M2AN. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
- MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

G. K. KULEV
D. D. BAINOV

Stability of the solutions of impulsive integro-differential equations in terms of two measures

M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 25, nº 1 (1991), p. 93-110
<http://www.numdam.org/item?id=M2AN_1991__25_1_93_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 1, 1991, p. 93 à 110)

STABILITY OF THE SOLUTIONS OF IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN TERMS OF TWO MEASURES (*)

G. K. KULEV [1], D. D. BAINOV [1]

Communicated by R. TEMAM

Abstract. — *In the present paper the stability of the solutions of impulsive systems of integro-differential equations of Volterra type with fixed moments of impulse effect in terms of two piecewise continuous measures is investigated. The investigations are carried out by means of piecewise continuous functions of the type of Lyapunov's functions using differential inequalities for piecewise continuous functions.*

Résumé. — *Stabilité des solutions d'équations intégro-différentielles impulsionnelles, en fonction de deux mesures.*
Dans ce papier on étudie la stabilité, en fonction de deux mesures continues par morceaux, des solutions de systèmes impulsionnels d'équations intégro-différentielles de type Volterra, avec effet impulsionnel à des moments fixés. Cette étude est menée au moyen de fonctions continues par morceaux, du type fonctions de Lyapunov, et utilise des inéquations différentielles pour des fonctions continues par morceaux.

1. INTRODUCTION

Impulsive differential and integro-differential equations represent an adequate mathematical model of many real processes and phenomena studied in physics, biology, technology, etc. Moreover, the mathematical theory of impulsive differential equations is much richer than the respective theory of ordinary differential equations. That is why in the recent years this theory develops very intensively [1]-[7].

The use of classical (continuous) Lyapunov's functions in the study of the stability of the solutions of impulsive systems of differential and integro-

(*) Received in June 1989.
(1) The present investigation is supported by the Ministry of Culture, Science and Education of People's Republic of Bulgaria under Grant 61.
P.O. Box 45, 1504 Sofia, Bulgaria.

differential equations by Lyapunov's direct method restricts the pliability of the method. The fact that the solutions of such systems are piecewise continuous functions shows that it is necessary to introduce some analogues of Lyapunov's functions which have discontinuities of the first kind. By means of such functions the application of Lyapunov's direct method to impulsive systems of differential and integro-differential equations is much more effective [3]-[7].

The advantages of the study of the stability of the solutions of differential and integro-differential equations by means of two different measures and the generality and the unification obtained by this approach are well known [7], [9].

In the present paper the question of stability of the solutions of a general class of impulsive systems of integro-differential equations of Volterra type with fixed moments of impulse effect in terms of two piecewise continuous measures is considered. The investigations are carried out by means of piecewise continuous functions which are analogues of Lyapunov's functions, and by means of the theory of differential inequalities for piecewise continuous functions. By this techniques, the study of the solutions of impulsive integro-differential systems is replaced by the study of the solutions of a scalar impulsive differential equation. For this purpose one chooses certain minimal subsets of an appropriate space of piecewise continuous functions, by the elements of which the derivatives of Lyapunov's functions are estimated [10].

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with a norm $\|.\|$ and $\mathbb{R}_+ = [0,\infty)$. Consider the following impulsive integro-differential system

$$\begin{aligned} x'(t) &= f\left(t, x(t), \int_{t_0}^{t} K(t, s, x(s))\, ds\right), \quad t \neq \tau_\kappa ; \\ \Delta x|_{t=\tau_\kappa} &= I_\kappa(x(\tau_\kappa)), \qquad (2.1) \\ x(t_0+0) &= x_0, \quad t_0 \in \mathbb{R}_+ \end{aligned}$$

where

$$f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad K : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, \quad I_\kappa : \mathbb{R}^n \to \mathbb{R}^n,$$

$$0 < \tau_\kappa < \tau_{\kappa+1}, \quad \kappa = 1, 2, \ldots, \quad \Delta x|_{t=\tau_\kappa} = x(\tau_\kappa + 0) - x(\tau_\kappa - 0).$$

Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$. Denote by $x(t\,; t_0, x_0)$ the solution of system (2.1) satisfying the initial condition $x(t_0 + 0\,; t_0, x_0) = x_0$. The solutions $x(t) = x(t\,; t_0, x_0)$ of system (2.1) are piecewise continuous functions with points of discontinuity of the first kind τ_κ, $\kappa = 1, 2, \ldots$, at which they are continuous from the left, i.e. at the moments of impulse effect τ_κ the following relations hold

$$x(\tau_\kappa - 0) = x(\tau_\kappa)\,; \quad x(\tau_\kappa + 0) = x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))\,.$$

Together with system (2.1) we shall consider the impulsive differential equation

$$\begin{aligned} &u' = g(t, u)\,, \quad t \neq \tau_\kappa\,; \quad \Delta u\,|_{t = \tau_\kappa} = B_\kappa(u(\tau_\kappa))\,, \\ &u(t_0 + 0) = u_0 \geqq 0\,, \quad t_0 \in \mathbb{R}_+\,, \end{aligned} \tag{2.2}$$

where

$$g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\,, \quad B_\kappa : \mathbb{R}_+ \to \mathbb{R}_+\,.$$

We shall introduce the class $\mathscr{V}_0$ of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions [3]-[6].

Let $\tau_0 = 0$. Introduce the sets

$$G_\kappa = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \tau_{\kappa-1} < t < \tau_\kappa\}\,, \quad G = \bigcup_{\kappa=1}^{\infty} G_\kappa\,.$$

DEFINITION 2.1 : *We shall say that the function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *belongs to the class* $\mathscr{V}_0$ *if* V *is continuous in* G, *locally Lipschitz continuous in* x *in each of the sets* G_κ *and for* $\kappa = 1, 2, \ldots$ *and* $x_0 \in \mathbb{R}^n$ *the following limits exist*

$$V(\tau_\kappa - 0, x_0) = \lim_{\substack{(t,x) \to (\tau_\kappa, x_0) \\ (t,x) \in G_\kappa}} V(t, x)\,, \quad V(\tau_\kappa + 0, x_0) = \lim_{\substack{(t,x) \to (\tau_\kappa, x_0) \\ (t,x) \in G_{\kappa+1}}} V(t, x)$$

and the equality $V(\tau_\kappa - 0, x_0) = V(\tau_\kappa, x_0)$ *holds.*

In the further considerations we shall also use the following classes of functions :

$\mathscr{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(.)$ is monotone increasing in $\mathbb{R}_+$ and $a(0) = 0\}$,
$C\mathscr{K} = \{a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+] : a(t, .)$ for any $t \in \mathbb{R}_+\}$,
$\mathscr{P}C[\mathbb{R}_+, \mathbb{R}^n] = \{x : \mathbb{R}_+ \to \mathbb{R}^n : x$ is piecewise continuous with points of discontinuity of the first kind τ_κ and $x(\tau_\kappa - 0) = x(\tau_\kappa)\}$,

$$\Gamma = \left\{h \in \mathscr{V}_0 : \inf_{x \in \mathbb{R}^n} h(t, x) = 0 \text{ for any } t \in \mathbb{R}_+\right\}\,,$$

$$\begin{aligned} E_A = \{x \in \mathscr{P}C[\mathbb{R}_+, \mathbb{R}^n] : &V(s, x(s))\,A(s) \\ &\leqq V(t, x(t))\,A(t),\ t_0 \leqq s \leqq t\}\,, \end{aligned}$$

$$E_1 = \{x \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] . V(s, x(s)) \leqq V(t, x(t)), t_0 \leqq s \leqq t\},$$

$$E_0 = \{x \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] : V(s, x(s)) \leqq \Phi(V(t, x(t))), t_1 \leqq s \leqq t, t_1 \geqq t_0\},$$

where

(i) $A(t) > 0$ is a continuous in $\mathbb{R}_+$ function,

(ii) $\Phi(u)$ is continuous and nondecreasing in $\mathbb{R}_+$ and $\Phi(u) > u$ for $u > 0$

Let $S(h, \rho) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t, x) < \rho\}$, $h \in \Gamma$, $\rho > 0$

We shall say that conditions (A) are satisfied if the following conditions hold

A1 $f \in C[S(h, \rho) \times \mathbb{R}^n, \mathbb{R}^n]$

A2 $K \in C[\mathbb{R}_+ \times S(h, \rho), \mathbb{R}^n]$

A3 $I_\kappa \in C[\mathbb{R}^n, \mathbb{R}^n]$, $\kappa = 1, 2,$

A4 $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{\kappa \to \infty} \tau_\kappa = \infty$

A5 $g \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}_+]$ and $g(t, 0) \equiv 0$, $t \in \mathbb{R}_+$

A6 $B_\kappa \in C[\mathbb{R}_+, \mathbb{R}_+]$, $B_\kappa(0) = 0$ and $\psi_\kappa(u) = u + B_\kappa(u)$, $\kappa = 1, 2,$ are nondecreasing in $\mathbb{R}_+$

A7 There exists ρ_0, $0 < \rho_0 < \rho$ such that $h(\tau_\kappa, x) < \rho_0$ implies $h(\tau_\kappa + 0, x + I_\kappa(x)) < \rho$, $\kappa = 1, 2,$

DEFINITION 2 2 *Let h_0, $h \in \Gamma$ We shall say that*

(a) *h_0 is finer than h if there exists a number $\delta > 0$ and a function $\varphi \in \mathcal{K}$ such that $h_0(t + 0, x) < \delta$ implies $h(t, x) \leqq \varphi(h_0(t + 0, x))$*

(b) *h_0 is weakly finer than h if there exists a number $\delta > 0$ and a function $\varphi \in C\mathcal{K}$ such that $h_0(t + 0, x) < \delta$ implies $h(t, x) \leqq \varphi(t, h_0(t + 0, x))$*

Let $V \in \mathcal{V}_0$, $t > t_0$, $t \neq \tau_\kappa$ and $x \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n]$ Introduce the function

$$D_- V(t, x(t)) = \liminf_{\sigma \to 0^-} \frac{1}{\sigma} \left[V(t + \sigma, x(t) + \sigma f\left(t, x(t), \int_{t_0}^t K(t, s, x(s))\, ds\right) - V(t, x(t))\right]$$

We shall give definitions of stability of system (2 1) in terms of two different measures, by which various classical notions of stability are generalized

DEFINITION 2.3 : *System* (2.1) *is said to be :*

(a) (h_0, h)*-stable if*

$$(\forall t_0 \in \mathbb{R}_+)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)(\forall x_0 \in \mathbb{R}^n,\\ h_0(t_0 + 0, x_0) < \delta)(\forall t > t_0) : h(t, x(t\,;\, t_0, x_0)) < \varepsilon\,.$$

(b) (h_0, h)*-uniformly stable if the number* δ *of* (a) *does not depend on* t_0.

(c) (h_0, h)*-equiattractive if*

$$(\forall t_0 \in \mathbb{R}_+)(\exists \delta_0 = \delta_0(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)(\forall x_0 \in \mathbb{R}^n,\\ h_0(t_0 + 0, x_0) < \delta_0)(\forall t > t_0 + T) : h(t, x(t\,;\, t_0, x_0)) < \varepsilon\,.$$

(d) (h_0, h)*-uniformly attractive if the numbers* δ_0 *and* T *of* (c) *do not depend on* t_0.

(e) (h_0, h)*-equiasymptotically stable if it is* (h_0, h)*-stable and* (h_0, h)*-equiattractive.*

(f) (h_0, h) *uniformly asymptotically stable if it is* (h_0, h)*-uniformly stable and* (h_0, h)*-uniformly attractive.*

For a concrete choice of the measures h_0 and h. Definition 2.3 is reduced to the following particular cases :

1) stability by Lyapunov of the zero solution of (2.1) if $h_0(t, x) = h(t, x) = \|x\|$;

2) stability with respect to part of the variables of the zero solution of (2.1) if

$$h_0(t, x) = \|x\|\,,\quad h(t, x) = \|x\|_\kappa = \sqrt{x_1^2 + \cdots + x_\kappa^2}\,,\quad 1 \leqq \kappa \leqq n\,;$$

3) stability by Lyapunov of a nonzero solution $x_0(t)$ of (2.1) if

$$h_0(t, x) = h(t, x) = \|x - x_0(t)\|\,;$$

4) stability of an invariant set $A \subset \mathbb{R}^n$ if

$$h_0(t, x) = h(t, x) = d(x, A)\,,$$

where d is the distance in $\mathbb{R}^n$;

5) stability of a set $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ if

$$h_0(t, x) = h(t, x) = d(x, M(t))\,,$$

where $M(t) = \{x \in \mathbb{R}^n : (t, x) \in M\} \neq \phi$;

6) stability of a conditionally invariant set B with respect to A where $A \subset B \subset \mathbb{R}^n$ if

$$h_0(t, x) = d(x, A), \quad h(t, x) = d(x, B).$$

DEFINITION 2.4 : *Let* h_0, $h \in \Gamma$ *and* $V \in \mathcal{V}_0$. *The function* V *is said to be :*

(a) *h-positively definite if there exists* $\delta > 0$ *and a function* $a \in \mathcal{K}$ *such that* $h(t, x) < \delta$ *implies* $V(t, x) \geqq a(h(t, x))$;

(b) h_0*-decrescent if there exists* $\delta > 0$ *and a function* $b \in \mathcal{K}$ *such that* $h_0(t+0, x) < \delta$ *implies* $V(t+0, x) \leqq b(h_0(t+0, x))$;

(c) *weakly* h_0*-decrescent if there exists* $\delta > 0$ *and a function* $b \in C\mathcal{K}$ *such that* $h_0(t+0, x) < \delta$ *implies* $V(t+0, x) \leqq b(t, h_0(t+0, x))$.

3. MAIN RESULTS

In the proof of the main theorems we shall use the following comparison lemmas :

LEMMA 3.1 : *Let the following conditions be fulfilled :*

1. *Conditions* (A1)-(A6) *hold.*
2. *The function* $V \in \mathcal{V}_0$, $V : S(h, \rho) \to \mathbb{R}_+$ *is such that for* $t > t_0 \geqq 0$ *and* $x \in E_1$

$$D_- V(t, x(t)) \leqq g(t, V(t, x(t))), \quad if \quad t \neq \tau_\kappa, \quad \kappa = 1, 2, \dots$$

$$V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq \psi_\kappa(V(\tau_\kappa, x(\tau_\kappa))), \quad if \quad t = \tau_\kappa. \quad (3.1)$$

3. *The solution* $x(t; t_0, x_0)$ *of system* (2.1) *is such that* $(t, x(t+0; t_0, x_0)) \in S(h, \rho)$ *for* $t \in [t_0, \beta]$ *where* $h \in \Gamma$.
4. *The maximal solution* $r(t; t_0, u_0)$, $u_0 \geqq V(t_0+0, x_0)$ *of equation* (2.2) *is defined on the interval* (t_0, ∞).

Then

$$V(t, x(t; t_0, x_0)) \leqq r(t; t_0, u_0) \quad for \quad t \in (t_0, \beta]. \quad (3.2)$$

Proof : The maximal solution $r(t; t_0, u_0)$ of equation (2.2) is defined by the equality

$$r(t; t_0, u_0) = \begin{cases} r_0(t; t_0, u_0^+), & t_0 < t \leqq \tau_1 \\ r_1(t; \tau_1, u_1^+), & \tau_1 < t \leqq \tau_2 \\ \dots\dots\dots & \\ r_k(t; \tau_\kappa, u_\kappa^+), & \tau_\kappa < t \leqq \tau_{\kappa+1} \\ \dots\dots\dots & \end{cases}$$

where $r_\kappa(t; \tau_\kappa, u_\kappa^+)$ is the maximal solution of the equation without impulses $u' = g(t, u)$ in the interval $[\tau_\kappa, \tau_{\kappa+1}]$, $\kappa = 0, 1, \dots$ for which $u_\kappa^+ = \psi_\kappa(r_{\kappa-1}(\tau_\kappa ; \tau_{\kappa-1}, u_{\kappa-1}^+))$, $\kappa = 1, 2, \dots$ and $u_0^+ = u_0$.

Let $t \in (t_0, \tau_1] \cap (t_0, \beta]$. From [10], Theorem 2.1 it follows that

$$V(t, x(t ; t_0, x_0)) \leqq r_0(t ; t_0, u_0) = r(t ; t_0, u_0)$$

i.e. inequality (3.2) holds for $t \in (t_0, \tau_1] \cap (t_0, \beta]$.

Assume that (3.2) holds for $t \in (\tau_{\kappa-1}, \tau_\kappa] \cap (t_0, \beta]$, $\kappa > 1$. Then, making use of (3.1) and of the fact that the function ψ_κ is nondecreasing, we obtain

$$V(\tau_\kappa + 0, x(\tau_\kappa + 0 ; t_0, x_0)) \leqq \psi_\kappa(V(\tau_\kappa, x(\tau_\kappa ; t_0, x_0))) \leqq$$
$$\leqq \psi_\kappa(r(\tau_\kappa ; t_0, u_0)) = \psi_\kappa(r_\kappa(\tau_\kappa ; \tau_{\kappa-1}, u_{\kappa-1}^+)) = u_\kappa^+ .$$

We apply again [10], Theorem 2.1 for $t \in (\tau_\kappa, \tau_{\kappa+1}] \cap (t_0, \beta]$ and obtain

$$V(t, x(t ; t_0, x_0)) \leqq r_\kappa(t ; \tau_\kappa, u_\kappa^+) = r(t ; t_0, u_0) ,$$

i.e. inequality (3.2) holds for $t \in (\tau_\kappa, \tau_{\kappa+1}] \cap [t_0, \beta]$.

This completes the proof of Lemma 3.1.

COROLLARY 3.1 : *Let the following conditions hold :*

1. *Conditions* (A1)-(A4) *are satisfied.*

2. *The function* $V \in \mathcal{V}_0$, $V : S(h, \rho) \to \mathbb{R}_+$ *is such that for* $t > t_0 \geqq 0$ *and* $x \in E_0$

$$D_- V(t, x(t)) \leqq 0 , \quad if \quad t \neq \tau_\kappa , \quad \kappa = 1, 2, \dots$$
$$V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq V(\tau_\kappa, x(\tau_\kappa)) , \quad if \quad t = \tau_\kappa .$$

3. *Condition* 3 *of Lemma* 3.1 *holds.*

Then

$$V(t, x(t ; t_0, x_0)) \leqq V(t_0 + 0, x_0) .$$

LEMMA 3.2 : *Let the following conditions hold :*

1. *Conditions* (A1)-(A6) *are satisfied.*

2. *The function* $V \in \mathcal{V}_0$, $V : S(h, \rho) \to \mathbb{R}_+$ *is such that for* $t > t_0 \geqq 0$ *and* $x \in E_A$

$$A(t) D_- V(t, x(t)) + V(t, x(t)) D_- A(t) \leqq$$
$$\leqq g(t, A(t) V(t, x(t))) , \quad if \quad t \neq \tau_\kappa , \quad \kappa = 1, 2, \dots , \quad (3.3)$$

$$A(\tau_\kappa + 0) V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq \psi_\kappa(A(\tau_\kappa) V(\tau_\kappa, x(\tau_\kappa))) ,$$
$$if \quad t = \tau_\kappa , \quad (3.4)$$

where $A(t) > 0$ *is a piecewise continuous in* $\mathbb{R}_+$ *function with points of* *discontinuity of the first kind* τ_κ *at which it is continuous from the left,* $A(\tau_\kappa + 0) > 0$ *and*

$$D_- A(t) = \liminf_{\sigma \to 0^-} \frac{1}{\sigma} [A(t+\sigma) - A(t)] .$$

3. *Condition* 3 *of Lemma* 3.1 *holds.*

4. *The maximal solution* $r(t; t_0, u_0)$, $u_0 \geqq A(t_0+0) V(t_0+0, x_0)$ *of equation* (2.2) *is defined on the interval* (t_0, ∞).

Then

$$A(t) V(t, x(t; t_0, x_0)) \leqq r(t; t_0, u_0) \quad for \quad t \in (t_0, \beta] . \tag{3.5}$$

Proof: Set

$$L(t, x(t)) = A(t) V(t, x(t)) .$$

Let $t > t_0$ and $x \in E_A$. For $t \neq \tau_\kappa$, $\kappa = 1, 2, \ldots$ and $\sigma < 0$ small enough we have

$$\begin{aligned} L\left(t+\sigma, x(t) + \sigma f\left(t, x(t), \int_{t_0}^{t} K(t, s, x(s))\, ds\right)\right) - \\ - L(t, x(t)) = V(t+\sigma, x(t) + \sigma f(t, x(t), \\ \int_{t_0}^{t} K(t, s, x(s))\, ds))[A(t+\sigma) - A(t)] \\ + A(t)[V(t+\sigma, x(t) + \sigma f(t, x(t), \\ \int_{t_0}^{t} K(t, s, x(s))\, ds)) - V(t, x(t))] . \end{aligned}$$

Then from (3.3) and (3.4) it follows that

$$\begin{aligned} &D_- L(t, x(t)) \leqq g(t, L(t, x(t))) , \quad \text{if} \quad t \neq \tau_\kappa , \quad \kappa = 1, 2, \ldots \\ &L(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq \psi_\kappa(L(\tau_\kappa, x(\tau_\kappa))) , \quad \text{if} \quad t = \tau_\kappa \end{aligned}$$

for $t > t_0 \geqq 0$ and $x \in E_1$ where E_1 is the class defined by $L(t, x)$ instead of $V(t, x)$.

Applying Lemma 3.1 for $L(t, x)$, we obtain that inequality (3.5) holds.

THEOREM 3.1 : *Let the following conditions hold:*

1. *Conditions* (A) *are satisfied.*
2. h_0, $h \in \Gamma$ *and* h_0 *is weakly finer than* h.
3. *The function* $V \in \mathcal{V}_0$, $V : S(h, \rho) \to \mathbb{R}_+$ *is h-positively definite in* $S(h, \rho)$ *and weakly* h_0*-decrescent.*

4. *For* $t > t_0 \geqq 0$ *and* $x \in E_1$

$$D_- V(t, x(t)) \leqq g(t, V(t, x(t))), \quad \text{if} \quad t \neq \tau_\kappa, \quad \kappa = 1, 2, \dots$$

$$V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq \psi_\kappa(V(\tau_\kappa, x(\tau_\kappa))), \quad \text{if} \quad t = \tau_\kappa .$$

Then

(a) *if the zero solution of* (2.2) *is stable, then system* (2.1) *is* (h_0, h)*-stable ;*

(b) *if the zero solution of* (2.2) *is equiasymptotically stable, then system* (2.1) *is equiasymptotically stable.*

Proof : (a) Since V is h-positively definite in $S(h, \rho)$, then there exists a function $a \in \mathcal{K}$ such that

$$V(t, x) \geqq a(h(t, x)), \quad (t, x) \in S(h, \rho). \tag{3.6}$$

Since V is weakly h_0-decrescent, then there exist $\delta_1 > 0$ and $b \in C\mathcal{K}$ such that

$$V(t + 0, x) \leqq b(t, h_0(t + 0, x)) \quad \text{for} \quad h_0(t + 0, x) < \delta_1 . \tag{3.7}$$

From condition 2 of Theorem 3.1 it follows that there exist $\delta_2 > 0$ and a function $\varphi \in C\mathcal{K}$ such that

$$h(t, x) \leqq \varphi(t, h_0(t + 0, x)) \quad \text{for} \quad h_0(t + 0, x) < \delta_2 . \tag{3.8}$$

Let $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$. From the properties of the functions b and φ it follows that there exist numbers $\delta_3 = \delta_3(t_0, \varepsilon)$, $0 < \delta_3 < \delta_1$ and $\delta_4 = \delta_4(t_0, \rho)$, $0 < \delta_4 < \delta_2$ such that

$$b(t_0, \delta_3) < \varepsilon \quad \text{and} \quad \varphi(t_0, \delta_4) < \rho . \tag{3.9}$$

From the stability of the zero solution of equation (2.2) it follows that there exists $\delta_5 = \delta_5(t_0, \varepsilon) > 0$ such that for $u_0 < \delta_5$

$$r(t ; t_0, u_0) < a(\varepsilon), \quad t > t_0 \tag{3.10}$$

where $r(t ; t_0, u_0)$ is the maximal solution of (2.2) for which $r(t_0 + 0 ; t_0, u_0) = u_0$.

Choose $\delta_6 = \delta_6(t_0, \varepsilon) > 0$ such that

$$b(t_0, \delta_6) < \delta_5 . \tag{3.11}$$

Let $\delta = \min(\delta_3, \delta_4, \delta_5, \delta_6)$. Then from (3.6), (3.7) and (3.9) it follows that if $h_0(t_0 + 0, x_0) < \delta$, then

$$a(h(t_0 + 0, x_0)) \leqq V(t_0 + 0, x_0) \leqq b(t_0, h_0(t_0 + 0, x_0)) < a(\varepsilon),$$

which shows that $h(t_0 + 0, x_0) < \varepsilon$.

Moreover, from (3.7) and (3.11) it follows that

$$V(t_0+0, x_0) < \delta_5 \quad \text{for} \quad h_0(t_0+0, x_0) < \delta . \tag{3.12}$$

Let $x(t) = x(t\,;t_0, x_0)$ be a solution of system (2.1) for which $h_0(t_0+0, x_0) < \delta$. We shall prove that

$$h(t, x(t)) < \varepsilon \quad \text{for} \quad t > t_0 .$$

Suppose that this is not true. Then there exists $t^* > t_0$ such that $\tau_\kappa < t^* \leqq \tau_{\kappa+1}$ for some positive integer κ for which

$$h(t^*, x(t^*)) \geqq \varepsilon \quad \text{and} \quad h(t, x(t)) < \varepsilon , \quad t_0 < t \leqq \tau_\kappa .$$

Since $0 < \varepsilon < \rho_0$, then from condition (A7) it follows that

$$h(\tau_\kappa + 0, x(\tau_\kappa + 0)) = h(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) < \rho .$$

Hence there exists t^0, $\tau_\kappa < t^0 \leqq t^*$ such that

$$\varepsilon \leqq h(t^0, x(t^0)) < \rho \quad \text{and} \quad h(t, x(t)) < \rho , \quad t_0 < t \leqq t^0 . \tag{3.13}$$

Applying Lemma 3.1 for the interval $(t_0, t^0]$, we obtain

$$V(t, x(t)) \leqq r(t\,; t_0, V(t_0+0, x_0)) , \quad t_0 < t \leqq t^0 . \tag{3.14}$$

But then from (3.13), (3.6), (3.14), (3.12) and (3.10) it follows that

$$a(\varepsilon) \leqq a(h(t^0, x(t^0))) \leqq V(t^0, x(t^0)) \leqq r(t^0\,; t_0, V(t_0+0, x_0)) < a(\varepsilon) .$$

The contradiction obtained shows that $h(t, x(t)) < \varepsilon$ for all $t > t_0$. Hence system (2.1) is (h_0, h)-stable.

(b) From assertion (a) of Theorem 3.1 it follows that system (2.1) is (h_0, h)-stable. Hence there exists $\delta_{01} = \delta_{01}(t_0, \rho) > 0$ such that for $h_0(t_0+0, x_0) < \delta_{01}$ we have $h(t, x(t\,; t_0, x_0)) < \rho$, $t > t_0$.

Let $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$. From the equiasymptotic stability of the zero solution of equation (2.2) it follows that there exist $\delta_{02} = \delta_{02}(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that for $u_0 < \delta_{02}$ we have

$$r(t\,; t_0, u_0) < a(\varepsilon) , \quad t > t_0 + T .$$

Choose $\delta_{03} = \delta_{03}(t_0, \varepsilon)$, $0 < \delta_{03} < \delta_{02}$ such that

$$b(t_0, \delta_{03}) < \delta_{02} . \tag{3.15}$$

Then from (3.7) and (3.15) it follows that if $h_0(t_0+0, x_0) < \delta_{03}$, then

$$V(t_0+0, x_0) \leqq b(t_0, h_0(t_0+0, x_0)) \leqq b(t_0, \delta_{03}) < \delta_{02}$$

hence

$$r(t\,;\,t_0,V(t_0+0,x_0))<a(\varepsilon)\,,\quad t>t_0+T\,. \tag{3.16}$$

Let $\delta_0=\min\,(\delta_{01},\delta_{02},\delta_{03})$ and let $h_0(t_0+0,x_0)<\delta_0$. From Lemma 3.1 it follows that if $x(t)=x(t\,;\,t_0,x_0)$ is a solution of system (2.1), then

$$V(t,x(t))\leqq r(t\,;\,t_0,V(t_0+0,x_0))\,,\quad t>t_0\,. \tag{3.17}$$

Then from (3.6), (3.17) and (3.16) we obtain that the inequalities

$$a(h(t,x(t)))\leqq V(t,x(t))\leqq r(t\,;\,t_0,V(t_0+0,x_0))<a(\varepsilon)$$

hold for $t>t_0+T$. Hence $h(t,x(t))<\varepsilon$, $t>t_0+T$, which shows that system (2.1) is (h_0,h)-equiattractive.

Theorem 3.1 is proved.

THEOREM 3.2 : *Let the following conditions be fulfilled :*

1. *Conditions* (A) *hold.*
2. h_0, $h\in\Gamma$ *and* h_0 *is finer than* h.
3. *The function* $V\in\mathcal{V}_0$, $V:S(h,\rho)\to\mathbb{R}_+$ *is h-positively definite and* h_0*-decrescent.*
4. *Condition* 4 *of Theorem* 3.1 *is satisfied.*

Then

(a) *if the zero solution of* (2.2) *is uniformly stable, then system* (2.1) *is* (h_0,h)*-uniformly stable ;*

(b) *if the zero solution of* (2.2) *is uniformly asymptotically stable, then system* (2.1) *is* (h_0,h)*-uniformly asymptotically stable.*

The proof of Theorem 3.2 is analogous to the proof of Theorem 3.1. We shall only note that in this case the numbers δ, δ_0 and T can be chosen independent of t_0.

THEOREM 3.3 : *Let the following conditions hold :*

1. *Conditions* (A1)-(A4) *and* (A7) *are satisfied.*
2. *Conditions* 2 *and* 3 *of Theorem* 3.2 *hold.*
3. *For* $t>t_0\geqq 0$ *and* $x\in E_0$

$$D_-\,V(t,x(t))\leqq 0\,,\quad if\quad t\neq\tau_\kappa\,,\quad \kappa=1,2,\dots$$

$$V(\tau_\kappa+0,x(\tau_\kappa)+I_\kappa(x(\tau_\kappa)))\leqq V(\tau_\kappa,x(\tau_\kappa))\,,\quad if\quad t=\tau_\kappa\,.$$

Then system (2.1) *is* (h_0,h)*-uniformly stable.*

The proof of Theorem 3.3 is carried out analogously to the proof of Theorem 3.1 (a). Corollary 3.1 is applied.

THEOREM 3.4 : *Let the following conditions hold :*

1. *Conditions* 1 *and* 2 *of Theorem* 3.3 *are satisfied.*
2. *For* $t > t_0 \geqq 0$ *and* $x \in E_0$

$$D_- V(t, x(t)) \leqq -c(h_0(t, x(t))), \text{ if } t \neq \tau_\kappa, \ \kappa = 1, 2, \ldots, \quad (c \in \mathcal{K}), \tag{3.18}$$

$$V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq V(\tau_\kappa, x(\tau_\kappa)), \quad \text{if} \quad t = \tau_\kappa. \tag{3.19}$$

Then system (2.1) *is* (h_0, h)*-uniformly asymptotically stable.*

Proof : From Theorem 3.3 it follows that system (2.1) is (h_0, h)-uniformly stable. Hence for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $h_0(t_0 + 0, x_0) < \delta$ we have

$$h(t, x(t; t_0, x_0)) = h(t, x(t)) < \varepsilon, \quad t > t_0.$$

Since V is h-positively definite in $S(h, \rho)$, then there exists a function $a \in \mathcal{K}$ such that

$$V(t, x(t)) \geqq a(h(t, x(t))), \quad (t, x) \in S(h, \rho). \tag{3.20}$$

Since V is h_0-decrescent, then there exists a number $\delta_1 > 0$ and a function $b \in \mathcal{K}$ such that

$$V(t + 0, x) \leqq b(h_0(t + 0, x)) \quad \text{for} \quad h_0(t + 0, x) < \delta_1. \tag{3.21}$$

Let $x(t) = x(t; t_0, x_0)$ be a solution of system (2.1) for which $h_0(t_0 + 0, x_0) < \delta_0$ where

$$\delta_0 = \min(\delta(\rho_0), \delta_1).$$

Then $h(t, x(t)) < \rho_0 < \rho$, $t > t_0$.

Choose η so that $0 < \eta \leqq \rho_0$. Then $a(\eta) \leqq b(\delta_0)$.

Let the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and nondecreasing in $\mathbb{R}_+$ and such that $\Phi(u) > u$ for $u > 0$. Set

$$\beta = \beta(\eta) = \min \{\Phi(u) - u : a(\eta) \leqq u \leqq b(\delta_0)\}.$$

Then

$$\Phi(u) > u + \beta \quad \text{for} \quad a(\eta) \leqq u \leqq b(\delta_0). \tag{3.22}$$

Choose the positive integer ν in such a way that

$$a(\eta) + \nu\beta > b(\delta_0). \tag{3.23}$$

If for some value of $t > t_0$ we have $V(t + 0, x(t + 0)) \geqq a(\eta)$, then

$$\begin{gathered} V(t, x(t)) \geqq V(t + 0, x(t + 0)) \geqq a(\eta), \\ b(h_0(t + 0; x(t + 0))) \geqq V(t + 0, x(t + 0)) \geqq a(\eta), \end{gathered} \tag{3.24}$$

hence

$$h_0(t+0, x(t+0)) \geqq b^{-1}(a(\eta)) = \delta_2(\eta) = \delta_2 .$$

Then

$$c(h_0(t+0, x(t+0))) \geqq c(\delta_2) = \delta_3(\eta) = \delta_3 . \tag{3.25}$$

Set

$$\xi_\kappa = \xi_\kappa(t_0, \eta) = t_0 + \kappa \frac{\beta}{\delta_3}, \quad \kappa = 0, 1, 2, \dots, \nu .$$

We shall prove that for any $\kappa = 0, 1, 2, \dots, \nu$

$$V(t, x(t)) < a(\eta) + (\nu - \kappa)\beta, \quad t \geqq \xi_\kappa . \tag{3.26}$$

Indeed, applying Corollary 3.1, (3.21) and (3.23), we obtain

$$V(t, x(t)) \leqq V(t_0+0, x_0) \leqq b(h_0(t_0+0, x_0)) \leqq b(\delta_0) < \\ < a(\eta) + \nu\beta, \quad t > t_0 = \xi_0 ,$$

which shows that (3.26) holds for $\kappa = 0$.

Let (3.26) hold for some positive integer κ, $0 < \kappa < \nu$, i.e.

$$V(s, x(s)) < a(\eta) + (\nu - \kappa)\beta, \quad t \geqq \xi_\kappa . \tag{3.27}$$

If we assume that the inequality

$$V(t, x(t)) \geqq a(\eta) + (\nu - \kappa - 1)\beta, \quad \xi_\kappa \leqq t \leqq \xi_{\kappa+1}$$

is possible, we obtain

$$a(\eta) \leqq V(t, x(t)) \leqq b(\delta_0), \quad \xi_\kappa \leqq t \leqq \xi_{k+1} .$$

Then from (3.22) and (3.27) it follows that

$$\Phi(V(t, x(t))) > V(t, x(t)) + \beta \geqq a(\eta) + (\nu - \kappa)\beta > \\ > V(s, x(s)), \quad \xi_\kappa \leqq s \leqq t, \quad t \in [\xi_\kappa, \xi_{\kappa+1}] .$$

This shows that $x(.) \in E_0$ for $\xi_\kappa \leqq s \leqq t$, $t \in [\xi_\kappa, \xi_{\kappa+1}]$. Then from condition 2 of Theorem 3.4 and from (3.25) we obtain

$$V(\xi_{\kappa+1}, x(\xi_{\kappa+1})) \leqq V(\xi_\kappa + 0, x(\xi_\kappa + 0)) - \int_{\xi_\kappa}^{\xi_{\kappa+1}} c(h_0(s, x(s)))\, ds < \\ < a(\eta) + (\nu - \kappa)\beta - \delta_3[\xi_{\kappa+1} - \xi_\kappa] = \\ = a(\eta) + (\nu - \kappa - 1)\beta < V(\xi_\kappa, x(\xi_\kappa)) ,$$

which contradicts the fact that $x(.) \in E_0$ for $\xi_\kappa \leqq s \leqq t$, $t \in [\xi_\kappa, \xi_{\kappa+1}]$.

Hence there exists t^*, $\xi_\kappa \leqq t^* \leqq \xi_{\kappa+1}$ such that

$$V(t^*, x(t^*)) < a(\eta) + (\nu - \kappa - 1)\,\beta\,,$$

and from (3.19) it follows that

$$V(t^* + 0, x(t^* + 0)) < a(\eta) + (\nu - \kappa - 1)\,\beta\,.$$

Now we shall prove that

$$V(t, x(t)) < a(\eta) + (\nu - \kappa - 1)\,\beta\,, \quad t \geqq t^*\,.$$

Suppose that this is not true and set

$$\xi = \inf\,\{t \geqq t^* : V(t, x(t)) \geqq a(\eta) + (\nu - \kappa - 1)\,\beta\}\,.$$

From (3.19) it follows that $\xi \neq \tau_\kappa$, $\kappa = 1, 2, \ldots$, hence

$$V(\xi, x(\xi)) = a(\eta) + (\nu - \kappa - 1)\,\beta\,.$$

Then for $\sigma < 0$ small enough the inequality

$$V(\xi + \sigma, x(\xi + \sigma)) < a(\eta) + (\nu - \kappa - 1)\,\beta\,,$$

holds which implies that

$$D_-\, V(\xi, x(\xi)) \geqq 0\,.$$

On the other hand, as above it can be proved that $x(.) \in E_0$ for $t^* \leqq s \leqq \xi$, hence

$$D_-\, V(\xi, x(\xi)) \leqq -\delta_3 < 0\,.$$

The contradiction obtained shows that

$$V(t, x(t)) < a(\eta) + (\nu - \kappa - 1)\,\beta\,, \quad t \geqq \xi_{\kappa+1}\,.$$

Hence (3.26) holds for any $\kappa = 0, 1, 2, \ldots, \nu$.

Let $T = T(\eta) = \nu \dfrac{\beta}{\delta_3}$. Then from (3.26) it follows that

$$V(t, x(t)) < a(\eta) \quad \text{for} \quad t \geqq t_0 + T(\eta)\,. \tag{3.28}$$

Finally, from (3.20) and (3.28) we obtain

$$a(h(t, x(t))) \leqq V(t, x(t)) < a(\eta)\,, \quad t \geqq t_0 + T(\eta)$$

and thus it is proved that system (2.1) is (h_0, h)-uniformly attractive.

Theorem 3.4 is proved.

THEOREM 3.5 : *Let the following conditions hold :*

1. *Conditions* (1)-(3) *of Theorem* 3.1 *are satisfied.*
2. *For* $t > t_0 \geqq 0$ *and* $x \in E_A$

$$A(t)\, D_-\ V(t, x(t)) + V(t, x(t))\, D_-\ A(t) \leqq \leqq g(t, A(t)\, V(t, x(t)))\,, \quad if \quad t \neq \tau_\kappa\,, \quad \kappa = 1, 2, \dots\,,$$

$$A(\tau_\kappa + 0)\ V(\tau_\kappa + 0, x(\tau_\kappa) + I_\kappa(x(\tau_\kappa))) \leqq \leqq \psi_\kappa(A(\tau_\kappa)\, V(\tau_\kappa, x(\tau_\kappa)))\,, \quad if \quad t = \tau_\kappa\,,$$

where $A(t) > 0$ *is a piecewise continuous in* $\mathbb{R}_+$ *function with points of discontinuity* τ_κ *at which it is continuous from the left,* $A(\tau_\kappa + 0) > 0$, $\kappa = 1, 2, \dots$ *and* $A(t) \to \infty$ *as* $t \to \infty$.

Then, if the zero solution of equation (2.2) *is stable, then system* (2.1) *is* (h_0, h)*-equiasymptotically stable.*

Proof : Let $\lambda = \inf_{t \in \mathbb{R}_+} A(t)$. From the properties of the function $A(t)$ it follows that $\lambda > 0$.

Since V is h-positively definite in $S(h, \rho)$, then there exists a function $a \in \mathcal{K}$ such that

$$V(t, x) \geqq a(h(t, x))\,, \quad (t, x) \in S(h, \rho)\,. \tag{3.29}$$

Since V is weakly h_0-decrescent, then there exists a number $\delta_1 > 0$ and a function of $b \in C\mathcal{K}$ such that

$$V(t + 0, x) \leqq b(t, h_0(t + 0, x)) \quad \text{for} \quad h_0(t + 0, x) < \delta_1\,. \tag{3.30}$$

Let $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$. Set $\varepsilon_1 = \lambda a(\varepsilon)$. From the stability of the zero solution of system (2.2) it follows that there exists $\delta^* = \delta^*(t_0, \varepsilon_1) > 0$ such that if $u_0 < \delta^*$, then $r(t\,; t_0, u_0) < \varepsilon_1$, $t > t_0$, where $r(t\,; t_0, u_0)$ is the maximal solution of (2.1) for which $r(t_0 + 0\,; t_0, u_0) = u_0$. Repeating the proof of Theorem 3.1 (a), replacing $a(\varepsilon)$ by ε_1 and $V(t_0 + 0, x_0)$ by $A(t_0 + 0)\ V(t_0 + 0, x_0)$, we obtain that system (2.1) is (h_0, h)-stable.

Hence there exists $\delta_0 = \delta_0(t_0, \rho) > 0$ such that if $h_0(t_0 + 0, x_0) < \delta_0$, then $h(t, x(t\,; t_0, x_0)) < \rho$ for $t > t_0$.

Let $\eta > 0$ and $t_0 \in \mathbb{R}_+$ be given. From the stability of the zero solution of (2.2) it follows that there exists $\delta_1 = \delta_1(t_0, \eta) > 0$ such that $u_0 < \delta_1$ implies $r(t\,; t_0, u_0) < \eta$ for $t > t_0$. We can assume that δ_1 is a continuous and strictly increasing function of η for t_0 fixed.

Choose the number η so that

$$A(t_0 + 0)\, b(t_0, \delta_0) = \delta_1(t_0, \eta)\,. \tag{3.31}$$

Let $x(t) = x(t\,;\,t_0, x_0)$ be a solution of (2.1) for which $h_0(t_0 + 0, x_0) < \delta_0$. From (3.30) and (3.31) it follows that

$$A(t_0 + 0)\,V(t_0 + 0, x_0) \leqq A(t_0 + 0)\,b(t_0, h_0(t_0 + 0, x_0)) < \\ < A(t_0 + 0)\,b(t_0, \delta_0) = \delta_1\,,$$

hence

$$r(t\,;\,t_0, A(t_0 + 0)\,V(t_0 + 0, x_0)) < \eta\,. \tag{3.32}$$

On the other hand, applying Lemma 3.2, we obtain that for $t > t_0$ the following inequality holds

$$A(t)\,V(t, x(t)) \leqq r(t\,;\,t_0, A(t_0 + 0)\,V(t_0 + 0, x_0))\,. \tag{3.33}$$

Then from (3.29), (3.33) and (3.32) it follows that

$$A(t)\,a(h(t, x(t))) \leqq A(t)\,V(t, x(t)) \leqq \\ \leqq r(t\,;\,t_0, A(t_0 + 0)\,V(t_0 + 0, x_0)) < \eta\,.$$

Hence $h(t, x(t)) < a^{-1}(\eta/A(t))$. From the condition $A(t) \to \infty$ as $t \to \infty$ it follows that there exists $T^* = T^*(t_0, \varepsilon) > 0$ such that

$$h(t, x(t)) < \varepsilon \quad \text{for} \quad t > T^*\,.$$

Set $T = T(t_0, \varepsilon) = T^*(t_0, \varepsilon) - t_0$. Then

$$h(t, x(t)) < \varepsilon \quad \text{for} \quad t > t_0 + T$$

and thus it is proved that system (2.1) is (h_0, h)-equiattractive.

Theorem 3.5 is proved.

4. AN EXAMPLE

Consider the linear impulsive integro-differential equation

$$\begin{aligned} x'(t) &= -ax(t) + \int_{t_0}^{t} K(t, s)\,x(s)\,ds\,, \quad t \neq \tau_\kappa\,; \\ \Delta x|_{t = \tau_\kappa} &= -\alpha_\kappa\,x(\tau_\kappa)\,, \end{aligned} \tag{4.1}$$

where $a > 0$, $0 \leqq \alpha_\kappa \leqq 2$, $K \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $0 < \tau_1 < \tau_2 < \cdots$ and $\tau_\kappa \to \infty$ as $\kappa \to \infty$.

Let $h_0(t, x) = h(t, x) = |x|$. Consider the functions $A(t) = e^{\alpha t}$, $\alpha > 0$; $V(t, x) = x^2$. Then the sets E_1 and E_A are defined by

$$E_1 = \{x \in \mathscr{P}C[\mathbb{R}_+, \mathbb{R}] : x^2(s) \leqq x^2(t), t_0 \leqq s \leqq t\},$$
$$E_A = \{x \in \mathscr{P}C[\mathbb{R}_+, \mathbb{R}] : x^2(s)\, e^{\alpha s} \leqq x^2(t)\, e^{\alpha t}, t_0 \leqq s \leqq t\}.$$

For $t > t_0 \geqq 0$ and $x \in E_1$

$$D_- V(t, x(t)) = -2ax^2(t) + 2x(t) \int_{t_0}^{t} K(t, s)\, x(s)\, ds \leqq$$
$$\leqq 2V(t, x(t)) \left[-a + \int_{t_0}^{t} K(t, s)\, ds \right], \quad \text{if} \quad t \neq \tau_\kappa, \quad \kappa = 1, 2, \ldots$$

and for $t > t_0 \geqq 0$ and $x \in E_A$

$$A(t)\, D_- V(t, x(t)) + V(t, x(t))\, D_- A(t) =$$
$$= \alpha\, e^{\alpha t} x^2(t) + 2x(t) \left[-ax(t) + \int_{t_0}^{t} K(t, s)\, ds \right] e^{\alpha t}$$
$$\leqq A(t)\, V(t, x(t)) \left[\alpha - 2a + 2 \int_{t_0}^{t} K(t, s)\, e^{\frac{\alpha}{2}(t-s)}\, ds \right],$$
$$\text{if} \quad t \neq \tau_\kappa, \quad \kappa = 1, 2, \ldots$$

Moreover,

$$V(\tau_\kappa + 0, x(\tau_\kappa) - \alpha_\kappa x(\tau_\kappa)) = (1 - \alpha_\kappa)^2 V(\tau_\kappa, x(\tau_\kappa)) \leqq$$
$$\leqq V(\tau_\kappa, x(\tau_\kappa)), \quad \tau_\kappa > t_0, \quad x \in E_1;$$
$$A(\tau_\kappa + 0)\, V(\tau_\kappa + 0, x(\tau_\kappa) - \alpha_\kappa x(\tau_\tau)) = (1 - \alpha_\kappa)^2 A(\tau_\kappa)\, V(\tau_\kappa, x(\tau_\kappa)) \leqq$$
$$\leqq A(\tau_\kappa)\, V(\tau_\kappa, x(\tau_\kappa)), \quad \tau_\kappa > t_0, \quad x \in E_A.$$

Let the following inequality hold

$$\int_{t_0}^{t} K(t, s)\, ds \leqq a.$$

Then, applying Theorem 3.2 (a) for $g(t, u) \equiv 0$ and $B_\kappa(u) \equiv 0$, we obtain that the zero solution of equation (4.1) is uniformly stable.

Let the following inequality hold

$$\int_{t_0}^{t} K(t, s)\, ds \leqq a - \varepsilon, \quad \varepsilon > 0.$$

Applying Theorem 3.2 (b), we obtain that the zero solution of equation (4.1) is uniformly asymptotically stable.

If the inequality

$$\int_{t_0}^{t} K(t,s)\, e^{\frac{\alpha}{2}(t-s)}\, ds \leqq \frac{2a-\alpha}{2},$$

holds, then the conditions of Theorem 3.5 are satisfied for $g(t,u) \equiv 0$ and $B_R(u) \equiv 0$. Hence the zero solution of equation (4.1) is equiasymptotically stable.

REFERENCES

[1] D. D. BAINOV, V. LAKSHMIKANTHAM, P. S. SIMEONOV, *Theory of Impulsive Differential Equations* (to appear).

[2] A. M. SAMOILENKO, N. A. PERESTYUK, *Differential Equations with Impulse Effect,* Višča Škola, Kiev, 1987 (in Russian).

[3] G. K. KULEV, D. D. BAINOV, *Appplication of Lyapunov's direct method in the investigation of the global stability of the solutions of systems with impulse effect,* Appl. Analysis, Vol. 26 (1988), 255-270.

[4] G. K. KULEV, D. D. BAINOV, *Second method of Lyapunov and comparison principle for systems with impulse effect,* J. of Computational and Applied Mathematics, 23 (1988), 305-321.

[5] G. K. KULEV, D. D. BAINOV, *Stability of sets for systems with impulses,* Bull. Inst. of Math. Academia Sinica, Vol. 17, No. 4 (to appear).

[6] G. K. KULEV, D. D. BAINOV, *On the asymptotic stability of systems with impulses by the direct method of Lyapunov,* J. Math. Anal. Appl., 138 (1989) (to appear).

[7] HINZHI LIU, V. LAKSHMIKANTHAM, *Stability for impulsive differential systems in terms of two measures* (to appear).

[8] V. LAKSHMIKANTHAM, *Some problems in integro-differential equations of Volterra type,* J. Integral Equations, 10 (1985), 137-146.

[9] V. LAKSHMIKANTHAM, S. LEELA, *Differential and Integral Inequalities,* Vol. I, Academic Press, New York, 1969.

[10] V. LAKSHMIKANTHAM, M. RAMA MOHANA RAO, *Integro-differential equations and extension of Lyapunov's method,* J. Math. Anal. Appl., 30 (1970), 435-447.