\,, \quad \forall t \geqslant 0\,, \tag{1.8}$$

so that there exists no absorbing set in H for the semi-group $S(t)$ defined by (1.6) (1.7). This difficulty is circumvented by making the semi-group operate in the set

$$H_\alpha = \{u \in H,\ |\bar{u}| \leqslant \alpha\}\ ,$$

where $\alpha \geqslant 0$ is fixed. It can be shown that the semi-group $S(t)$ possesses absorbing sets in H_α and $H_\alpha \cap H^1(\Omega)$ and a global attractor $\mathscr{A}_\alpha$ in H_α. The regularity and the dimension of the attractor are also studied in [11] to which the reader is referred. We will only recall here a time uniform estimate which will be used later. Let u_0 be given in a ball $B(0, R)$ of H_α of center 0 and of radius R. Then there exists a time t_0 depending only on (Ω, f), on α and on R such that

$$\|u\|_{H^2(\Omega)} \leqslant \kappa_0\,, \quad \forall t \geqslant t_0\,, \tag{1.9}$$

where κ_0 denotes a constant depending only on (Ω, f) and α. Alternatively, (1.9) expresses that the ball of center 0 and of radius κ_0 is an absorbing set in $H_\alpha \cap H^2(\Omega)$ for the semi-group $S(t)$.

We conclude this section by stating some time uniform estimates on the time derivatives of u. We set

$$u^{(j)} = \frac{d^j u}{dt^j}\,, \quad j \in \mathcal{N}\,.$$

PROPOSITION 1.1 : *Assume* (1.2) (1.3) *hold. Then the solution u of* (1.6) (1.7) *satisfies*

$$\|u^{(j)}\|_{H^2(\Omega)} \leqslant \kappa_j\,, \quad \forall t \geqslant t_j\,, \quad j \in \mathcal{N}\,, \tag{1.10}$$

where κ_j, $j \in \mathcal{N}$, *depend on* (Ω, f, α) ; t_j, $j \in \mathcal{N}$, *depend on* (Ω, f, α) *and on* R *when* $|u_0| \leqslant R$.

The proof of this Proposition is given in Appendix A.

2. FAST DECAY OF SMALL STRUCTURES

We denote by $\{w_j\}$ the basis of H consisting of the eigenvectors of A

$$A w_j = \lambda_j w_j\,, \quad j = 1, \ldots$$

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \ldots\,; \quad \lambda_j \to +\infty \quad \text{as} \quad j \to +\infty\,.$$

We have $\operatorname{Ker} A = \mathscr{R}$ and the corresponding projection is

$$u \to \bar{u}\,, \qquad \bar{u} \text{ given by (1 8)} \tag{2 1}$$

Let m denote a fixed integer To such an m, we associate the orthogonal projector $P = P_m$ in H onto the space spanned by the first m eigenvectors of A, w_1, , w_m (note that P_1 is given by (2 1)) We set also $Q = Q_m = I - P_m$ and we have an orthogonal decomposition of H

$$H = P_m H \oplus Q_m H$$

For the sake of simplicity, we set

$$\lambda = \lambda_m\,, \quad \Lambda = \lambda_{m+1}$$

and we introduce

$$\delta = \lambda_2/\lambda_{m+1}$$

We associate to any orbit u of (1 6) (1 7) in H its projections

$$p = Pu\,, \quad q = Qu$$

Here p represents a superposition of « large structures » of size larger than $\lambda_m^{-1/2}$ and q represents « small structures » of size smaller than $\lambda_{m+1}^{-1/2}$

We now project (1 6) on PH and QH Since P, Q commute with A and A^2, we obtain a coupled system for p, q

$$\frac{dp}{dt} + A^2 p + PAf(p+q) = 0\,, \tag{2 2}$$

$$\frac{dq}{dt} + A^2 q + QAf(p+q) = 0 \tag{2 3}$$

Hereafter, we denote by κ any constant which depends only on (Ω, f) and α Our goal in this section is to prove that q remains small for large t We will give two results the first one concerns q itself, and the second one deals with its time derivatives

We first derive the

PROPOSITION 2 1 *Assume that* (1 2) (1 3) *hold and let u be a solution of* (1 6) (1 7) *lying in* H_α *Then, for t sufficiently large,* $t \geq t_0'$, *the small structures component of* u, $q = Qu$, *is small in the following sense*

$$|q| \leq \kappa\delta^2\,, \quad |Aq| \leq \kappa\delta\,, \tag{2 4}$$

where t_0' *depends on* (Ω, f, α) *and on* R *when* $|u_0| \leq R$

Proof: Taking the scalar product of (2.3) with $A^2 q$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Aq|^2 + |A^2 q|^2 &= -(Af(p+q), A^2 q), \\ &\leqslant |Af(p+q)| \, |A^2 q| .\end{aligned} \tag{2.5}$$

Since $H^2(\Omega)$ is a multiplicative algebra for $n \leqslant 3$, we infer from (1.2) that

$$|Af(u)| \leqslant \sum_{j=1}^{2p-1} |a_j| \, c_1^j \|u\|_{H^2(\Omega)}^j ,$$

where c_1 denotes a constant depending only on Ω. Combining this inequality with (1.9), we have, for t sufficiently large, $t \geqslant t_0$,

$$|Af(u)| \leqslant \sum_{j=1}^{2p-1} |a_j| \, c_1^j \kappa_0^j = \kappa_1 .$$

Hence, coming back to (2.5),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Aq|^2 + |A^2 q|^2 &\leqslant \kappa_1 |A^2 q| , \quad \forall t \geqslant t_0 , \\ &\leqslant \frac{1}{2}|A^2 q|^2 + \frac{\kappa_1^2}{2} .\end{aligned}$$

$$\frac{d}{dt}|Aq|^2 + |A^2 q|^2 \leqslant \kappa_1^2 , \quad \forall t \geqslant t_0 . \tag{2.6}$$

Due to the definition of Q, we have

$$|A^2 v| \geqslant \Lambda |Av| , \quad |Av| \geqslant \Lambda |v| , \quad \forall v \in QD(A^2) . \tag{2.7}$$

Therefore, (2.6) gives

$$\frac{d}{dt}|Aq|^2 + \Lambda^2 |Aq|^2 \leqslant \kappa_1^2 , \quad \forall t \geqslant t_0 ,$$

and by integration

$$|Aq(t)|^2 \leqslant |Aq(t_0)|^2 \exp(-\Lambda^2(t-t_0)) + \frac{\kappa_1^2}{\Lambda^2} , \quad \forall t \geqslant t_0 .$$

Hence, using (1.9)

$$\begin{aligned}|Aq(t)|^2 &\leqslant \kappa_0^2 \exp(-\Lambda^2(t-t_0)) + \frac{\kappa_1^2}{\Lambda^2} , \quad \forall t \geqslant t_0 , \\ &\leqslant \frac{2\kappa_1^2}{\Lambda^2} , \quad \forall t \geqslant \max\left(t_0, t_0 + \frac{2}{\Lambda^2}\log\frac{\Lambda\kappa_0}{\kappa_1}\right) .\end{aligned}$$

Recalling that $\Lambda = \lambda_{m+1}$ and setting

$$t_0' = \max_{m \in \mathcal{N}} \max \left(t_0, t_0 + \frac{2}{\lambda_{m+1}^2} \log \frac{\lambda_{m+1}\, \kappa_0}{\kappa_1} \right),$$

we have

$$|Aq(t)|^2 \leqslant \frac{2\,\kappa_1^2}{\Lambda^2}, \quad \forall t \geqslant t_0',$$

which implies thanks to (2.7)

$$|q(t)|^2 \leqslant \frac{2\,\kappa_1^2}{\Lambda^4}, \quad \forall t \geqslant t_0'.$$

This shows (2.4) and concludes the proof of Proposition 2.1.

We now show that the time derivatives of q become also small for large t, with the same order of magnitude as q. We set

$$q^{(j)} = \frac{d^j q}{dt^j}, \quad j \geqslant 1.$$

PROPOSITION 2.2 : *The assumptions are* (1.2) (1.3) *and we let* $j \geqslant 1$. *Let* u *be a solution of* (1.6) (1.7) *lying in* H_α. *Then, for* t *sufficiently large,* $t \geqslant t_j'$, $q = Qu$ *is such that*

$$|q^{(j)}| \leqslant \kappa_j\, \delta^2, \; |Aq^{(j)}| \leqslant \kappa_j\, \delta, \tag{2.8}$$

where t_j' *depends on* j, *on* (Ω, f, α) *and on* R *when* $|u_0| \leqslant R$.

Proof: The function $u^{(j)} = d^j u/dt^j$ satisfies an equation of the form

$$\frac{du^{(j)}}{dt} + A^2 u^{(j)} + A(f'(u)\, u^{(j)} + F(u, u^{(1)}, \ldots, u^{(j-1)})) = 0, \tag{2.9}$$

where $F : \mathcal{R}^j \to \mathcal{R}$ is a polynomial. By projection of (2.9) on QH, we obtain

$$\frac{dq^{(j)}}{dt} + A^2 q^{(j)} + AQ(f'(u)\, u^{(j)} + F(u, u^{(1)}, \ldots, u^{(j-1)})) = 0.$$

We take the scalar product of this equation with $A^2 q^{(j)}$ in H:

$$\frac{1}{2}\frac{d}{dt}|Aq^{(j)}|^2 + |A^2 q^{(j)}|^2 =$$

$$= -(A(f'(u)\, u^{(j)} + F(u, u^{(1)}, \ldots, u^{(j-1)})), A^2 q^{(j)}),$$

$$\leqslant |A(f'(u)\, u^{(j)} + F(u, u^{(1)}, \ldots, u^{(j-1)}))|\, |A^2 q^{(j)}|. \tag{2.10}$$

We now assume t sufficiently large, $t \geq t'$, so that (1.9) and (1.10) hold for u and its first j time derivatives $u^{(i)}$, $1 \leq i \leq j$. Then, thanks to the algebra property of $H^2(\Omega)$, it is easy to check that

$$|A(f'(u)\,u^{(j)} + F(u, u^{(1)}, \dots, u^{(j-1)}))| \leq \kappa\,, \quad \forall t \geq t'\,. \tag{2.11}$$

Hence, coming back to (2.10), we have for $t \geq t'$,

$$\frac{1}{2}\frac{d}{dt}|Aq^{(j)}|^2 + |A^2 q^{(j)}|^2 \leq \kappa |A^2 q^{(j)}|\,,$$

$$\leq \frac{1}{2}|A^2 q^{(j)}|^2 + \frac{\kappa^2}{2}\,.$$

$$\frac{d}{dt}|Aq^{(j)}|^2 + |A^2 q^{(j)}|^2 \leq \kappa^2\,. \tag{2.12}$$

The inequality (2.12) is similar to (2.6) and we conclude the proof of Proposition 2.2 by computations similar to the ones following (2.6) in the proof of Proposition 2.1. The details are omitted.

3. THE APPROXIMATE MANIFOLDS $\mathscr{M}_1$ AND $\mathscr{M}_2$

3.1 The approximate manifold $\mathscr{M}_1$

As mentioned in the Introduction, the different approximate inertial manifolds are constructed by introducing simplified forms of equation (2.3). The first manifold $\mathscr{M}_1$ corresponds to the simplest approximation. Thanks to Propositions 2.1 and 2.2, we know that q and $q' = dq/dt$ are small for large time. Therefore, we can expect that $AQf(p)$ is a good approximation of $AQf(p+q)$, while q' can be neglected. This leads us to replace (2.3) by the following approximate equation

$$A^2 q + AQf(p) = 0\,. \tag{3.1}$$

A rigorous proof of this heuristical argument will be given below.

For p given in PH, the resolution of (3.1) is easy and we denote by q_1 its solution

$$q_1 = \Phi_1(p)\,. \tag{3.2}$$

The graph of the function $\Phi_1 : PH \to QD(A^2)$ defines a smooth (analytic) manifold $\mathscr{M}_1$ in H of dimension m. Our aim here is to show that the solutions of (1.6) (1.7) are attracted by a thin neighbourhood of $\mathscr{M}_1$.

THEOREM 3.1 : *Assume that* (1.2) (1.3) *hold. Then for t sufficiently large,* $t \geq t_1^*$, *any orbit of* (1.6) (1.7) *remains at a distance in H of* $\mathscr{M}_1$ *bounded by*

$\kappa\delta^3$, κ *an appropriate constant, the constant* κ *depends on* (Ω, f, α) *and* t_1^* *depends on* (Ω, f, α) *and on* R *when* $|\bar{u}_0| \leq \alpha$, *and* $|u_0| \leq R$

Remark 3 2 The constant κ in Theorem 3 1 depends only on (Ω, f, α), it is in particular independent of u_0 in H_α and m Therefore, the orbits enter a neighbourhood of $\mathcal{M}_1$ that can be made arbitrarily thin by choosing m sufficiently large (i e λ_2/λ_{m+1} sufficiently small) Note also that the transient time t^* is independent of m

Remark 3 3 In view of Proposition 2 1 and Theorem 3 1, the distance of the orbits to $\mathcal{M}_1$ is of better order than their distance to the linear space $q = 0$ This suggests that $\mathcal{M}_1$ gives a better approximation to the orbits than PH This remarks leads to the introduction of new numerical schemes [9]

Proof of Theorem 3 1 Let $u = p + q$ be an orbit of (1 6) (1 7) lying in H_α For every $t > 0$, we define $q_1(t) = \Phi_1(p(t))$ Then, $p(t) + q_1(t)$ lies in $\mathcal{M}_1$ and

$$\begin{aligned} \operatorname{dist}(u(t), \mathcal{M}_1) &= \inf_{v \in \mathcal{M}_1} |u(t) - v|, \\ &\leq |q(t) - q_1(t)| \end{aligned}$$

Therefore, it suffices to evaluate the norm of

$$\chi_1(t) = q_1(t) - q(t)$$

Substracting (2 3) from (3 1) with $q = q_1$ we find

$$A^2 \chi_1 = QA(f(p+q) - f(p)) + q',$$

and

$$|A^2 \chi_1| \leq |A(f(p+q) - f(p))| + |q'| \tag{3 3}$$

We have

$$\begin{aligned} f(p+q) - f(p) &= \int_0^1 f'(p + \theta q)\, q \, d\theta, \\ &= q \int_0^1 f'(p + \theta q)\, d\theta \end{aligned}$$

Next, since $H^2(\Omega)$ is a multiplicative algebra, we obtain

$$|A(f(p+q) - f(p))| \leq c_1 \|q\|_{H^2(\Omega)} \int_0^1 \|f'(p + \theta q)\|_{H^2(\Omega)}\, d\theta \tag{3 4}$$

as well as

$$\|f'(p + \theta q)\|_{H^2(\Omega)} \leq \sum_{i=0}^{2(p-1)} c_1^i (i+1) |a_{i+1}| \, \|p + \theta q\|^i_{H^2(\Omega)}, \ \theta \in \,]0, 1[\tag{3 5}$$

Since p, q are bounded in $H^2(\Omega)$ for large t by constants depending only on (Ω, f, α) (see (1.9)), (3.5) implies that, for large t,

$$\| f'(p + \theta q) \|_{H^2(\Omega)} \leqslant \kappa , \quad \forall \theta \in \,]0, 1[\,.$$

Hence, coming back to (3.4)

$$|A(f(p+q) - f(p))| \leqslant \kappa \|q\|_{H^2(\Omega)} \quad \text{for large } t\,. \tag{3.6}$$

But, $q \in QH$ has zero mean value

$$q \in H_0 = \left\{ v \in H, \int_\Omega v(x)\,dx = 0 \right\} ,$$

and it is classical that, on $D(A) \cap H_0$, $|A\,.\,|$ is a norm equivalent to the one induced by $H^2(\Omega)$: there exists a constant c_2 depending only on Ω such that

$$\|v\|_{H^2(\Omega)} \leqslant c_2 |Av| \,, \quad \forall v \in H_0 \cap D(A)\,. \tag{3.7}$$

Thus, (3.6) gives

$$|A(f(p+q) - f(p))| \leqslant \kappa c_2 |Aq|\,. \tag{3.8}$$

Inserting this inequality in (3.3), we obtain finally

$$\begin{aligned} |A^2 \chi_1| &\leqslant \kappa c_2 |Aq| + |q'|\,, \\ &\leqslant \text{(thanks to (2.4) and (2.8))}\,, \\ &\leqslant \frac{\kappa}{\Lambda} + \frac{\kappa}{\Lambda^2} \leqslant \frac{\kappa}{\Lambda}\,, \end{aligned} \tag{3.9}$$

which yields, since $\chi_1 \in QD(A^2)$,

$$|A\chi_1| \leqslant \frac{\kappa}{\Lambda^2}, \quad |\chi_1| \leqslant \frac{\kappa}{\Lambda^3}\,. \tag{3.10}$$

Theorem 3.1 is proved.

3.2 The approximate manifold $\mathcal{M}_2$

Theorem 3.1 above provides the existence of a manifold $\mathcal{M}_1$ such that the orbits enter a neighbourhood of $\mathcal{M}_1$ of thickness $\kappa\delta^3$. Looking at its proof, we see that this bound on the thickness of the neighbourhood is related to the approximation of the nonlinear term in (2.3) (see (3.9) where the contribution of the nonlinear term is of the order of κ/Λ while the contribution of the time derivative is of the order of κ/Λ^2). By improving the approximation of this term, we now construct a second manifold $\mathcal{M}_2$ which provides a better order approximation to the orbits. Indeed, taking advantage of $q_1(t) = \Phi_1(p(t))$, we can now approximate $QAf(p+q)$ by $QAf(p+q_1)$, and introduce the following simplified form of equation (2.3)

$$A^2 q + QAf(p+q_1) = 0 . \tag{3.11}$$

This leads to the following definition of $\mathcal{M}_2$. For $p \in PH$, we define, as in Section 3.1, q_1 by (3.2). Then, the resolution of (3.11) gives

$$q_2 = \Phi_2(p) . \tag{3.12}$$

The graph of $\Phi_2 : PH \to QD(A^2)$ defines an analytic manifold $\mathcal{M}_2$ of dimension m in H and we can state

THEOREM 3.4 : *Assume that* (1.2) (1.3) *hold. Then, for t sufficiently large,* $t \geq t_2^*$, *any orbit of* (1.6) (1.7) *remains at a distance in H of* $\mathcal{M}_2$ *bounded by* $\kappa\delta^4$, κ *an appropriate constant; the constant* κ *depends on* (Ω, f, α) *and* t_2^* *depends on* (Ω, f, α) *and on R when* $|\bar{u}_0| \leq \alpha$ *and* $|u_0| \leq R$.

Remark 3.5 : A remark similar to Remark 3.2 can be made here. The constants κ and t_2^* are independent of m. The orbits enter a neighbourhood of $\mathcal{M}_2$ that can be made arbitrarily thin by choosing m sufficiently large. Moreover, their distance to $\mathcal{M}_2$ is of order better than to $\mathcal{M}_1$ by a factor δ. Their distance to $\mathcal{M}_2$ is of order better than to PH by a factor δ^2.

Proof of Theorem 3.4 : The proof follows the same steps as that of Theorem 3.1 and is only sketched.

Let $u(t) = p(t) + q(t)$ be an orbit of (1.6) (1.7) lying in H_α. We define $q_1(t)$ by (3.2) and $q_2(t)$ by (3.12) and we aim to estimate $|\chi_2|$ where $\chi_2(t) = q_2(t) - q(t)$. We have

$$A^2\chi_2 = QA(f(p+q) - f(p+q_1)) + q' .$$
$$|A^2\chi_2| \leq A(f(p+q) - f(p+q_1)) + |q'| .$$

Then, as for (3.8), one obtains

$$|A(f(p+q) - f(p+q_1))| \leq \kappa |A(q-q_1)| = \kappa |A\chi_1| .$$

Hence

$$|A^2\chi_2| \leq \kappa |A\chi_1| + |q'| . \tag{3.13}$$

Making use of (3.10) (2.8), we infer from (3.13)

$$|A^2\chi_2| \leq \frac{\kappa}{\Lambda^2} ,$$

which implies, since $\chi_2 \in QD(A^2)$,

$$|A\chi_2| \leq \frac{\kappa}{\Lambda^3}, \quad |\chi_2| \leq \frac{\kappa}{\Lambda^4}. \tag{3.14}$$

This shows Theorem 3.4.

4. THE APPROXIMATE MANIFOLDS $\mathcal{M}_3$ AND $\mathcal{M}_4$

The aim of this Section is to construct two manifolds $\mathcal{M}_3$ and $\mathcal{M}_4$ which provide better order approximations to the orbits than $\mathcal{M}_1$ and $\mathcal{M}_2$. This will in particular be obtained by introducing convenient approximations of the first order time derivatives which were previously neglected in the construction of $\mathcal{M}_1$ and $\mathcal{M}_2$.

4.1 The approximate manifold $\mathcal{M}_3$

The simplified form of equation (2.3) for the manifold $\mathcal{M}_3$ is obtained by approximating $QAf(p+q)$ by $QAf(p+q_2)$, q_2 given by (3.12), and q' by $\bar{q}_1'$ defined as follows. By differentiating (2.3) with respect to t, we find

$$q'' + A^2 q' + QAf'(p+q)(p'+q') = 0. \tag{4.1}$$

In (4.1), p' given by (2.2) is approximated by

$$\bar{p}_1' = -A^2 p - PA(p+q_1), \quad q_1 \text{ given by (3.2)}; \tag{4.2}$$

also q'' is neglected and the nonlinear term $QAf'(p+q)(p'+q')$ is approximated by $QAf'(p)\bar{p}_1'$; the approximate value $\bar{q}_1'$ is given by

$$A^2\bar{q}_1' + QAf'(p)\bar{p}_1' = 0. \tag{4.3}$$

Hence, (2.3) is now replaced by the approximate equation

$$\bar{q}_1' + A^2 q + QAf(p+q_2) = 0. \tag{4.4}$$

The manifold $\mathcal{M}_3$ is therefore defined as follows. For $p \in PH$, we define as in Section 3, q_1 and q_2 by (3.2) and (3.12). Then, we define $\bar{p}_1'$ by (4.2) and the resolution of (4.3) gives $\bar{q}_1'$. Finally, by solving (4.4), we obtain

$$q_3 = \Phi_3(p). \tag{4.5}$$

The graph of the function $\Phi_3 : PH \to QD(A^2)$ defines an analytic manifold $\mathcal{M}_3$ of dimension m in H. This manifold provides a better order approximation to the orbits than $\mathcal{M}_2$ and this is stated in

THEOREM 4.1 : *Assume that* (1.2) (1.3) *hold. Then, for t sufficiently large, $t \geq t_3^*$, any orbit of* (1.6) (1.7) *remains at a distance in H of $\mathcal{M}_3$ bounded by $\kappa\delta^5$, κ an appropriate constant ; the constant κ depends on (Ω, f, α) and t_3^* depends on (Ω, f, α) and on R when $|\bar{u}_0| \leq \alpha$ and $|u_0| \leq R$.*

Remark 4.2 : The constants κ and t_3^* are independent of m. The orbits enter a neighbourhood of $\mathcal{M}_3$ that can be made arbitrarily thin, by choosing m sufficiently large. Moreover, their distance to $\mathcal{M}_3$ is of order better than to $\mathcal{M}_2$ by a factor δ. Their distance to $\mathcal{M}_3$ is of order better than to PH by a factor δ^3.

Proof of Theorem 4.1 : Let $u(t) = p(t) + q(t)$ be an orbit of (1.6) (1.7) lying in H_α. We define $q_1(t)$ and $q_2(t)$ by (3.2) and (3.12), $\bar{p}_t'(t)$ and $\bar{q}_1'(t)$ by (4.2) and (4.3), $q_3(t)$ by the resolution of (4.4). In order to evaluate the distance of $u(t)$ to $\mathcal{M}_3$, it suffices to estimate $|\chi_3|$ where $\chi_3(t) = q_3(t) - q(t)$. Substracting (2.3) from (4.4) with $q = q_3$, we obtain

$$A^2 \chi_3 = q' - \bar{q}_1' + QA(f(p+q) - f(p+q_2)) . \tag{4.6}$$

Since p, q and q_2 are bounded in $H^2(\Omega)$ for large t by constants depending only on (Ω, f, α), using the algebra property of $H^2(\Omega)$, one can show as for (3.8) that, for large t,

$$\begin{aligned} |QA(f(p+q) - f(p+q_2))| &\leq \kappa |A(q-q_2)| = \kappa |A\chi_2| , \\ &\leq (\text{using } (3.14)) , \\ &\leq \frac{\kappa}{\Lambda^3} . \end{aligned} \tag{4.7}$$

Next, we claim that

$$|q' - \bar{q}_1'| \leq \frac{\kappa}{\Lambda^3} , \quad \text{for large } t . \tag{4.8}$$

Indeed, substracting (4.1) from (4.3), we obtain

$$A^2(\bar{q}_1' - q') = q'' + QA[f'(p+q)(p'+q') - f'(p)\bar{p}_1'] .$$

Hence

$$\begin{aligned} |A^2(\bar{q}_1' - q')| \leq |q''| &+ |Af'(p+q)q'| + \\ &+ |A(f'(p+q) - f'(p))p'| + |Af'(p)(p' - \bar{p}_1')| . \end{aligned} \tag{4.9}$$

We now majorize the different terms in the right-hand side of (4.9). By (2.8), we have

$$|q''| \leq \frac{\kappa}{\Lambda^2} , \quad \text{for large } t . \tag{4.10}$$

Then, the algebra property of $H^2(\Omega)$ implies

$$\begin{aligned} |Af'(p+q)\,q'| &\leq c_1 \|f'(p+q)\|_{H^2(\Omega)} \|q'\|_{H^2(\Omega)} \\ &\leq (\text{by } (1.9)) \\ &\leq \kappa \|q'\|_{H^2(\Omega)} . \end{aligned} \tag{4.11}$$

Since $q' \in H_0$, using (3.7), we infer from (4.11)

$$\begin{aligned} |Af'(p+q)\,q'| &\leq c_2\,\kappa\,|Aq'| \;, \\ &\leq (\text{thanks to } (2.8)) \;, \\ &\leq \frac{\kappa}{\Lambda} . \end{aligned} \tag{4.12}$$

Next, similar arguments give successively

$$\begin{aligned} |A(f'(p+q) - f'(p))\,p'| &\leq c_1 \|f'(p+q) - f'(p)\|_{H^2(\Omega)} \|p'\|_{H^2(\Omega)} \,, \\ &\leq c_1\,\kappa \|q\|_{H^2(\Omega)} \|p'\|_{H^2(\Omega)} \,, \\ &\leq c_1\, c_2^2\, \kappa |Aq|\; |Ap'| \;, \\ &\leq c_1\, c_2^2\, \kappa |Aq|\; |Au'| \;, \end{aligned}$$

which yields, along with (2.4) (1.10)

$$|A(f'(p+q) - f'(p))\,p'| \leq \frac{\kappa}{\Lambda} . \tag{4.13}$$

Finally, for the last term in the right-hand side of (4.9)

$$\begin{aligned} |Af'(p)(p' - \bar{p}_1')| &\leq c_1 \|f'(p)\|_{H^2(\Omega)} \|p' - \bar{p}_1'\|_{H^2(\Omega)} \,, \\ &\leq \kappa \|p' - \bar{p}_1'\|_{H^2(\Omega)} . \end{aligned} \tag{4.14}$$

Substracting (4.2) from (2.2), we obtain

$$\begin{aligned} p' - \bar{p}_1' &= PA[f(p+q_1) - f(p+q)] \\ |p' - \bar{p}_1'| &\leq |A(f(p+q_1) - f(p+q))| \;, \\ &\leq \kappa |A(q_1 - q)| = \kappa |A\chi_1| \;, \\ &\leq (\text{by } (3.10)) \;, \\ &\leq \frac{\kappa}{\Lambda^2} . \end{aligned} \tag{4.15}$$

Due to the definition of PH, we have that

$$|A^2 v| \leq \lambda |Av| \;, \quad |Av| \leq \lambda |v| \;, \quad \forall v \in PH . \tag{4.16}$$

Also, since $\{|\,.\,|^2 + |A\,.\,|^2\}^{1/2}$ is on $D(A)$ a norm equivalent to the one induced by $H^2(\Omega)$, there exists a constant c_3 depending only on Ω such that

$$\|v\|_{H^2(\Omega)} \leq c_3\{|v|^2 + |Av|^2\}^{1/2}, \quad \forall v \in D(A). \tag{4.17}$$

Therefore, combining (4.17) (4.16) (4.15), we obtain

$$\begin{aligned}\|p' - \bar{p}_1'\|_{H^2(\Omega)} &\leq c_3\{|p' - \bar{p}_1'|^2 + |A(p' - \bar{p}_1')|^2\}^{1/2}, \\ &\leq c_3(1 + \lambda^2)^{1/2}\,|p' - \bar{p}_1'| \\ &\leq \frac{\kappa}{\Lambda}.\end{aligned} \tag{4.18}$$

This gives, coming back to (4.14)

$$|Af'(p)(p' - \bar{p}_1')| \leq \frac{\kappa}{\Lambda}. \tag{4.19}$$

To conclude, by combining (4.9) and the estimates (4.10) (4.12) (4.13) (4.19), we obtain that

$$|A^2(q' - \bar{q}_1')| \leq \frac{\kappa}{\Lambda} \tag{4.20}$$

which gives (4.8), since $q' - \bar{q}_1' \in QD(A^2)$. Finally, it follows from (4.6) (4.7) (4.8) that

$$|A^2\chi_3| \leq \frac{\kappa}{\Lambda^3},$$

which yields since $\chi_3 \in QD(A^2)$,

$$|A\chi_3| \leq \frac{\kappa}{\Lambda^4}, \quad |\chi_3| \leq \frac{\kappa}{\Lambda^5}, \tag{4.21}$$

Theorem 4.1 is proved.

4.2 The approximate manifold $\mathcal{M}_4$

This new manifold will give better order approximation to the orbits thanks to improved approximations of the nonlinear term and of q' in (2.3) (while q'' is still neglected). Making use of $q_3 = \Phi_3(p)$, $QAf(p+q)$ is now approximated by $QAf(p+q_3)$. We also define a new approximate value of p', namely $\bar{p}_2'$, by

$$\bar{p}_2' = -A^2p - PAf(p+q_2), \qquad q_2 \text{ given by (3.12)}, \tag{4.22}$$

and a new approximate value of q', namely $\bar{q}_2'$, by

$$A^2\bar{q}_2' + QAf'(p+q_1)(\bar{p}_2' + \bar{q}_1') = 0, \quad q_1, \bar{q}_1' \text{ given by (3.2) (4.3)}. \tag{4.23}$$

The simplified form of (2 3) is given by

$$\bar{q}_2' + A^2 q + QAf(p + q_3) = 0 \tag{4 24}$$

To p given in PH, we associate q_1 by (3 2), q_2 by (3 12), $\bar{q}_1'$, q_3 by (4 3) (4 5) We then define $\bar{p}_2'$ and $\bar{q}_2'$ by (4 22) (4 23) Finally the resolution of (4 24) gives

$$q_4 = \Phi_4(p) \tag{4 25}$$

The graph of Φ_4 $PH \to QD(A^2)$ defines an analytic manifold $\mathscr{M}_4$ of dimension m in H The orbits enter a thin neighbourhood of $\mathscr{M}_4$, as shown in

THEOREM 4 3 *Assume that* (1 2) (1 3) *hold Then for t sufficiently large, $t \geq t_4^*$, any orbit of* (1 6) (1 7) *remains at a distance in H of $\mathscr{M}_4$ bounded by $\kappa\delta^6$, κ an appropriate constant, the constant κ depends on (Ω, f, α) and t_4^* depends on (Ω, f, α) and on R when $|\bar{u}_0| \leq \alpha$ and $|u_0| \leq R$*

Remark 4 4 The constants κ and t_4^* are independent of m The orbits enter a neighbourhood of $\mathscr{M}_4$ that can be made arbitrarily thin, by choosing m sufficiently large Their distance to $\mathscr{M}_4$ is of order better than to $\mathscr{M}_3$ by a factor δ and to $\mathscr{M}_4$ than to PH by a factor δ^4

Proof of Theorem 4 3 The proof follows the same steps as that of Theorem 4 1 and we only give here the main estimates Let $u(t) = p(t) + q(t)$ be an orbit of (1 6) (1 7) lying in H_α We define $q_1(t)$ by (3 2), $q_2(t)$ by (3 12), $\bar{q}_1'(t)$, $q_3(t)$ by (4 3) (4 5), $\bar{p}_2'(t)$, $\bar{q}_2'(t)$ by (4 22) (4 23) , $q_4(t)$ is given by the resolution of (4 24) and we aim to estimate $\chi_4(t) = q_4(t) - q(t)$ We have

$$\begin{aligned} A^2\chi_4 &= q' - \bar{q}_2' + QA(f(p + q_3) - f(p + q)) \\ |A^2\chi_4| &\leq |q' - \bar{q}_2'| + |A(f(p + q_3) - f(p + q))| \\ &\leq |q' - \bar{q}_2'| + \kappa|A(q_3 - q)| \\ &\leq \text{(thanks to (4 21))} \\ &\leq |q' - \bar{q}_2'| + \frac{\kappa}{\Lambda^4} \end{aligned} \tag{4 26}$$

Then, by substracting (4 1) from (4 23), we obtain

$$\begin{aligned} A^2(\bar{q}_2' - q') &= q'' + QA\,[f'(p + q)(p' + q') - f'(p + q_1)(\bar{p}_2' + \bar{q}_1')] \\ &= q'' + QA\,[(f'(p + q) - f'(p + q_1))(p' + q') + \\ &\quad + f'(p + q_1)(q' - \bar{q}_1') + f'(p + q_1)(p' - \bar{p}_2')]\,, \end{aligned}$$

which yields for large t,

$$|A^2(\bar{q}_2' - q')| \leqslant |q''| + \kappa|A(q - q_1)| + \kappa|A(q' - \bar{q}_1')| + \\ + \kappa\|p' - \bar{p}_2'\|_{H^2(\Omega)} . \quad (4.27)$$

Then, using (2.8) (3.10) (4.20), we infer from (4.27)

$$|A^2(\bar{q}_2' - q')| \leqslant \frac{\kappa}{\Lambda^2} + \kappa\|p' - \bar{p}_2'\|_{H^2(\Omega)} . \quad (4.28)$$

From (2.2) (4.22), we have

$$p' - \bar{p}_2' = PA(f(p + q_2) - f(p + q))$$

which, using also (3.14), implies for large t,

$$\begin{aligned} |p' - \bar{p}_2'| &\leqslant \kappa|A(q_2 - q)| , \\ &\leqslant \frac{\kappa}{\Lambda^3} . \end{aligned} \quad (4.29)$$

Hence

$$\|p' - \bar{p}_2'\|_{H^2(\Omega)} \leqslant \frac{\kappa}{\Lambda^2} . \quad (4.30)$$

Combining (4.30) with (4.28), we obtain

$$\begin{aligned} &|A^2(\bar{q}_2' - q')| \leqslant \frac{\kappa}{\Lambda^2} , \\ &|A(\bar{q}_2' - q')| \leqslant \frac{\kappa}{\Lambda^3} , \ |\bar{q}_2' - q'| \leqslant \frac{\kappa}{\Lambda^4} \end{aligned} \quad (4.31)$$

and, coming back to (4.26),

$$|A^2 \chi_4| \leqslant \frac{\kappa}{\Lambda^4} ,$$

$$|A\chi_4| \leqslant \frac{\kappa}{\Lambda^5} , \quad |\chi_4| \leqslant \frac{\kappa}{\Lambda^6} . \quad (4.32)$$

Theorem 4.3 is proved.

5. THE APPROXIMATE MANIFOLDS $\mathcal{M}_5$ AND $\mathcal{M}_6$

The goal of this section is to derive the existence of two more manifolds $\mathcal{M}_5$ and $\mathcal{M}_6$ which give better order approximations to the orbits than $\mathcal{M}_3$ and $\mathcal{M}_4$. These manifolds will be constructed by considering in particular approximations of the second order time derivative of q (which up to now was neglected).

5.1 The approximate manifold $\mathcal{M}_5$

The equation for q'' reads

$$q''' + A^2 q'' + QA\,[f'(p+q)(p''+q'') + f''(p+q)(p'+q')^2] = 0 \quad (5.1)$$

and p'' is given by

$$p'' = -A^2 p' - PAf'(p+q)(p'+q')\,. \quad (5.2)$$

We first define a new approximation of p', namely $\bar{p}_3'$, by

$$\bar{p}_3' = -A^2 p - PAf(p+q_3)\,, \quad (5.3)$$

and an approximation of p'', namely $\bar{p}_1''$, by

$$\bar{p}_1'' = -A^2\bar{p}_3' - PAf'(p+q_1)(\bar{p}_2' + \bar{q}_1')\,. \quad (5.4)$$

Then, in (5.1), q''' is neglected and the approximate value of q'', namely $\bar{q}_1''$, is given by the resolution of

$$A^2\bar{q}_1'' + QA\,[f'(p)\,\bar{p}_1'' + f''(p)(\bar{p}_1')^2] = 0\,. \quad (5.5)$$

Finally, the new approximation of q', namely $\bar{q}_3'$, is defined by

$$\bar{q}_1'' + A^2\bar{q}_3' + QAf'(p+q_2)(\bar{p}_3' + \bar{q}_2') = 0\,, \quad (5.6)$$

and the approximate form of (2.3) is

$$\bar{q}_3' + A^2 q + QAf(p+q_4) = 0\,. \quad (5.7)$$

Note that the formulas (5.3) (5.7) are similar to (4.22) (4.24), while (5.6) differs mainly from (4.23) by the introduction of the approximation of q''.

This leads to the following definition of $\mathcal{M}_5$. For $p \in PH$, we define q_1 by (3.2), q_2 by (3.12), $\bar{p}_1'$, $\bar{q}_1'$, q_3 by (4.2) (4.3) (4.5), $\bar{p}_2'$, $\bar{q}_2'$, q_4 by (4.22) (4.23) (4.25), $\bar{p}_3'$, $\bar{p}_1''$, $\bar{q}_1''$, $\bar{q}_3'$, by (5.3)-(5.6) and the resolution of (5.7) gives finally

$$q_5 = \Phi_5(p)\,. \quad (5.8)$$

The graph of $\Phi_5 : PH \to QD(A^2)$ defines an analytic manifold $\mathcal{M}_5$ of dimension m in H and we have

THEOREM 5.1 : *Assume that* (1.2) (1.3) *hold. Then for t sufficiently large,* $t \geq t_5^*$, *any orbit of* (1.6) (1.7) *remains at a distance in H of* $\mathcal{M}_5$ *bounded by*

$\kappa\delta^7$, κ *an appropriate constant; the constant* κ *depends on* (Ω, f, α) *and* t_5^* *depends on* (Ω, f, α) *and on* R *when* $|\bar{u}_0| \leq \alpha$ *and,* $|u_0| \leq R$.

Remark 5.2 : The constant κ and t_5^* are independent of m. The orbits enter a neibourhood of $\mathcal{M}_5$ that can be made arbitrarily thin, by choosing m sufficiently large. Their distance to $\mathcal{M}_5$ is of order better than to $\mathcal{M}_4$ by a factor δ and to $\mathcal{M}_5$ than to PH by a factor δ^5.

Proof of Theorem 5.1 : Let $u(t) = p(t) + q(t)$ be an orbit of (1.6) (1.7) lying in H_α. We define $q_1(t)$ by (3.2), $q_2(t)$ by (3.12), $\bar{p}_1'(t)$, $\bar{q}_1'(t)$, $q_3(t)$ by (4.2) (4.3) (4.5), $\bar{p}_2'(t)$, $\bar{q}_2'(t)$, $q_4(t)$ by (4.22) (4.23) (4.25), $\bar{p}_3'(t)$, $\bar{p}_1''(t)$, $\bar{q}_1''(t)$, $\bar{q}_3'(t)$ by (5.3)-(5.6) and $q_5(t)$ by (5.8). We aim to estimate the norm of $\chi_5(t) = q_5(t) - q(t)$.

We start by proving the following Lemma which gives the order of the different approximations introduced above.

LEMMA 5.3 : *For sufficiently large t,* $t \geq t^*$, *we have*

$$|p' - \bar{p}_3'| \leq \frac{\kappa}{\Lambda^4}, \tag{5.9}$$

$$|p'' - \bar{p}_1''| \leq \frac{\kappa}{\Lambda^2}, \tag{5.10}$$

$$|q'' - \bar{q}_1''| \leq \frac{\kappa}{\Lambda^3}, \tag{5.11}$$

$$|q' - \bar{q}_3'| \leq \frac{\kappa}{\Lambda^5}. \tag{5.12}$$

Proof: Substracting (2.2) from (5.3) we have

$$\bar{p}_3' - p' = PA(f(p+q) - PAf(p+q_3)). \tag{5.13}$$

Since, p, q and q_3 are bounded in $H^2(\Omega)$ for large t by constants depending only on (Ω, f, α), it follows from (5.13) by using the algebra property of $H^2(\Omega)$

$$\begin{aligned} |\bar{p}_3' - p'| &\leq \kappa \|q - q_3\|_{H^2(\Omega)}, \\ &\leq \text{(thanks to (3.7))}, \\ &\leq \kappa |A(q - q_3)| = \kappa |A\chi_3|, \end{aligned}$$

which, along with (4.21), gives (5.9).

Then, (5.2) and (5.4) imply

$$\bar{p}_1'' - p'' = A^2(p' - \bar{p}_3') + PA(f'(p+q)(p'+q') - f'(p+q_1)(\bar{p}_2' + \bar{q}_1')). \tag{5.14}$$

$$\begin{aligned} |\bar{p}_1'' - p''| \leq{} & |A^2(p' - \bar{p}_3')| + |A(f'(p+q) - f'(p+q_1))(p'+q')| + \\ & + |Af'(p+q_1)(p' - \bar{p}_2')| + |Af'(p+q_1)(q' - \bar{q}_1')|. \end{aligned}$$

Making use of (4.16) and (5.9), we have

$$\begin{aligned} |A^2(p' - \bar{p}_3')| &\leq \lambda^2 |p' - \bar{p}_3'| \\ &\leq \frac{\kappa}{\Lambda^2}. \end{aligned} \tag{5.15}$$

Also,

$$\begin{aligned} |A(f'(p+q) - f'(p+q_1))| &\leq \kappa |A(q-q_1)| , \\ &\leq (\text{by } (3.10)) , \\ &\leq \frac{\kappa}{\Lambda^2}. \end{aligned} \tag{5.16}$$

Next

$$|Af'(p+q_1)(p' - \bar{p}_2')| \leq \kappa \|p' - \bar{p}_2'\|_{H^2(\Omega)} ,$$

which yields thanks to (4.17) (4.16) (4.30)

$$\begin{aligned} |Af'(p+q_1)(p' - \bar{p}_2')| &\leq \kappa c_3 (1 + \lambda^2)^{1/2} |p' - \bar{p}_2'| , \\ &\leq \frac{\kappa}{\Lambda^2}. \end{aligned} \tag{5.17}$$

Finally, for the last term in the right-hand side of (5.14)

$$\begin{aligned} |Af'(p+q_1)(q' - \bar{q}_1')| &\leq \kappa |A(q' - \bar{q}_1')| , \\ &\leq (\text{thanks to } (4.20)) , \\ &\leq \frac{\kappa}{\Lambda^2}. \end{aligned} \tag{5.18}$$

Combining (5.14) and the estimates (5.15)-(5.18) provides

$$|\bar{p}_1'' - p''| \leq \frac{\kappa}{\Lambda^2} ,$$

i.e. (5.10).

We now aim to show (5.11). By (5.1) (5.5), we have

$$\begin{aligned} A^2(\bar{q}_1'' - q'') = q''' + QA [f'(p+q)(p'' + q'') + \\ + f''(p+q)(p' + q')^2 - f'(p) \bar{p}_1'' - f''(p)(\bar{p}_1')^2] . \end{aligned} \tag{5.19}$$

Using again the algebra property of $H^2(\Omega)$ and (3.7), it follows from (5.19)

$$\begin{aligned} |A^2(\bar{q}_1'' - q'')| \leq |q'''| + \kappa \{ |Aq| + |Aq''| + \|p'' - \bar{p}_1''\|_{H^2(\Omega)} + \\ + \|p' - \bar{p}_1'\|_{H^2(\Omega)} + |Aq'| \} \end{aligned}$$

which along with (2.8) (2.4) (5.10) (4.18) gives

$$|A^2(\bar{q}_1'' - q'')| \leqslant \frac{\kappa}{\Lambda},$$

hence (5.11).

Finally, substracting (4.1) from (5.6), we have

$$A^2(\bar{q}_3' - q') = q'' - \bar{q}_1'' + QA(f'(p+q)(p'+q') - f'(p+q_2)(\bar{p}_3' + \bar{q}_2')).$$

This yields

$$|A^2(\bar{q}_3' - q')| \leqslant$$
$$\leqslant |q'' - \bar{q}_1''| + \kappa\{|A(q-q_2)| + \|p' - \bar{p}_3'\|_{H^2(\Omega)} + |A(q' - \bar{q}_2')|\},$$

and by virtue of (5.11) (3.14) (5.9) (4.31)

$$|A^2(\bar{q}_3' - q')| \leqslant \frac{\kappa}{\Lambda^3},$$

hence (5.12).

The proof of Lemma 5.3 is complete.

It is now easy to conclude the proof of Theorem 5.1. Substracting (2.3) from (5.7) with $q = q_5$, we obtain

$$\begin{aligned} A^2\chi_5 &= q' - \bar{q}_3' + QA(f(p+q) - f(p+q_4)). \\ |A^2\chi_5| &\leqslant |q' - \bar{q}_3'| + |A(f(p+q) - f(p+q_4))| \\ &\leqslant |q' - \bar{q}_3'| + \kappa|A(q-q_4)|. \end{aligned} \tag{5.20}$$

Therefore, using (5.12) (4.32) we infer from (5.20)

$$|A^2\chi_5| \leqslant \frac{\kappa}{\Lambda^5}.$$

which gives

$$|A\chi_5| \leqslant \frac{\kappa}{\Lambda^6}, \qquad |\chi_5| \leqslant \frac{\kappa}{\Lambda^7}.$$

Theorem 5.1 is proved.

5.2 The approximate manifold $\mathcal{M}_6$

This manifold is constructed by improving the different approximations of Section 5.1. In (5.1), we now approximate p' by

$$\bar{p}_4' = -A^2p - PAf(p+q_4), \tag{5.21}$$

and p'' by

$$\bar{p}_2'' = -A^2 \bar{p}_4' - PAf'(p + q_2)(\bar{p}_3' + \bar{q}_2') . \tag{5.22}$$

Then, the new approximate value of q'', namely $\bar{q}_2''$, is given by

$$A^2 \bar{q}_2'' + QA[f'(p + q_1)(\bar{p}_2'' + \bar{q}_1'') + f''(p + q_1)(\bar{p}_2' + \bar{q}_1')^2] = 0 , \tag{5.23}$$

and the new approximate value of q', namely $\bar{q}_4'$, by

$$\bar{q}_2'' + A^2 \bar{q}_4' + QAf'(p + q_3)(\bar{p}_4' + \bar{q}_3') = 0 . \tag{5.24}$$

The simplified form of (2.3) is here

$$\bar{q}_4' + A^2 q + QAf(p + q_5) = 0 . \tag{5.25}$$

The manifold $\mathscr{M}_6$ is defined as follows. To $p \in PH$, we associate q_i, $1 \leqslant i \leqslant 5$, $\bar{p}_i'$, $i = 2, 3$, $\bar{q}_i'$, $1 \leqslant i \leqslant 3$, $\bar{q}_1''$, defined in the previous Sections. Then (5.21)-(5.24) give $\bar{p}_4'$, $\bar{p}_2''$, $\bar{q}_2''$, $\bar{q}_4'$ and the resolution of (5.25) provides

$$q_6 = \Phi_6(p) . \tag{5.26}$$

The graph of $\Phi_6 : PH \to QD(A^2)$ defines an analytic manifold $\mathscr{M}_6$ of dimension m in PH and we have

THEOREM 5.4 : *Assume that* (1.2) (1.3) *hold. Then for t sufficiently large, $t \geqslant t_6^*$, any orbit of* (1.6) (1.7) *remains at a distance in H of $\mathscr{M}_6$ bounded by $\kappa\delta^8$, κ an appropriate constant ; the constant κ depends on (Ω, f, α) and t_6^* depends on (Ω, f, α) and on R when $|\bar{u}_0| \leqslant \alpha$, $|u_0| \leqslant R$.*

Remark 5.5 : The constants κ and t_6^* are independent of m. The orbits enter a neighbourhood of $\mathscr{M}_6$ that can be made arbitrarily thin, by choosing m sufficiently large. Their distance to $\mathscr{M}_6$ is of order better than to $\mathscr{M}_5$ by a factor δ and to $\mathscr{M}_6$ than to PH by a factor δ^6.

The proof of Theorem 5.4 follows the same steps as that of Theorem 5.1 and is omitted. We only note here that the analogs of the estimates of Lemma 5.3 are

$$|p' - \bar{p}_4'| \leqslant \frac{\kappa}{\Lambda^5} , \qquad |p'' - \bar{p}_2''| \leqslant \frac{\kappa}{\Lambda^3} ,$$

$$|q' - \bar{q}_4'| \leqslant \frac{\kappa}{\Lambda^6} , \qquad |q'' - \bar{q}_2''| \leqslant \frac{\kappa}{\Lambda^4} .$$

APPENDIX A

PROOF OF PROPOSITION 1.1

We prove (1.10) using an induction argument. More precisely, we will derive by induction on $j \in \mathcal{N}$ the existence of t_j depending on j, (Ω, f, α) and R such that

$$\|u^{(i)}\|_{H^2(\Omega)} \leqslant \kappa , \qquad \text{for} \qquad i = 0 ,\dots, j , \quad \forall t \geqslant t_j , \qquad (A.1)_j$$

$$\int_t^{t+1} |u^{(j+1)}|^2 \, ds \leqslant \kappa , \quad \forall t \geqslant t_j . \qquad (A.2)_j$$

i) *Initialization of the induction* $(j = 0)$. The estimate $(A.1)_0$ is (1.9), while $(A.2)_0$ is also proved in [11] to which the reader is referred.

ii) *The induction argument.* We now assume that $(A.1)_j$, $(A.2)_j$ hold for some $j \in \mathcal{N}$ and we prove that the same is true for $(j + 1)$.

The function $u^{(j+1)}$ satisfies an equation of the form

$$\frac{du^{(j+1)}}{dt} + A^2 u^{(j+1)} + A(f'(u)\, u^{(j+1)} + F(u, u^{(1)}, \dots, u^{(j)})) = 0 , \quad (A.3)$$

where $F : \mathcal{R}^{j+1} \to \mathcal{R}$ is a polynomial. Taking the scalar product of (A.3) by $u^{(j+1)}$ in H, we obtain

$$\frac{1}{2}\frac{d}{dt} |u^{(j+1)}|^2 + |Au^{(j+1)}|^2 =$$

$$= - (f'(u)\, u^{(j+1)} + F(u, u^{(1)}, \dots, u^{(j)}), Au^{(j+1)})$$

$$\leqslant \frac{1}{2} |Au^{(j+1)}|^2 + \frac{1}{2} |f'(u)\, u^{(j+1)} + F(u, u^{(1)}, \dots, u^{(j)})|^2$$

$$\frac{d}{dt} |u^{(j+1)}|^2 + |Au^{(j+1)}|^2 \leqslant |f'(u)\, u^{(j+1)} + F(u, u^{(1)}, \dots, u^{(j)})|^2 .$$

For $n \leqslant 3$, we have $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and the induction assumption $(A.1)_j$ implies that

$$|f'(u)\, u^{(j+1)} + F(u, u^{(1)}, \dots, u^{(j)})|^2 \leqslant \kappa(1 + |u^{(j+1)}|^2) , \quad \forall t \geqslant t_j .$$

Hence

$$\frac{d}{dt} |u^{(j+1)}|^2 + |Au^{(j+1)}|^2 \leqslant \kappa(1 + |u^{(j+1)}|^2) , \quad \forall t \geqslant t_j . \quad (A.4)$$

Then, using the induction assumption $(A.2)_j$, we can apply the uniform Gronwall Lemma (see [14] for instance) to (A.4) and this gives successively

$$|u^{(j+1)}|^2 \leqslant \kappa , \quad \forall t \geqslant t_j + 1 , \tag{A.5}$$

$$\int_t^{t+1} |Au^{(j+1)}|^2 \, ds \leqslant \kappa , \quad \forall t \geqslant t_j + 1 . \tag{A.6}$$

We now multiply (A.3) by $A^2 u^{(j+1)}$ in H and we have

$$\frac{1}{2}\frac{d}{dt} |Au^{(j+1)}|^2 + |A^2 u^{(j+1)}|^2 = \\ = -\,(A(f'(u)\, u^{(j+1)} + F(u, \ldots, u^{(j)})), A^2 u^{(j+1)}) ,$$

$$\frac{d}{dt} |Au^{(j+1)}|^2 + |A^2 u^{(j+1)}|^2 \leqslant |A(f'(u)\, u^{(j+1)} + F(u, \ldots, u^{(j)}))|^2 . \tag{A.7}$$

Since $H^2(\Omega)$ is a multiplicative algebra, we infer from $(A.1)_j$:

$$|A(f'(u)\, u^{(j+1)} + F(u, \ldots, u^{(j)}))|^2 \leqslant \kappa(1 + \|u^{(j+1)}\|^2_{H^2(\Omega)}) ,$$

which yields

$$\begin{aligned} \frac{d}{dt} |Au^{(j+1)}|^2 + |A^2 u^{(j+1)}|^2 &\leqslant \kappa(1 + \|u^{(j+1)}\|^2_{H^2(\Omega)}) \\ &\leqslant (\text{by } (4.17)) \\ &\leqslant \kappa\left\{1 + c_3(|u^{(j+1)}|^2 + |Au^{(j+1)}|^2)\right\} . \end{aligned} \tag{A.8}$$

Making use of (A.5) (A.6), we can apply the uniform Gronwall Lemma to (A.8) and obtain

$$|Au^{(j+1)}|^2 \leqslant \kappa , \quad \forall t \geqslant t_j + 2 , \tag{A.9}$$

$$\int_t^{t+1} |A^2 u^{(j+1)}|^2 \, ds \leqslant \kappa , \quad \forall t \geqslant t_j + 2 . \tag{A.10}$$

This concludes the induction argument since (A.5) (A.9) give $(A.1)_{j+1}$, while equation (A.3) combined with (A.10) yields

$$\int_t^{t+1} |u^{(j+2)}|^2 \, ds \leqslant \kappa , \quad \forall t \geqslant t_j + 2 ,$$

i.e. $(A.2)_{j+1}$.

Proposition 1.1 is proved.

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