M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

Rob Stevenson

Discrete Sobolev spaces and regularity of elliptic difference schemes

M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 25, nº 5 (1991), p. 607-640

<http://www.numdam.org/item?id=M2AN_1991__25_5_607_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 5, 1991, p. 607 à 640)

DISCRETE SOBOLEV SPACES AND REGULARITY OF ELLIPTIC DIFFERENCE SCHEMES (*)

Rob STEVENSON (1)

Communicated by V. THOMÉE

Abstract. — *This paper is concerned with the regularity of elliptic finite difference schemes with respect to discrete (fractional order) Sobolev spaces. For schemes arising from discretisations that are from the same « type » at the boundary as in the interior, it proves the discrete equivalent of Nečas' regularity theorem for differential operators on Lipschitz regions. A different proof was given by Hackbusch. However, the proof here is shorter and more transparent. In case of a curved boundary, usually different discretisations are applied in points near the boundary. For schemes of this kind, it is shown by using Nečas' theorem for the corresponding « unperturbed » scheme, that « minimal » regularity implies the stronger regularity from Nečas' theorem. Finally, conditions sufficient for minimal regularity are given.*

Résumé. — *Dans cet article on s'occupe de la régularité des schémas de différences finies elliptiques relativement à des espaces de Sobolev discrets (d'ordre fractionnaire). Pour des schémas provenant de discrétisations qui sont du même « type » à la frontière que dans l'intérieur, on démontre l'équivalent discret du théorème de régularité de Nečas pour les opérateurs différentiels sur les domaines à frontière lipschitzienne. Une démonstration différente a été donnée par Hackbusch. Toutefois, la démonstration donnée ici est plus courte et plus transparente. Dans le cas d'une frontière courbée, d'habitude on utilise des discrétisations différentes aux points près de la frontière. Pour les schémas de ce type on démontre, en utilisant le théorème de Nečas pour le schéma « imperturbé » correspondant, que la régularité « minimale » implique la régularité plus forte du théorème de Nečas. Enfin, on donne des conditions suffisantes pour la régularité minimale.*

(*) Received March 1990.

(1) Mathematical Institute, University of Utrecht, Budapestlaan 6, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands.

Subject : article ms 291 « Discrete Sobolev Spaces and Regularity of Elliptic Difference Schemes » by R. P. Stevenson.

M2AN Modélisation mathématique et Analyse numérique 0764-583X/91/05/606/34/$5.40
Mathematical Modelling and Numerical Analysis

1. INTRODUCTION

In this paper, we define spaces of grid functions, which are discrete versions of the Sobolev spaces $H^s(\mathbb{R}^d)$ and $H_0^s(\Omega)$ respectively. We study finite difference discretisations L_h of the homogeneous boundary value problem

$$Lu = f \quad u \in H_0^m(\Omega),$$

where

$$L = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta$$

is an strongly elliptic differential operator of order $2m$. We are interested in regularity properties of L_h with respect to the « discrete Sobolev spaces », which are uniform with respect to the mesh-width h. « Discrete regularity » of L_h is a useful property, for instance, to give sharp error estimates for the solution u_h of the discretized boundary value problem (*cf.* [5], [9] § 9.2) or to prove convergence of multi-grid methods to approximate u_h (*cf.* [4, 5], [8] § 6.3.2.2).

For Ω a « Lipschitz domain » and L_h arising from discretisations which are from the same « type » at the boundary as in the interior, a very important regularity result has been proved by Hackbusch in [5]. In this proof, he mentions and uses many results for the discrete Sobolev spaces.

The purpose of our paper is to give

- a shorter and more transparent proof of Hackbusch's theorem in [5]
- a more systematic and complete account of properties of the discrete Sobolev spaces
- easy-to-check conditions for the regularity of « general L_h », i.e. L_h arising from discretisations which are possibly of a different type at the boundary as in the interior
- easy-to-check conditions for the regularity of « scaled » general L_h ; the nature and purpose of this scaling will be explained in § 3.5.

The paper is organized as follows. In § 2, we define the discrete Sobolev spaces and show a number properties of these spaces, most of them being equivalents of well-known properties of the continuous Sobolev spaces. The properties proved in this section can also be found, possibly in slightly different forms, in the existing literature ([5] mainly, [12], [13], [14]), but there the results are either less general or stated without (satisfactory) proof. In particular, the author has never found proofs of the lemmas 2.4 and 2.6 (for $s \notin \mathbb{N}$) in the literature. Lemma 2.10 generalizes lemma 2.2 (ii) in [5]. Moreover, the proof of that lemma given in [5] does not cover all situations which have been considered there. The proof of theorem 2.12 (given in § 4) is based on [5], but two errors have been corrected.

In § 3, we concentrate on the question of the regularity of L_h. It is well known that, under very weak conditions concerning the smoothness of the coefficients, for the (generalized) L, it holds that

$$(L + \lambda I)^{-1} : H^{-m}(\Omega) \to H_0^m(\Omega) \text{ is bounded}$$

for $\lambda \geqslant 0$ large enough (Gårding inequality). Nečas ([10]) has proved that, for Ω a Lipschitz domain and for « sufficiently smooth » coefficients, it even holds that

$$(L + \lambda I)^{-1} : H^{-m+\theta}(\Omega) \to H_0^{m+\theta}(\Omega) \text{ is bounded },$$

the so-called $m + \theta$ regularity of $L + \lambda I$, for all $|\theta| < \theta_0$, $\theta_0 \leqslant 1/2$ small enough (precise formulations in § 3.1). These problems are often called « less regular », because for smoother coefficients and for $\Omega \in C^{2m}$ or Ω bounded, convex and $m = 1$, it holds that

$$(L + \lambda I)^{-1} : L^2(\Omega) \to H_0^m(\Omega) \cap H^{2m}(\Omega) \text{ is bounded}$$

(the « standard » $2m$ regularity), while even for still smoother problems the operator

$$(L + \lambda I)^{-1} : H^s(\Omega) \to H_0^m(\Omega) \cap H^{2m+s}(\Omega)$$

can be proved to be bounded for $s > 0$ (« higher order » $2m + s$ regularity).

In §§ 3.2-3.3, we consider only L_h arising from discretisations which are of the same type at the boundary as in the interior. We prove two regularity results, namely discrete versions of theorems of Gårding (§ 3.2) and Nečas (§ 3.3). Our discrete version of Gårding's theorem is a generalization of a result obtained by Stummel in [12]. A quite different proof of the discrete Nečas theorem can already be found in [5]. However, in contrast with [5], our proof makes use of the $m + \theta$ regularity of the corresponding $L + \lambda I$. This technique has been developed by Hackbusch in [7] to prove standard and higher order regularity for a number of « smoother problems ».

In § 3.4, we consider general L_h. We prove that m regularity of $L_h + \lambda I_h$ in combination with $m + \theta$ regularity of the operator induced by $L_h + \lambda I_h$ without the « discretisations of the different type » at the boundary (discrete Nečas applies) implies $m + \theta$ regularity of $L_h + \lambda I_h$. For given L_h, the reduced problem of checking m regularity is much easier to solve than proving $m + \theta$ regularity in the more direct manner of [5]. It will turn out that we are able to take $\lambda = 0$ in all above regularity results if L_h is stable with respect to the Euclidian norm.

In § 3.5, we discuss the regularity of « scaled » general L_h. We give some easy to check conditions sufficient for the m regularity of scaled and unscaled versions of L_h. Finally, as an illustration, we use the obtained results to investigate regularity of two popular discretisation schemes.

Notations 1.1. Ω is a domain (i.e. open and connected) in $\mathbb{R}^d$, $\Gamma = \partial\Omega = \bar{\Omega}\backslash\Omega$. For any $q \in \mathbb{N}$ of interest, we equip both $\mathbb{C}^q$ and $\mathbb{R}^q$ with standard basis $\{e_i\}_{1 \leq i \leq q}$, where $e_i = (0, \dots, 1, \dots, 0\ ) \in \mathbb{R}^q$, and norm $|a| = \left(\sum_{i=1}^{q} |a_i|^2\right)^{1/2}$.

$h \in (0, h_0]$ is the mesh-width of the grid $h\mathbb{Z}^d$. The constant h_0 is always assumed to be « small enough ». We consider families of grids $(\Omega_h)_{h \in (0, h_0]}$ with the property that there exists a $D > 0$ such that for all $h \in (0, h_0]$

$$\{x \in h\mathbb{Z}^d : \text{dist }(x, \mathbb{R}^d\backslash\Omega) \geq Dh\} \subset \Omega_h \subset \{x \in h\mathbb{Z}^d : \text{dist }(x, \Omega) \leq Dh\}\ ,$$

where for $A \subset \mathbb{R}^d$, dist $(x, A) = \inf\ \{|x - y| : y \in A\}$. An example of such a family is $(\Omega \cap h\mathbb{Z}^d)_{h \in (0, h_0]}$.

In this paper, we investigate operators, grids etc., which depend on the mesh-width h. In most cases, the obtained results are only significant since they hold « uniformly in h ». In order to reduce the number of clauses as « uniformly in h », « for all h » etc., we use in this paper the convention that c, c', C etc. stand for positive constants not necessarily the same throughout the text, but which are always *independent of* $h \in (0, h_0]$. Furthermore in the sequel, we also use notations as L_h, Ω_h etc., where formally seen $(L_h)_{h \in (0, h_0]}$, $(\Omega_h)_{h \in (0, h_0]}$ etc. would be more correct.

For $A \subset \mathbb{R}^d$, $\eta > 0$, we denote $\{x \in \mathbb{R}^d : \text{dist }(x, A) < \eta\}$ by $A(\eta)$. For d-tuples $\alpha \in \mathbb{N}^d$, we put $|\alpha| = \sum_{i=1}^{d} |\alpha_i|$. All inequalities for matrices should be understood in terms of their elements.

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $C^n(\bar{\Omega})$ is the space of complex valued functions u with uniformly continuous and bounded derivatives $D^\alpha u(|\alpha| \leq n)$ on Ω, with norm

$$\|u\|_{C^n(\Omega)} := \sup\ \{|D^\alpha u(x)| : |\alpha| \leq n, x \in \Omega\}\ .$$

$\bigcap_{n=0}^{\infty} C^n(\bar{\Omega})$ is denoted by $C^\infty(\bar{\Omega})$.

For $\lambda \in (0, 1]$, we define

$$C^{n,\lambda}(\bar{\Omega}) = \left\{u \in C^n(\bar{\Omega}) : \|u\|_{C^{n,\lambda}(\Omega)} < \infty\right\}\ ,$$

where

$$\|u\|_{C^{n,\lambda}(\Omega)} = \max\ \left\{\|D^\alpha u\|_{C^{0,\lambda}(\Omega)} : |\alpha| \leq n\right\}$$

and

$$\|v\|_{C^{0,\lambda}(\Omega)} = \max\left\{\|v\|_{C^0(\Omega)}, \sup\left\{\frac{|v(x)-v(y)|}{|x-y|^\lambda} : x, y \in \Omega, x \neq y\right\}\right\}.$$

Finally, we put $C^{n,0}(\bar{\Omega}) = C^n(\bar{\Omega})$ and $\|\cdot\|_{C^{n,0}(\Omega)} = \|\cdot\|_{C^n(\Omega)}$.

2. DISCRETE SOBOLEV SPACES

2.1. Definitions and basic properties

In this subsection the discrete counterparts of $H^s(\mathbb{R}^d)$ and $H_0^s(\Omega)$ are defined. For the notation of the spaces $H^s(\mathbb{R}^d)$ and $H_0^s(\Omega)$ see, for instance, [1]. For each $h \in (0, h_0]$, we consider the space $\mathbb{G}(h\mathbb{Z}^d)$ of complex valued grid functions u_h on $h\mathbb{Z}^d$ with $\sum_{j \in \mathbb{Z}^d} |u_h(jh)|^2 < \infty$. $\mathbb{G}(h\mathbb{Z}^d)$ is a Hilbert space with the standard scalar product

$$\langle u_h, v_h\rangle = h^d \sum_{j \in \mathbb{Z}^d} u_h(jh)\,\overline{v_h(jh)},$$

which defines the norm

$$\|u_h\|_0 = \sqrt{\langle u_h, u_h\rangle}.$$

In order to define an s-scalar product on $\mathbb{G}(h\mathbb{Z}^d)$ corresponding to the scalar product of $H^s(\mathbb{R}^d)$, we use the discrete Fourier transform

$$\hat{u}_h(\xi) = \left(\frac{1}{2\pi}\right)^{d/2} \sum_{j \in \mathbb{Z}^d} u_h(jh)\, e^{-ij\cdot\xi} \quad (\xi \in T^d = [-\pi, \pi)^d)$$

with back transformation formula

$$u_h(jh) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{T^d} \hat{u}_h(\xi)\, e^{ij\cdot\xi}\, d\xi.$$

Note that $\langle u_h, v_h\rangle = h^d \langle \hat{u}_h, \hat{v}_h\rangle_{L^2(T^d)}$ (Parseval relation).

We define for $s \in \mathbb{R}$

$$\langle u_h, v_h\rangle_s = h^d \int_{T^d} \left(1 + 4h^{-2} \sum_{j=1}^d \sin^2\frac{\xi_j}{2}\right)^s \hat{u}_h(\xi)\,\overline{\hat{v}_h(\xi)}\, d\xi$$

and

$$\|u_h\|_s = \sqrt{\langle u_h, u_h\rangle_s}.$$

For $s \in \mathbb{R}$, it holds that

$$\|u_h\|_{-s} = \sup \left\{ \frac{|\langle u_h, v_h \rangle|}{\|v_h\|_s} : 0 \neq v_h \in \mathbb{G}(h\mathbb{Z}^d) \right\} . \tag{2.1}$$

For each h, all s-norms are equivalent, but the equivalence does not hold uniformly in h.

Note that for $s < t$, $\|\cdot\|_t \leq c(t-s)\, h^{s-t} \|\cdot\|_s$.

The above definition of the s-norm is natural since for $s = n = 1$ it corresponds to the usual definition

$$\|u_h\|_n^* = \left(\sum_{|\alpha| \leq n, \alpha \in \mathbb{N}_0^d} \|\partial_h^\alpha u_h\|_0^2 \right)^{1/2} \quad (n \in \mathbb{N}_0) , \tag{2.2}$$

where

$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \dots \partial_{h,d}^{\alpha_d} \ (\alpha \in \mathbb{N}_0^d) , \quad T_h^\gamma = T_{h,1}^{\gamma_1} \dots T_{h,d}^{\gamma_d} \quad (\gamma \in \mathbb{Z}^d) ,$$

$$\partial_{h,j} = h^{-1}(I_h - T_{h,j}^{-1}) \quad \text{and} \quad T_{h,j}\, u_h(x) = u_h(x + e_j h) ,$$

while for general $n = s \in \mathbb{N}_0$, c, $C > 0$ exist with

$$c\|\cdot\|_n^* \leq \|\cdot\|_n \leq C\|\cdot\|_n^* . \tag{2.3}$$

We now consider discrete analogues of $H_0^s(\Omega)$. For each $h \in (0, h_0]$, let $\mathbb{G}(\Omega_h)$ be the space of the grid functions u_h on Ω_h (*cf.* notation 1.1) with $\sum_{x \in \Omega_h} |u_h(x)|^2 < \infty$. Define $\omega_h : \mathbb{G}(\Omega_h) \to \mathbb{G}(h\mathbb{Z}^d)$ as the extension with zero. $\mathbb{G}(\Omega_h)$ becomes a Hilbert space with the scalar product

$$\langle\!\langle u_h, v_h \rangle\!\rangle = \langle \omega_h u_h, \omega_h v_h \rangle .$$

We define norms on $\mathbb{G}(\Omega_h)$ by

$$\|u_h\|_{s,0} = \|\omega_h u_h\|_s \quad (s \geq 0)$$

and

$$\|u_h\|_{-s,0} = \sup \left\{ \frac{|\langle\!\langle u_h, v_h \rangle\!\rangle|}{\|v_h\|_{s,0}} : 0 \neq v_h \in \mathbb{G}(\Omega_h) \right\} \quad (s > 0) .$$

Note that $\|\cdot\|_{-s,0} \neq \|\omega_h \cdot\|_{-s}$ (unless $\Omega_h = h\mathbb{Z}^d$).

Remark 2.1 : Sometimes, we will apply norms $\|\cdot\|_s$ and $\|\cdot\|_{s,0}$ ($s \in \mathbb{R}$) also to non-grid functions. In that case, $\|\cdot\|_s$ stands for the norm on

$H^s(\mathbb{R}^d)$ and $\|\cdot\|_{s,0}$ for the norm on $H_0^s(\Omega)$ (or $H^s(\Omega) = (H_0^{-s}(\Omega))'$, if $s < 0$).

2.2. Further properties

For the discrete Sobolev spaces we state a couple of properties, which, except for lemma 2.10, have well-known equivalents in the corresponding continuous spaces.

LEMMA 2.2 : *For any $h > 0$, $t > 0$, $\{(\mathbb{G}(h\mathbb{Z}^d), \langle \cdot, \cdot \rangle_s) : -t \leq s \leq t\}$, with pivot space $(\mathbb{G}(h\mathbb{Z}^d), \langle \cdot, \cdot \rangle)$, is a so-called Hilbert scale (cf.* e.g. [8] § 1.4.4).

Proof : Define

$$\mathscr{A}_h = \mathscr{I}_h - \sum_{j=1}^{d} T_{h,j}(\partial_{h,j})^2 : \mathbb{G}(h\mathbb{Z}^d) \to \mathbb{G}(h\mathbb{Z}^d). \tag{2.4}$$

For any $h > 0$, $\mathscr{A}_h$ is bounded, and because of $(\partial_h^\alpha)^* = (-1)^{|\alpha|} T_h^\alpha \partial_h^\alpha$, $(T_h^\alpha)^* = T_h^{-\alpha}$, self-adjoint and positive definite, so $\mathscr{A}_h^r$ exists for all $r \in \mathbb{R}$. One can verify that

$$\|\cdot\|_s = \|\mathscr{A}_h^{s/2} \cdot\|_0, \tag{2.5}$$

which implies the Hilbert scale property. □

Remark 2.3 : The Hilbert scale property makes it possible to use « interpolation » (*cf.* [8] § 1.4.4).

LEMMA 2.4 : *$\forall p, q, r \in \mathbb{R}$, with $p < q < r$ and $\varepsilon > 0$ there is a $C(\varepsilon)$ such that*

$$\|u_h\|_q \leq \varepsilon \|u_h\|_r + C(\varepsilon)\|u_h\|_p \quad \textit{for all } u_h \in \mathbb{G}(h\mathbb{Z}^d).$$

Proof :

$$\begin{aligned}
\|u_h\|_q &= \langle \mathscr{A}_h^{q/2} u_h, \mathscr{A}_h^{q/2} u_h \rangle^{1/2} = \langle \mathscr{A}_h^{r/2} u_h, \mathscr{A}_h^{q-r/2} u_h \rangle^{1/2} \\
&\leq (\|u_h\|_r \|u_h\|_{2q-r})^{1/2} \\
&\leq \frac{\varepsilon}{2} \|u_h\|_r + \frac{1}{\varepsilon} \|u_h\|_{2q-r}.
\end{aligned}$$

The repetition of this argument at most a finite number of times will complete the proof. □

DEFINITION 2.5 : *For $u_h \in \mathbb{G}(h\mathbb{Z}^d)$, we define*

$$\|u_h\|_\infty = \sup \{|u_h(jh)| : j \in \mathbb{Z}^d\}.$$

LEMMA 2.6 : *$\forall s > \frac{d}{2}\ \exists c(s)$ such that*

$\|u_h\|_\infty \leq c(s)\|u_h\|_s$ *for all $u_h \in \mathbb{G}(h\mathbb{Z}^d)$* « discrete Sobolev inequality » .

Proof : From the back-transformation formula and the Schwarz inequality it follows that

$$\|u_h\|_\infty \leq \left(\frac{1}{2\pi}\right)^{d/2} \int_{T^d} |\hat{u}_h(\xi)|\, d\xi \leq \left(\frac{1}{2\pi}\right)^{d/2} \times$$
$$\times \left(\int_{T^d} \left(1 + 4h^{-2}\sum_{j=1}^{d} \sin^2\frac{\xi_j}{2}\right)^{-s} d\xi\right)^{1/2} \times h^{-d/2}\|u_h\|_s .$$

For given h, define the continuous function f by $f(x) = 1$ for $x \in [-h, h]$ and $f(x) = h^{-2}x^2$ for $x \in T\backslash[-h, h]$. $\forall \xi \in T^d$, it holds that $\sum_{j=1}^{d} f(\xi_j) \leq c\left(1 + 4h^{-2}\sum_{j=1}^{d}\sin^2\frac{\xi_j}{2}\right)$, where $c = \max\left(d, \frac{\pi^2}{4}\right)$. It follows from this inequality that for $s > d/2$

$$\int_{T^d}\left(1 + 4h^{-2}\sum_{j=1}^{d}\sin^2\frac{\xi_j}{2}\right)^{-s} d\xi \leq c^s \int_{T^d}\left(\sum_{j=1}^{d} f(\xi_j)\right)^{-s} d\xi$$
$$\leq c^s \int_{T^d}\left(\prod_{j=1}^{d} f(\xi_j)^{-s/d}\right) d\xi$$
$$\leq h^d c^s \left(\frac{4s}{2s-d}\right)^d ,$$

which gives the proof of the lemma. □

LEMMA 2.7 ([14] lemma 2.4) : *For $\Omega \subset [a, b] \times \mathbb{R}^{d-1}$, $p \in \mathbb{N}_0$, $\exists C(b-a, p)$ such that*

$$\|u_h\|_{p,0} \leq C(b-a, p)\left(\sum_{|\alpha| = p} \|\partial_h^\alpha \omega_h u_h\|_0^2\right)^{1/2} \textit{for all } u_h \in \mathbb{G}(\Omega_h)$$

« Discrete Poincaré inequality ».

DEFINITION 2.8 : *For every $h \in (0, h_0]$, let A_h be a subset of $h\mathbb{Z}^d$. Then A_h (more exactly the family $(A_h)_{h\in(0,h_0]}$) is said to have the* discrete cone property, *hereafter abbreviated by* d.c.p., *if*

$\forall k \in \mathbb{N}\quad \exists M > 0, h_1 \in (0, h_0]\quad \forall h \in (0, h_1], x \in A_h\quad \exists \alpha \in \mathbb{Z}^d,\ |\alpha| \leq M$

with $\{x + \alpha h + \delta h : \delta \in \mathbb{Z}^d,\ |\delta| < k\} \subset A_h$.

DEFINITION 2.9 : *For* $\ell \in \mathbb{N}$, *we denote* $\{x \in \Omega_h : \text{dist}\ (x, h\mathbb{Z}^d \backslash \Omega_h) \leq \ell h\}$ *by* $\Gamma_h(\ell)$.

We define $\gamma_h(\ell) : \mathbb{G}(\Omega_h) \to \mathbb{G}(\Omega_h)$ by

$$\begin{aligned}(\gamma_h(\ell)\, u_h)(x) &= u_h(x) \quad x \in \Gamma_h(\ell) \\ &= 0 \qquad\quad x \in \Omega_h \backslash \Gamma_h(\ell)\,.\end{aligned}$$

The following lemma plays a crucial role in many of the proofs given in sections 3 and 4.

LEMMA 2.10 : *Let* Ω_h *such that* $h\mathbb{Z}^d \backslash \Omega_h$ *has the d.c.p.. Then for all* ℓ, $k \in \mathbb{N}$, *there exists a* $c > 0$ *such that*

$$\|\gamma_h(\ell)\|_{t, 0 \leftarrow s, 0} \leq c h^{s-t} \quad \textit{for all } s, t \in [-k, k]\,.$$

Proof: Since $\exists c$ such that $\|\cdot\|_{p,0} \leq c h^{q-p} \|\cdot\|_{q,0}$ for all q, $p \in [-k, k], p > q$, it holds that

$$\|\gamma_h(\ell)\|_{t, 0 \leftarrow s, 0} \leq c h^{-k-t} \|\gamma_h(\ell)\|_{-k, 0 \leftarrow 0, 0} \|\gamma_h(\ell)\|_{0, 0 \leftarrow k, 0}\, c h^{s-k}\,.$$

Furthermore because $\gamma_h(\ell)^* = \gamma_h(\ell)$, $\|\gamma_h(\ell)\|_{-k, 0 \leftarrow 0, 0}$ equals $\|\gamma_h(\ell)\|_{0, 0 \leftarrow k, 0}$ (1), so it suffices to give the proof of the lemma for the case $s = k$ and $t = 0$.

Now let h_1, M as in definition 2.8, $h \in (0, h_1]$, $u_h \in \mathbb{G}(\Omega_h)$ and $x \in \Gamma_h(\ell)$. There exists an $\alpha \in \mathbb{Z}^d$, with $|\alpha| \leq \ell + M$, such that

$$\{x + \alpha h + \delta h : \delta \in \mathbb{Z}^d,\ |\delta| < k\} \subset h\mathbb{Z}^d \backslash \Omega_h\,.\ (2)$$

(1) For any fixed $h \in (0, h_0]$, s, $t \in \mathbb{R}$ and any linear operator C_h on $\mathbb{G}(\Omega_h)$, bounded with respect to an $\|\cdot\|_{r,0}$-norm, it holds that $\|C_h\|_{t,0 \leftarrow s,0} = \|C_h^*\|_{-s,0 \leftarrow -t,0}$. This follows from the following observations : Since all $\|\cdot\|_{r,0}$-norms are equivalent (h fixed), the vectorspace of bounded linear functionals on $\mathbb{G}(\Omega_h)$, denoted by $\mathbb{G}(\Omega_h)'$, does not depend on r. Moreover $f_h \in \mathbb{G}(\Omega_h)'$ can be written as $f_h = \langle\!\langle \cdot\,, R_h(f_h) \rangle\!\rangle$, where R_h is the Riesz operator with respect to the standard inner product. It holds that $C_h^* = R_h C_h^{\times} R_h^{-1}$ where $C_h^{\times} : \mathbb{G}(\Omega_h)' \to \mathbb{G}(\Omega_h)'$, the dual of C_h. The Hahn-Banach theorem implies that the operator norm of $C_h^{\times} : (\mathbb{G}(\Omega_h), \|\cdot\|_{t,0})' \to (\mathbb{G}(\Omega_h), \|\cdot\|_{s,0})'$, with respect to the dual norms, equals $\|C_h\|_{t,0 \leftarrow s,0}$. Finally, from the definition of the $\|\cdot\|_{r,0}$-norm, we find that $R_h : (\mathbb{G}(\Omega_h), \|\cdot\|_{r,0})' \to (\mathbb{G}(\Omega_h), \|\cdot\|_{-r,0})$ is an isomorphism for all $r \in \mathbb{R}$.

(2) The proof of [5] lemma 2.2 (ii), a lemma which corresponds with our lemma 2.10, requires $\alpha = \alpha_i e_i$. Also for the more restrictive class of Ω_h considered in [5], this is clearly not always possible.

Firstly we assume that $\alpha = \alpha(x) \geq 0$. We state the *discrete Taylor formula* (*cf.* [14] lemma 2.1):
There exist constants $r_{\alpha\beta\gamma}$, with indices $\beta, \gamma \in \mathbb{N}_0^d$, such that

$$T_h^{-\alpha} = (I_h - h\,\partial_h)^\alpha = \sum_{\substack{|\beta| < k \\ \beta \leq \alpha}} (-h)^{|\beta|} \binom{\alpha}{\beta} \partial_h^\beta + h^k \sum_{\substack{|\beta| = k \\ |\gamma| \leq |\alpha| - k}} r_{\alpha\beta\gamma}\, T_h^{-\gamma}\, \partial_h^\beta ,$$

where $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$. Consequently

$$u_h(x) = T_h^{-\alpha}\, \omega_h\, u_h(x + \alpha h) = h^k \sum_{\substack{|\beta| = k \\ |\gamma| \leq |\alpha| - k}} r_{\alpha\beta\gamma}\, T_h^{-\gamma}\, \partial_h^\beta\, \omega_h\, u_h(x + \alpha h) .$$

It is clear that for $\alpha \ngeq 0$ a similar relation can be obtained. For example for $\alpha \leq 0$, write $T_h^{-\alpha} = (I_h + hT_h\,\partial_h)^{-\alpha}$.

Since every $y (= x + (\alpha - \gamma)\, h) \in h\mathbb{Z}^d$ is involved only in at most finitely many (only dependent on ℓ and M, and thus on ℓ and k) of such sums and $\|\cdot\|_k$ and $\|\cdot\|_k^*$ are equivalent norms, one can now conclude the proof by summing over $x \in \Gamma_h(\ell)$ and applying the Schwarz inequality. □

Remark 2.11 : Let Ω_h such as always be related to Ω as explained in notations 1.1. Then if $\mathbb{R}^d \backslash \Omega$ has the *cone property (c.p.)* (*cf.* [1] p. 66), $h\mathbb{Z}^d \backslash \Omega_h$ has the d.c.p. The converse of this statement does not hold. This can be seen by noticing that the « discrete cones » shrink if h goes to zero.

We will now state the theorem that, for Ω with « sufficiently smooth » boundary, the $\|\cdot\|_{s,0}$-norms on $\mathbb{G}(\Omega_h)$ are equivalent to the corresponding norms of a Hilbert scale (*cf.* lemma 2.2). The proof of this theorem 2.12, which is rather lengthy and technical, is not given until section 4.

THEOREM 2.12 : *Let Ω have the strong local Lipschitz property* (see definition 2.14 below) *and let $k \in \mathbb{N}$ be given. Denote $\mathscr{A}_h^k$ by $\mathscr{B}_h$, where* $\mathscr{A}_h = \mathscr{I}_h - \sum_{j=1}^{d} T_{h,j}(\partial_{h,j})^2$ (see (2.4)), *and $\omega_h^*\, \mathscr{B}_h\, \omega_h : \mathbb{G}(\Omega_h) \to \mathbb{G}(\Omega_h)$ by B_h. Then c, $C > 0$ exist with*

$$c\,\|u_h\|_{s,0} \leq \|B_h^{s/2k} u_h\|_{0,0} \leq C\,\|u_h\|_{s,0} \quad \textit{for all } s \in [-k, k],\ u_h \in \mathbb{G}(\Omega_h) .$$

(recall that, by convention, c, C are independent of h).

Remark 2.13 : Since B_h is bounded (fixed h), self-adjoint and positive definite, one can verify that for any $h > 0$

$$\{(\mathbb{G}(\Omega_h),\ \langle\!\langle B_h^{s/2k}\cdot, B_h^{s/2k}\cdot \rangle\!\rangle) : -k \leq s \leq k\} ,$$

with pivot space $(\mathbb{G}(\Omega_h), \langle\!\langle \cdot, \cdot \rangle\!\rangle)$, is a Hilbert scale (*cf.* proof lemma 2.2). Consequently interpolation can now also be applied to the $\| \cdot \|_{s,0}$-norms.

This result is not needed in the remainder of this paper, but can be used for example for proving the so-called smoothing property (*cf.* [8]) $(\| L_h S_h^{\nu} \|_{q,0 \leftarrow p,0} \leq \eta(\nu) h^{-\alpha}$ with $\alpha = (2 - p + q) m > 0)$ for two(multi)-grid methods applied to difference schemes if $\alpha < 2m$ (*cf.* [8] § 6.2.4.4, [4]). We will be interested in this case when dealing with multi-grid methods applied to « less regular » difference schemes, such as considered in section 3.

The definition of the strong local Lipschitz property is given in [1] chapter IV. Since we will refer to constants which appear in the definition we recall this definition here :

DEFINITION 2.14 : *A domain Ω has the* strong local Lipschitz property *(hereafter abbreviated by* s.l.L.p) *provided there exist positive numbers δ and L, a countable open cover $\{U_j\}$ of $\Gamma = \partial\Omega$, and for each U_j a real-valued function f_j of $d-1$ real variables, such that the following conditions hold :*

(i) *For some finite R, every collection of $R+1$ of the sets U_j has an empty intersection.*

(ii) *For every pair of points $x, y \in \Gamma(\delta)$* (3) (cf. *notation* 1.1) *such that $|x - y| < \delta$, there exists j such that*

$$x, y \in V_j = \{x \in U_j : \text{dist}\ (x, \partial U_j) > \delta\} .$$

(iii) *Each function f_j satisfies a Lipschitz condition with constant L :*

$$|f(\xi_1, \dots, \xi_{d-1}) - f(\eta_1, \dots, \eta_{d-1})| \leq L\, |(\xi_1 - \eta_1, \dots, \xi_{d-1} - \eta_{d-1})| .$$

(iv) *For some Cartesian coordinate system $(\xi_{j,1}, \dots, \xi_{j,d})$ in U_j the set $\Omega \cap U_j$ is represented by the inequality*

$$\xi_{j,d} < f_j(\xi_{j,1}, \dots, \xi_{j,d-1}) .$$

Remarks 2.15 : If Ω has the s.l.L.p., then Ω (and $\mathbb{R}^d \backslash \Omega$) has the c.p. For bounded Ω, Ω has the s.l.L.p. if and only if each point $x \in \partial\Omega$ has a neighbourhood U such that the set $U \cap \Omega$ is represented by the inequality $\xi_d < f(\xi_1, \dots, \xi_{d-1})$ for some Cartesian coordinate system, where function f satisfies a Lipschitz condition.

Remark 2.16 : In the proof of theorem 2.12 we use the following property : Let Ω be a domain with the s.l.L.p. Then there is an $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0]$, $\Omega(\eta)$ has the s.l.L.p. with the same « L » and « R » as Ω has.

(3) In [1], condition (ii) is imposed for $x, y \in \Gamma(\delta) \cap \Omega$ only. We have adapted this definition slightly in order to make the s.l.L.p. symmetric in the sense that if Ω has the s.l.L.p., then $\mathbb{R}^d \backslash \Omega$ also has this property.

3. REGULARITY OF ELLIPTIC DIFFERENCE SCHEMES

3.1. Introduction

In this section, we study the regularity of difference operators $L_h : \mathbb{G}(\Omega_h) \to \mathbb{G}(\Omega_h)$. In doing so, we will have to distinguish between (general) L_h arising from discretisations which are possibly of a different « type » at the boundary than in the interior (§ 3.4) and L_h arising from discretisations which are of the same « type » everywhere on Ω_h (§§ 3.2-3.3). Hereafter, difference operators of the second kind will be denoted as L_h'.

For L_h', we will prove discrete versions of the Gårding inequality (theorem 3.1) and Nečas' theorem (theorem 3.3) stated below; these theorems concern the homogeneous boundary value problem $Lu = f$, $u \in H_0^m(\Omega)$. These results for L_h' will be used to show regularity of general L_h. Finally in § 3.5, we discuss the regularity of a scaling of a general L_h; the purpose of the scaling being to keep the coefficients of the corresponding « difference star » $\mathcal{O}(h^{-2m})$ also at the boundary. Furthermore, in this subsection we discuss two examples.

THEOREM 3.1 (Gårding inequality) : *Let*

$$L = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^{\alpha} a_{\alpha\beta}(\,.\,) D^{\beta},$$

with $m \in \mathbb{N}$, $a_{\alpha\beta} \in L^{\infty}(\Omega)$ *for all* α, β, $a_{\alpha\beta} \in C^0(\bar{\Omega})$ *(thus uniformly continuous) for* $|\alpha| = |\beta| = m$, *and* L strongly elliptic, *which means*

$$\exists \varepsilon > 0 \text{ with } \operatorname{Re}\left(\sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta} \right) \geq$$

$$\geq \varepsilon \|\xi\|^{2m} \text{ for all } \xi \in \mathbb{R}^d, x \in \Omega .$$

Then the sesquilinear form associated with L *is* $H_0^m(\Omega)$*-coercive :*

$$\exists \lambda_0 \geq 0,\ c > 0\ \forall u \in H_0^m(\Omega)$$

$$\operatorname{Re} \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} a_{\alpha\beta}(x)\, D^{\beta} u(x)\, \overline{D^{\alpha} u(x)}\, dx \geq c \|u\|_m^2 - \lambda_0 \|u\|_0^2 .$$

The proof of this theorem can be found in every textbook about elliptic partial differential equations ([4]).

([4]) In many monographs the condition Ω bounded is added. However, this condition is not needed because one can adapt the proof such that the Poincaré inequality has to be applied only to functions with uniformly bounded support, also in case Ω is unbounded.

COROLLARY 3.2 : *Since the sesquilinear form is bounded too, it follows from the Lax-Milgram lemma that*

$$\forall \lambda \geqslant \lambda_0 \ (L + \lambda I)^{-1} : H^{-m}(\Omega)(= (H_0^m(\Omega))') \to H_0^m(\Omega) \text{ bounded}.$$

THEOREM 3.3 (Nečas [10] théorème 3) : *Consider the situation of theorem* 3.1, *but now let* Ω *be bounded and have the* s.l.L.p.

Put $\sigma = \sup \left\{\chi \mid \forall \alpha, \beta \, |\alpha| = m, a_{\alpha\beta} \in C^{0,\chi}(\bar{\Omega})\right\}$,

$\tau = \sup \left\{\chi \mid \forall \alpha, \beta \, |\beta| = m, a_{\alpha\beta} \in C^{0,\chi}(\bar{\Omega})\right\}$,

where by convention $\sup \varnothing = 0$.

Then there is a $\theta_0 \in (0, 1/2]$, *such that for all*

$$\theta \in (-\theta_0, \theta_0) \cap (-\tau, \sigma) \cup \{0\},$$

there exists a $\lambda_0 \geqslant 0$ *with*

$$\forall \lambda \geqslant \lambda_0 \ (L + \lambda I)^{-1} : H^{-m+\theta}(\Omega) \to H_0^{m+\theta}(\Omega) \text{ bounded}.$$

If $a_{\alpha\beta} + a_{\beta\alpha}$ *is a real-valued for* $|\alpha| = |\beta| = m$, *the theorem holds with* $\theta_0 = 1/2$.

Remark 3.4 : In § 3.4, it will turn out that we are able to take $\lambda_0 = 0$ in the discrete analogues of the above theorems, if $\|L_h^{-1}\|_{0,0 \leftarrow 0,0} \leqslant c$ (*stability*).

3.2. The difference operator L_h' and the discrete Gårding inequality.

DEFINITIONS 3.5 : *We consider difference operators* $\mathscr{L}_h$ *on* $h\mathbb{Z}^d$ *of the form*

$$\mathscr{L}_h = \sum_{|\alpha|, |\beta| \leqslant m} \sum_{\gamma, \delta} (-1)^{|\alpha|} \partial_h^\alpha T_h^\gamma c_{\alpha\beta\gamma\delta}(\cdot, h) T_h^\delta \partial_h^\beta, \quad m \in \mathbb{N},$$

with for $\alpha, \beta \in \mathbb{N}_0^d$, $\gamma, \delta \in \mathbb{Z}^d$, $c_{\alpha\beta\gamma\delta}$ bounded functions *on* $\mathbb{R}^d \times [0, h_0]$, *of which finitely many are non-zero* (5).

(5) Let $\mathscr{L} = \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta}(\cdot) D^\beta$. Consider a discretisation of $\mathscr{L}$ of the form

$$\sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\alpha|} \mathscr{P}_{h\alpha\beta} \mathscr{Q}_{h\alpha\beta} \mathscr{R}_{h\alpha\beta},$$

with finite difference operators $\mathscr{P}_{h\alpha\beta} = h^{-|\alpha|} \sum_\mu p_{\mu\alpha\beta}(h) T_h^\mu$, $\mathscr{Q}_{h\alpha\beta} = \sum_\mu q_{\mu\alpha\beta}(\cdot, h) T_h^\mu$ and $\mathscr{R}_{h\alpha\beta} = h^{-|\beta|} \sum_\mu r_{\mu\alpha\beta}(h) T_h^\mu$, which are consistent discretisations of D^α, $a_{\alpha\beta}(\cdot) I$ and D^β respectively. Assume that $p_{\mu\alpha\beta} \in C^{|\alpha|-1,1}(\overline{(0, h_0)})$ if $\alpha \neq 0$, $p_{\mu\alpha\beta}$ bounded on $[0, h_0]$ if $\alpha = 0$, $q_{\mu\alpha\beta}$ bounded on $\mathbb{R}^d \times [0, h_0]$, $r_{\mu\alpha\beta} \in C^{|\beta|-1,1}(\overline{(0, h_0)})$ if $\beta \neq 0$, $r_{\mu\alpha\beta}$ bounded on $[0, h_0]$ if $\beta = 0$. Lemma 2.2 of [14] shows that such a discretisation can be written in the form of $\mathscr{L}_h$, with $\sum_{\gamma, \delta} c_{\alpha\beta\gamma\delta}(x, 0) = a_{\alpha\beta}(x)$.

Corresponding to an $\mathscr{L}_h$, *we define*

(*a*) $\mathscr{L}_h^{(p)} = \sum_{|\alpha| = |\beta| = m} \sum_{\gamma, \delta} (-1)^{|\alpha|} \partial_h^\alpha T_h^\gamma c_{\alpha\beta\gamma\delta}(\cdot, h) T_h^\delta \partial_h^\beta$.

(*b*) $\mathscr{L}_h^{(r)} = \mathscr{L}_h - \mathscr{L}_h^{(p)}$.

(*c*) *for* $x \in \mathbb{R}^d$,

$$\mathscr{L}_{h,x}^{(p)} = \sum_{|\alpha| = |\beta| = m} \sum_{\gamma, \delta} (-1)^{|\alpha|} \partial_h^\alpha T_h^\gamma c_{\alpha\beta\gamma\delta}(x, 0) T_h^\delta \partial_h^\beta ;$$

that is, $\mathscr{L}_h^{(p)}$ *« frozen at »* $(x, 0)$.

(*d*) *the* symbol

$$\rho(x, \xi) = \sum_{|\alpha| = |\beta| = m} \sum_{\gamma, \delta} c_{\alpha\beta\gamma\delta}(x, 0) e^{i\xi \cdot (\delta + \gamma - \alpha)} \prod_{j=1}^{d} (1 - e^{-i\xi_j})^{\beta_j} (1 - e^{i\xi_j})^{\alpha_j} .$$

(*e*) $L_h' = \omega_h^* \mathscr{L}_h \omega_h$ *a difference operator on* Ω_h.

L_h' *is said to be* strongly elliptic *if* $\exists \varepsilon > 0$ *such that*

$$\operatorname{Re} \rho(x, \xi) \geq \varepsilon \left(\sum_{j=1}^{d} 4 \sin^2 \frac{\xi_j}{2} \right)^m \quad \text{for all} \quad x \in \Omega, \quad \xi \in T^d .$$

THEOREM 3.6 (discrete Gårding inequality) : *Let* L_h' *be strongly elliptic, with* $c_{\alpha\beta\gamma\delta} \in C^0(\overline{\mathbb{R}^d \times (0, h_0)})$ *for* $|\alpha| = |\beta| = m$ (6).

Then L_h' *is* $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercive :

$$\exists \lambda_0 \geq 0, \ c > 0, \quad \forall h \in (0, h_0], \quad u_h \in \mathbb{G}(\Omega_h)$$

$$\operatorname{Re} \langle\!\langle L_h' u_h, u_h \rangle\!\rangle \geq c \|u_h\|_{m,0}^2 - \lambda_0 \|u_h\|_{0,0}^2 .$$

In particular, for all $\lambda \geq \lambda_0$ $\|(L_h' + \lambda I_h)^{-1}\|_{m,0 \leftarrow -m,0} \leq 1/c$.

Before proving this theorem, we state two lemmas in order to treat variable coefficients using a partition of unity. Except for the trivial part *b* of lemma 3.10, these lemmas are special cases of lemmas in [5] and proofs can be found there.

LEMMA 3.7 : (a) *Let* $\{g_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ *be a sequence of functions with the following properties :*

(1) *For all* $K > 0$, *there is an* $N(K) > 0$ *such that for all* $x^* \in \mathbb{R}^d$ *at most* $N(K)$ *functions ar not identically zero on the ball*

$$S_K(x^*) = \{x \in \mathbb{R}^d : \|x - x^*\| \leq K\}.$$

(6) In fact, it is sufficient if there is an $\eta > 0$ such that for $|\alpha| = |\beta| = m$, $c_{\alpha\beta\gamma\delta}$, restricted to $\Omega(\eta) \times (0, \eta_0)$, is an element of $C^0(\overline{\Omega(\eta) \times (0, h_0)})$. However, by [11] Ch. VI § 2, for all open sets $A \subset \mathbb{R}^n$, $n \in \mathbb{N}_0$, $\lambda \in [0, 1]$, there exists a bounded extension $\mathscr{E} : C^{n,\lambda}(\bar{A}) \to C^{n,\lambda}(\mathbb{R}^n)$. (Analogous remarks hold for proposition 3.14 and theorem 3.15).

(2) *The diameters of the supports of the* g_k *are uniformly bounded by a* $\rho > 0$.

(3) $\forall p \in \mathbb{N}_0$, $\exists c_1 (= c_1(p))$ *with for all* k, $\|g_k\|_{C^p(\mathbb{R}^d)} \leq c_1$.

Then $\forall p \in \mathbb{N}_0$, $\exists C (= C(\{g_k\}, p))$, $\forall u_h \in \mathbb{G}(h\mathbb{Z}^d)$, $\sum_k \|g_k u_h\|_p^2 \leq C \|u_h\|_p^2$.

(*b*) *If in addition to the properties* (1)-(3), *a sequence has also the property*

(4) $0 \leq g_k \leq 1$ $(k \in \mathbb{N})$, $\sum_k g_k^2 = 1$,

then $\forall p \in \mathbb{N}_0$, $\exists c (= c(\{g_k\}, p))$, $C (= C(\{g_k\}, p))$, $\forall u_h \in \mathbb{G}(h\mathbb{Z}^d)$

$$c \|u_h\|_p^2 \leq \sum_k \|g_k u_h\|_p^2 \leq C \|u_h\|_p^2 .$$

DEFINITION 3.8 : *A sequence* $\{e_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ *with the properties* (1)-(4) *of lemma* 3.7 *is called a* partition of unity.

Remark 3.9 : For each $\rho > 0$, there exist a partition of unity $\{e_k\}$ with $\sup\{\text{diam}(\text{supp}(e_k)) : k \in \mathbb{N}\} \leq \rho$ (see e.g. [1] theorem 3.14).

LEMMA 3.10. $\forall \alpha \in \mathbb{N}_0^d$, $\alpha \neq 0$, $\exists C$ *such that for all* $u_h \in \mathbb{G}(h\mathbb{Z}^d)$

(*a*) $\|g\, \partial_h^\alpha u_h - \partial_h^\alpha (g u_h)\|_0 \leq c \|g\|_{C^{|\alpha|-1,1}(\mathbb{R}^d)} \|u_h\|_{|\alpha|-1}$

$$(g \in C^{|\alpha|-1,1}(\overline{\mathbb{R}^d}))$$

and

(*b*) $\|g T_h^\alpha u_h - T_h^\alpha (g u_h)\|_0 \leq ch \|g\|_{C^{0,1}(\mathbb{R}^d)} \|u_h\|_0 \qquad (g \in C^{0,1}(\overline{\mathbb{R}^d}))$.

Proof of theorem 3.6 : Consider a partition of unity $\{e_k\}_{k \in \mathbb{N}}$. Denote $\sup\{\text{diam}(\text{supp}(e_k)) : k \in \mathbb{N}\}$ by ρ. Choose $\{g_k\}_{k \in \mathbb{N}}$, so that it satisfies the conditions (1)-(3) of lemma 3.7, and such that there is a $\delta > 0$ with $\forall k \in \mathbb{N}$, $g_k \equiv 1$ on $\{x \in \mathbb{R}^d : \text{dist}(x, \text{supp}(e_k)) < \delta\}$. Furthermore select for those k for which $\text{dist}(\text{supp}(e_k), \Omega) < \rho$ an $x_k \in \Omega$ with $\text{dist}(\text{supp}(e_k), x_k) < \rho$. By summing over these k only (h_0 « small enough ») we get

$$\langle\!\langle L_h' u_h, u_h \rangle\!\rangle = \sum_k \langle e_k \mathscr{L}_h^{(p)} \omega_h u_h, e_k \omega_h u_h \rangle + \langle \mathscr{L}_h^{(r)} \omega_h u_h, \omega_h u_h \rangle$$

$$= \sum_k \left\langle \mathscr{L}_{h, x_k}^{(p)} e_k \omega_h u_h, e_k \omega_h u_h \right\rangle \qquad (a)$$

$$+ \sum_k \left\langle (\mathscr{L}_h^{(p)} - \mathscr{L}_{h, x_k}^{(p)}) e_k \omega_h u_h, e_k \omega_h u_h \right\rangle \qquad (b)$$

$$+ \sum_k \left\langle (e_k \mathscr{L}_h^{(p)} - \mathscr{L}_h^{(p)} e_k)\, g_k\, \omega_h\, u_h,\, e_k\, \omega_h\, u_h \right\rangle \qquad (c)$$

$$+ \left\langle \mathscr{L}_h^{(r)}\, \omega_h\, u_h,\, \omega_h\, u_h \right\rangle . \qquad (d)$$

For (a), straightforward calculations give

$$\operatorname{Re} \sum_k \left\langle \mathscr{L}_{h, x_k}^{(p)}\, e_k\, \omega_h\, u_h,\, e_k\, \omega_h\, u_h \right\rangle =$$

$$= \operatorname{Re} \sum_k h^{-2m+d} \int_{T^d} \rho(x_k, \xi)\, \left| (e_k\, \omega_h\, u_h)^{\frown}(\xi) \right|^2 d\xi$$

$$\geqslant_{(L_h' \text{ strongly elliptic})} \sum_k \varepsilon h^{-2m+d} \int_{T^d} \left(\sum_{j=1}^d 4 \sin^2 \frac{\xi_j}{2} \right)^m \times$$

$$\times \left| (e_k\, \omega_h\, u_h)^{\frown}(\xi) \right|^2 d\xi$$

$$\geqslant \varepsilon h^{-2m+d} \sum_k \int_{T^d} \sum_{|\alpha| = m} \prod_{j=1}^d \left(4 \sin^2 \frac{\xi_j}{2} \right)^{\alpha_j} \left| (e_k\, \omega_h\, u_h)^{\frown}(\xi) \right|^2 d\xi$$

$$= \varepsilon \sum_k \sum_{|\alpha| = m} \left\| \partial_h^\alpha\, e_k\, \omega_h\, u_h \right\|_0^2 .$$

If we take ρ (*cf.* remark 3.9) and h_0 « small enough », the uniform continuity of $c_{\alpha\beta\gamma\delta}$ for $|\alpha| = |\beta| = m$ implies that we can majorize the absolute value of (b) by $\frac{\varepsilon}{2} \sum_k \sum_{|\alpha| = m} \left\| \partial_h^\alpha e_k\, \omega_h\, u_h \right\|_0^2$ $(h \in (0, h_0])$.

For (a) and (b) combined, we thus have

$$\operatorname{Re} (\dots) \geqslant$$

$$\geqslant \frac{\varepsilon}{2} \sum_k \sum_{|\alpha| = m} \left\| \partial_h^\alpha e_k\, \omega_h\, u_h \right\|_0^2 \geqslant_{(\text{lemma 2.7, } cf. \text{ footnote 4})} \tilde{\varepsilon} \sum_k \left\| e_k\, \omega_h\, u_h \right\|_m^2$$

$$\geqslant_{(\text{lemma 3.7(b)})} \tilde{\tilde{\varepsilon}} \left\| u_h \right\|_{m,0}^2 .$$

By writing out (c) we get

$$\sum_k \sum_{|\alpha| = |\beta| = m} \sum_{\gamma, \delta} \left\{ \left\langle (-1)^{|\alpha|} (e_k\, \partial_h^\alpha T_h^\gamma - \partial_h^\alpha T_h^\gamma\, e_k) \times \right. \right.$$

$$\left. \times\, c_{\alpha\beta\gamma\delta}(\cdot\,, h\,)\, T_h^\delta\, \partial_h^\beta g_k\, \omega_h\, u_h,\, e_k\, \omega_h\, u_h \right\rangle$$

$$\left. + \left\langle (-1)^{|\alpha|}\, \partial_h^\alpha T_h^\gamma\, c_{\alpha\beta\gamma\delta}(\cdot\,, h\,)\, (e_k\, T_h^\delta\, \partial_h^\beta - T_h^\delta\, \partial_h^\beta e_k)\, g_k\, \omega_h\, u_h,\, e_k\, \omega_h\, u_h \right\rangle \right\}$$

$$= \sum_k \sum_{|\alpha| = |\beta| = m} \sum_{\gamma, \delta} \left\{ \left\langle c_{\alpha\beta\gamma\delta}(\cdot, h\,)\, T_h^\delta\, \partial_h^\beta g_k\, \omega_h\, u_h\, , \right. \right.$$

$$\left. (T_h^{\alpha - \gamma}\, \partial_h^\alpha e_k - e_k\, T_h^{\alpha - \gamma}\, \partial_h^\alpha)\, e_k\, \omega_h\, u_h \right\rangle$$

$$\left. + \left\langle (e_k\, T_h^\delta\, \partial_h^\beta - T_h^\delta\, \partial_h^\beta e_k)\, g_k\, \omega_h\, u_h\, ,\, \overline{c_{\alpha\beta\gamma\delta(\cdot\,, h)}}\, T_h^{\alpha - \gamma}\, \partial_h^\alpha e_k\, \omega_h\, u_h \right\rangle \right\} .$$

By lemma 3.10, the absolute value of this expression can be majorized by

$$c \sum_k \left\{ \|g_k \omega_h u_h\|_m \|e_k \omega_h u_h\|_{m-1} + \|g_k \omega_h u_h\|_{m-1} \|e_k \omega_h u_h\|_m \right\} \leqslant$$

$$\leqslant c' \|u_h\|_{m,0} \|u_h\|_{m-1,0} \text{ (the Schwarz inequality and lemma 3.7 } (a)) .$$

Finally, because the $c_{\alpha\beta\gamma\delta}$ are bounded, (d) can be estimated on $c\|u_h\|_{m,0}\|u_h\|_{m-1,0}$.

By combining the results for (a)-(d) we get

$$\text{Re } \langle\!\langle L_h' u_h, u_h \rangle\!\rangle \geqslant \tilde{\tilde{\varepsilon}} \|u_h\|_{m,0}^2 - \tilde{c} \|u_h\|_{m,0} \|u_h\|_{m-1,0} .$$

Using lemma 2.4, the desired inequality can now be obtained easily. □

3.3. Discrete version of Nečas' theorem

In order to prove a discrete version of Nečas' theorem, we will make use of the results of (the continuous version of) Nečas' theorem (theorem 3.3). For this purpose, we will need restrictions and prolongations between the discrete and the continuous Sobolev spaces (*cf.* definitions 3.11 and 3.13).

DEFINITIONS 3.11 : *For $\alpha \in \mathbb{N}_0^d$, we define σ_h^α, $\sigma_h^{*\alpha} : C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^d)$ by*

$$\sigma_h^\alpha = \sum_{i=1}^d (\sigma_h^{e_i})^{\alpha_i} , \quad \sigma_h^{*\alpha} = \prod_{i=1}^d (\sigma_h^{*e_i})^{\alpha_i} ,$$

where $\sigma_h^{e_i}$, $\sigma_h^{*e_i}$ are given by

$$(\sigma_h^{e_i} u)(x) = \int_{-1}^0 u(x + e_i \xi h) \, d\xi , \quad (\sigma_h^{*e_i} u)(x) = \int_0^1 u(x + e_i \xi h) \, d\xi .$$

For $s \in \mathbb{R}$, we denote $(s, \ldots, s) \in \mathbb{R}^d$ by (s).

For $\alpha \in \mathbb{N}_0^d$, we define $\mathcal{R}_h^\alpha : C_0^\infty(\mathbb{R}^d) \to \mathbb{G}(h\mathbb{Z}^d)$ by $\mathcal{R}_h^\alpha = \mathcal{R}_h^0 \sigma_h^\alpha$, where $\mathcal{R}_h^{(0)}$ is given by $(\mathcal{R}_h^{(0)} u)(jh) = (\sigma_h^{(1)} u)(jh)$.

Since the proof of the following lemma is straightforward, it is left to the reader (except (9)).

LEMMA 3.12 : *It holds that*

(1) $\|\mathcal{R}_h^{(0)}\|_{0 \leftarrow 0} \leqslant c$ (2a) $\|\sigma_h^{e_i}\|_{0 \leftarrow 0} \leqslant c$ (2b) $\|\sigma_h^{*e_i}\|_{0 \leftarrow 0} \leqslant c$

(3a) $\sigma_h^{e_i} D^{e_i} = D^{e_i} \sigma_h^{e_i} = \partial_{h,i}$ (3b) $\sigma_h^{*e_i} D^{e_i} = D^{e_i} \sigma_h^{*e_i} = T_{h,i} \, \partial_{h,i}$

(4) $\|\mathcal{R}_h^\alpha - \mathcal{R}_h^\beta\|_{0 \leftarrow 1} \leqslant ch$ $(\alpha, \beta \in \mathbb{N}_0^d)$

(5) $\|a\mathcal{R}_h^\alpha - \mathcal{R}_h^\alpha a\|_{0 \leftarrow 0} \leqslant ch^\chi \|a\|_{C^{0,\chi}(\mathbb{R}^d)}$ $(\chi \in [0, 1], \alpha \in \mathbb{N}_0^d)$.

Because of (1), (2), $\mathcal{R}_h^\alpha$ can be extended as a map $L^2(\mathbb{R}^d) \to \mathbb{G}(h\mathbb{Z}^d)$ and $\sigma_h^{*e_i} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ both with the preservation of norm.

DEFINITION 3.13 : *For* $\alpha \in \mathbb{N}_0^d$, *we define* $\mathscr{P}_h^\alpha : \mathbb{G}(h\mathbb{Z}^d) \to L^2(\mathbb{R}^d)$ *by* $\mathscr{P}_h^\alpha = \sigma_h^{*\alpha} \mathscr{P}_h^{(0)}$, *where* $\mathscr{P}_h^{(0)}$ *is defined by* $(\mathscr{P}_h^{(0)} u_h)(x) = u_h(jh)$ *if* $jh - (1) h < x \leq jh$.

LEMMA 3.12 (continued) : *For all* $\alpha \in \mathbb{N}_0^d$, *it holds that*

(6) $(\mathscr{R}_h^\alpha)^* = \mathscr{P}_h^\alpha$

(7) $\mathscr{R}_h^\alpha$ *and* $\mathscr{P}_h^\alpha$ *are* local, *i.e.* $\exists c$ *such that*

$$\operatorname{supp}(\mathscr{R}_h^\alpha u) \subset \{x \in h\mathbb{Z}^d : \operatorname{dist}(x, \operatorname{supp}(u)) < ch\} \quad (u \in C_0^\infty(\mathbb{R}^d)),$$

$$\operatorname{supp}(\mathscr{P}_h^\alpha u_h) \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x, \operatorname{supp}(u_h)) < ch\} \quad (u_h \in \mathbb{G}(h\mathbb{Z}^d)).$$

From (3) and (6), it appears that for $\alpha \in \mathbb{N}_0^d$, $s \in \mathbb{Z}$ with $(s) \leq \alpha$, $\mathscr{R}_h^\alpha$ can be regarded as a bounded map $H^{-s}(\mathbb{R}^d) \to (\mathbb{G}(h\mathbb{Z}^d), \|\cdot\|_{-s})$, and $\mathscr{P}_h^\alpha$ as a bounded map $(\mathbb{G}(h\mathbb{Z}^d), \|\cdot\|_s) \to H^s(\mathbb{R}^d)$. By applying lemma 2.2 and an interpolation theorem (e.g. [8] lemma 1.4.3), one can obtain the same results in case $s \in \mathbb{R} \backslash \mathbb{Z}$.

LEMMA 3.12 (continued) :

(8) $\forall \alpha \in \mathbb{N}_0^d$, $s \in \mathbb{R}$ *with* $(s) \leq \alpha$, $\exists c$ *such that* $\|\mathscr{R}_h^\alpha\|_{-s \leftarrow -s} \leq c$, $\|\mathscr{P}_h^\alpha\|_{s \leftarrow s} \leq c$.

(9) $\forall k \in \mathbb{N}_0$, $\exists c$, $\forall s, t \in [k-1, k]$ $\|\mathscr{R}_h^{(0)} \mathscr{P}_h^{(k)} - \mathscr{I}_h\|_{s \leftarrow t} \leq c h^{t-s}$.

Proof of (9) : It holds that $\|\mathscr{R}_h^{(0)} \mathscr{P}_h^{(k)} - \mathscr{I}_h\|_{k \leftarrow k} \leq c$. So by interpolation it is sufficient to show that $\|\mathscr{R}_h^{(0)} \mathscr{P}_h^{(k)} - \mathscr{I}_h\|_{k-1 \leftarrow k} \leq ch$. $\mathscr{R}_h^{(0)} \mathscr{P}_h^{(k)}$ can be written as $\sum_{\mu \in \mathbb{Z}^d} a_\mu T_h^\mu$, in which finitely many a_μ are non-zero and $\sum_\mu a_\mu = 1$, and thus also as $\mathscr{I}_h - h \sum_{|\beta| = 1, \gamma} b_{\beta\gamma} T_h^\gamma \partial_h^\beta$, in which finitely many $b_{\beta\gamma}$ are non-zero ([14] lemma 2.1). This last notation directly yields the desired estimate. □

PROPOSITION 3.14 : *Consider an* L_h' *of the form given in definition* 3.5 (*e*). *Let* $\theta \in \mathbb{R}$ *such that for all* α, β, γ, δ, *for which*

$$s := \max\{|\alpha| - m + \theta, \ |\beta| - m - \theta\} > 0,$$

there exist $n \in \mathbb{N}_0$, $\lambda \in [0, 1]$ *with* $n + \lambda \geq s$ *if* $s \in \mathbb{N}$ *and* $n + \lambda > s$ *if* $s \notin \mathbb{N}$, *such that* $c_{\alpha\beta\gamma\delta}(\cdot, h) \in C^{n,\lambda}(\overline{\mathbb{R}^d})$, *with norm which is uniformly bounded in* $h \in (0, h_0]$.

Then $c(\theta)$ *exist with* $\|L_h'\|_{-m+\theta, 0 \leftarrow m+\theta, 0} \leq c(\theta)$.

Proof: It is easy to verify that it suffices to show that $\forall s > 0$, $n \in \mathbb{N}_0$, $\lambda \in [0, 1]$ with $n + \lambda \geq s$ if $s \in \mathbb{N}$ and $n + \lambda > s$ if $s \notin \mathbb{N}$, $\exists c$ such that

$$\|g u_h\|_s \leq c \|g\|_{C^{n,\lambda}(\mathbb{R}^d)} \|u_h\|_s \quad (u_h \in \mathbb{G}(h\mathbb{Z}^d), g \in C^{n,\lambda}(\overline{\mathbb{R}^d})). \quad (3.1)$$

Since $C^{n_1,\lambda_1}(\overline{\mathbb{R}^d})$ is continuously embeddable in $C^{n_2,\lambda_2}(\overline{\mathbb{R}^d})$, whenever $n_1+\lambda_1 \geq n_2+\lambda_2$ and $n_1 \geq n_2$ (see e.g. [1] theorem 1.31), it suffices to show (3.1) for $n = s-1$, $\lambda = 1$ if $s \in \mathbb{N}$ and for $n = [s]$, $\lambda \in (s-[s], 1]$ if $s \notin \mathbb{N}$.

For $s \in \mathbb{N}$, $n = s-1$, $\lambda = 1$, (3.1) follows directly from lemma 3.10(*a*).

Now let $s \notin \mathbb{N}$, $n = [s]$ and $\lambda \in (s-[s], 1]$. Write $s = n + \omega$. There exist $c(= c(n))$, $C(= C(n)) > 0$ with

$$c\|\cdot\|^*_{n+\omega} \leq \|\cdot\|_{n+\omega} \leq C\|\cdot\|^*_{n+\omega},$$

where

$$\|u_h\|^*_{n+\omega} := \left(\sum_{|\alpha| \leq n} \|\partial_h^\alpha u_h\|^2_\omega\right)^{1/2} (cf.\ (2.3)).$$

For $g \in C^{n,\lambda}(\overline{\mathbb{R}^d})$, write

$$\partial_h^\alpha(g u_h) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (T_h^{-\beta}\, \partial_h^{\alpha-\beta} g) \cdot (\partial_h^\beta u_h).$$

Since for $|\alpha| \leq n$, $\beta \leq \alpha$, $T_h^{-\beta}\, \partial_h^{\alpha-\beta} g \in C^{0,\lambda}(\overline{\mathbb{R}^d})$, with

$$\|T_h^{-\beta}\, \partial_h^{\alpha-\beta} g\|_{C^{0,\lambda}(\mathbb{R}^d)} \leq \|g\|_{C^{n,\lambda}(\mathbb{R}^d)} \quad (h > 0),$$

it is sufficient to show (3.1) for $s \in (0, 1)$.

Finally, let $s \in (0, 1)$ and $\lambda \in (s, 1]$. For any $g \in C^{0,\lambda}(\overline{\mathbb{R}^d})$, denote the map $\mathbb{G}(h\mathbb{Z}^d) \to \mathbb{G}(h\mathbb{Z}^d) : u_h \to g u_h$ by g_h and the map $C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^d) : u \to g u$ by g. It is well-known that $\exists c$ such that

$$\|g\|_{s \leftarrow s} \leq c\|g\|_{C^{0,\lambda}(\mathbb{R}^d)}\ (g \in C^{0,\lambda}(\overline{\mathbb{R}^d}))$$ (see e.g. [15] Hilfsatz 4.3 and 4.5).

Write

$$g_h = \mathcal{R}_h^{(0)} g \mathcal{P}_h^{(1)} + (g_h \mathcal{R}_h^{(0)} - \mathcal{R}_h^{(0)} g)\, \mathcal{P}_h^{(1)} - g_h(\mathcal{R}_h^{(0)} \mathcal{P}_h^{(1)} - \mathcal{I}_h).$$

By lemma 3.12 (8), (5), (9), it follows that

$$\|\mathcal{R}_h^{(0)} g \mathcal{P}_h^{(1)}\|_{s \leftarrow s} \leq c\|g\|_{C^{0,\lambda}(\mathbb{R}^d)} \text{ and } \|g_h - \mathcal{R}_h^{(0)} g \mathcal{P}_h^{(1)}\|_{s \leftarrow s} \leq c\|g\|_{C^{0,s}(\mathbb{R}^d)}$$

and thus

$$\|g_h\|_{s \leftarrow s} \leq c\|g\|_{C^{0,\lambda}(\mathbb{R}^d)}\ (g \in C^{0,\lambda}(\overline{\mathbb{R}^d})).$$ □

THEOREM 3.15 (discrete Nečas theorem) : *Consider the situation of theorem* 3.6, *but now let* Ω *be bounded and have the s.l.L.p.*

Put

$$\sigma = \sup \left\{\chi \mid \forall \alpha, \beta, \gamma, \delta \ |\alpha| = m, c_{\alpha\beta\gamma\delta} \in C^{0,\chi}(\overline{\mathbb{R}^d \times (0, h_0)})\right\}$$

and

$$\tau = \sup \left\{\chi \mid \forall \alpha, \beta, \gamma, \delta \ |\beta| = m, c_{\alpha\beta\gamma\delta} \in C^{0,\chi}(\overline{\mathbb{R}^d \times (0, h_0)})\right\},$$

where by convention $\sup \emptyset = 0$.

Then, there is a $\theta_0 \in (0, 1/2]$, *such that for all*

$$\theta \in (-\theta_0, \theta_0) \cap (-\tau, \sigma) \cup \{0\},$$

there exists a $\lambda_0 \geq 0$ *with*

$$\forall \lambda \geq \lambda_0 \ \| (L_h' + \lambda I_h)^{-1} \|_{m+\theta, 0 \leftarrow -m+\theta, 0} \leq c(= c(\theta, \lambda)).$$

If $\sum_{\gamma,\delta} c_{\alpha\beta\gamma\delta} + c_{\beta\alpha\gamma\delta}$ *is real-valued for* $|\alpha| = |\beta| = m$, *the theorem holds with* $\theta_0 = 1/2$.

Proof: The proof consists of the steps (α)-(ε).

(α) Since

$$\| (L_h' + \lambda I_h)^{-1} \|_{m+\theta, 0 \leftarrow -m+\theta, 0} = \| (L_h'^{*} + \lambda I_h)^{-1} \|_{m-\theta, 0 \leftarrow -m-\theta, 0}$$

and

$$L_h'^{*} = \omega_h^{*} \left(\sum_{|\alpha|, |\beta| \leq m} \sum_{\gamma, \delta} (-1)^{|\alpha|} \partial_h^{\alpha} T_h^{\gamma} \bar{c}_{\beta, \alpha, \beta-\delta, \alpha-\gamma} T_h^{\delta} \partial_h^{\beta} \right) \omega_h,$$

it suffices to prove the theorem for $\theta \geq 0$.

(β) Put

$$L_h'^{(1)} = \omega_h^{*} \left(\sum_{|\alpha| = m, \beta} \sum_{\gamma, \delta} (-1)^{|\alpha|} \partial_h^{\alpha} T_h^{\gamma} c_{\alpha\beta\gamma\delta} T_h^{\delta} \partial_h^{\beta} \right) \omega_h$$

and $L_h'^{(2)} = L_h' - L_h'^{(1)}$. Then $\|L_h'^{(2)}\|_{-m+1, 0 \leftarrow m, 0} \leq c$. By the discrete Gårding inequality (theorem 3.6) $\exists \lambda_0$ with

$$\forall \lambda \geq \lambda_0 \ \| (L_h' + \lambda I_h)^{-1} \|_{m, 0 \leftarrow -m, 0} \leq c.$$

Suppose that for certain $\theta > 0$, $\exists \lambda_1$ with

$$\forall \lambda \geq \lambda_1 \ \| (L_h'^{(1)} + \lambda I_h)^{-1} \|_{m+\theta, 0 \leftarrow -m+\theta, 0} \leq c(= c(\theta, \lambda)). \quad (3.2)$$

Since $\|I_h\|_{-m+\theta, 0 \leftarrow -m+1, 0} \leq c$, $\|I_h\|_{-m, 0 \leftarrow -m+\theta, 0} \leq c$ and

$$(L_h' + \lambda I_h)^{-1} = (L_h'^{(1)} + \lambda I_h)^{-1} - (L_h'^{(1)} + \lambda I_h)^{-1} I_h L_h'^{(2)} (L_h' + \lambda I_h)^{-1} I_h,$$

we see that (3.2) implies that

$$\forall \lambda \geqslant \max(\lambda_0, \lambda_1),\ \|(L_h' + \lambda I_h)^{-1}\|_{m+\theta, 0 \leftarrow -m+\theta, 0} \leqslant c (= c(\theta, \lambda)) .$$

Therefore, without loss of generality, we can assume hereafter that $c_{\alpha\beta\gamma\delta} = 0$ whenever $|\alpha| \neq m$.

(γ) Define $\omega : C_0^\infty(\Omega) \to C_0^\infty(\mathbb{R}^d)$ (both spaces with L^2 scalar products) as the extension with zero,

$$\mathscr{L} = \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta ,$$

where $a_{\alpha\beta} = \sum_{\gamma, \delta} c_{\alpha\beta\gamma\delta}(\,.\,, 0\,)$, and $L = \omega^* \mathscr{L}\omega$.

The strong ellipticity of L_h' implies the strong ellipticity of L, so the (continuous) Nečas theorem (theorem 3.3) is applicable to L.

Now let $\theta \in (0, \theta_0) \cap (0, \sigma) \cup \{0\}$, where θ_0 is taken from theorem 3.3. It follows from the theorems 3.3 and 3.6, that there is a $\lambda_0 \geqslant 0$ such that

$$\forall \lambda \geqslant \lambda_0 \quad (L + \lambda I)^{-1} : H^{-m+\theta}(\Omega) \to H_0^{m+\theta}(\Omega) \text{ bounded, and} \tag{3.3}$$

$$\forall \lambda \geqslant \lambda_0 \ \|(L_h' + \lambda I_h)^{-1}\|_{m, 0 \leftarrow -m, 0} \leqslant c . \tag{3.4}$$

Let $\lambda \geqslant \lambda_0$ be fixed. Using (3.3) and (3.4), we will show that

$$\|(L_h' + \lambda I_h)^{-1}\|_{m+\theta, 0 \leftarrow -m+\theta, 0} \leqslant c .$$

For notational convenience, we assume that $\lambda = 0$; if $\lambda \neq 0$, consider $c_{0000} - \lambda$ instead of c_{0000}. Since we have assumed earlier on that $c_{\alpha\beta\gamma\delta} = 0$ whenever $|\alpha| \neq m$, c_{0000} is now a constant.

(δ) Put $R_h^\alpha = \omega_h^* \mathscr{R}_h^\alpha \omega$ and $P_h^\alpha = \omega^* \mathscr{P}_h^\alpha \omega_h$ ($\alpha \in \mathbb{N}_0^d$) (*cf.* definitions 3.11 and 3.13) and write

$$(L_h')^{-1} = R_h^{(0)} L^{-1} P_h^{(0)} - I_h (L_h')^{-1} \times \\ \times \{(L_h' R_h^{(0)} - R_h^{(m)} L) L^{-1} P_h^{(0)} + (R_h^{(m)} P_h^{(0)} - I_h)\} .$$

We will show that

$$(a)\ \|R_h^{(0)}\|_{m+\theta, 0 \leftarrow m+\theta, 0} \leqslant c \quad (b)\ \|P_h^{(0)}\|_{-m+\theta, 0 \leftarrow -m+\theta, 0} \leqslant c$$

$$(c)\ \|L_h' R_h^{(0)} - R_h^{(m)} L\|_{-m, 0 \leftarrow m+\theta, 0} \leqslant ch^\theta \text{ (consistency)}$$

$$(d)\ \|R_h^{(m)} P_h^{(0)} - I_h\|_{-m, 0 \leftarrow -m+\theta, 0} \leqslant ch^\theta \text{ (interpolation error)} .$$

Because of (3.3), (3.4) and $\|I_h\|_{m+\theta, 0 \leftarrow m, 0} \leqslant ch^{-\theta}$, (*a*)-(*d*) imply $\|L_h'^{-1}\|_{m+\theta, 0 \leftarrow -m+\theta, 0} \leqslant c$.

The properties (*a*), (*b*) and (*d*) follow easily from lemma 3.12 (7), (8), (9), (6). In order to demonstrate the type of arguments, we show (*a*) and leave the proofs of (*b*) and (*d*) to the reader. Write

$$\omega_h R_h^{(0)} = \mathcal{R}_h^{(0)} \omega + (\omega_h \omega_h^* - \mathcal{I}_h) \mathcal{R}_h^{(0)} \omega .$$

Since $\mathcal{R}_h^{(0)}$ is local ((7)), it holds that

$$\operatorname{supp} (\mathcal{R}_h^{(0)} \omega u) \subset \Omega_h' := \{x \in h\mathbb{Z}^d : \operatorname{dist}(x, \Omega) < ch\} \quad (u \in C_0^\infty(\Omega)) .$$

Now remark 2.11, lemma 2.10 (applied to Ω_h'), and (8) show that $\| (\omega_h \omega_h^* - \mathcal{I}_h) \mathcal{R}_h^{(0)} \omega \|_{m+\theta \leftarrow m+\theta, 0} \leqslant c$. This result and the estimate $\| \mathcal{R}_h^{(0)} \omega \|_{m+\theta \leftarrow m+\theta, 0} \leqslant c$ (see (8)) imply (*a*).

In (ε), we will show (c') :

$$\| \mathcal{L}_h \mathcal{R}_h^{(0)} - \mathcal{R}_h^{(m)} \mathcal{L} \|_{-m \leftarrow m+\theta} \leqslant ch^\theta .$$

From (c') and lemma 3.12 (7) (8), (*c*) follows easily.

(ε) In order to prove (c'), we have to show that $\exists c$ such that

$$| \langle (\mathcal{L}_h \mathcal{R}_h^{(0)} - \mathcal{R}_h^{(m)} \mathcal{L}) u, v_h \rangle | \leqslant ch^\theta \| u \|_{m+\theta} \| v_h \|_m$$
$$(u \in C_0^\infty(\mathbb{R}^d), v_h \in \mathbb{G}(h\mathbb{Z}^d)) :$$

$$| \langle \ldots, \ldots \rangle | \leqslant \sum_{|\alpha| = m, \beta} \sum_{\gamma, \delta} | \langle \{ (-1)^{|\alpha|} \partial_h^\alpha T_h^\gamma c_{\alpha\beta\gamma\delta}(\,.\,, h) T_h^\delta \partial_h^\beta \mathcal{R}_h^{(0)} -$$
$$- \mathcal{R}_h^{(m)} (-1)^{|\alpha|} D^\alpha c_{\alpha\beta\gamma\delta}(\,.\,, 0) D^\beta \} u, v_h \rangle |$$
$$+ | c_{0000} | \, | \langle (\mathcal{R}_h^{(0)} - \mathcal{R}_h^{(m)}) u, v_h \rangle | .$$

By using lemma 3.12 (8) ($s = 0$), (4) and interpolation, one can estimate the second part on the right by $ch^\theta \| u \|_\theta \| v_h \|_0$. We now consider one term of the double sum of the first part and write d instead of $c_{\alpha\beta\gamma\delta}$:

$$| \langle \{ \ldots \} u, v_h \rangle | =_{(\text{lemma } 3.12\,(3)\,a)}$$
$$| \langle \{ T_h^\gamma d(\,.\,, h) T_h^\delta \mathcal{R}_h^\beta - \mathcal{R}_h^{(m)-\alpha} d(\,.\,, 0) \} D^\beta u, T_h^\alpha \partial_h^\alpha v_h \rangle | \leqslant$$
$$\leqslant \| T_h^\gamma d(\,.\,, h) T_h^\delta \mathcal{R}_h^\beta - \mathcal{R}_h^{(m)-\alpha} d(\,.\,, 0) \|_{0 \leftarrow \theta} \| u \|_{m+\theta} \| v_h \|_m .$$

Write

$$T_h^\gamma d(\,.\,, h) T_h^\delta \mathcal{R}_h^\beta - \mathcal{R}_h^{(m)-\alpha} d(\,.\,, 0) = d(\,.\,, 0)(T_h^{\gamma+\delta} \mathcal{R}_h^\beta - \mathcal{R}_h^{(m)-\alpha}) +$$
$$+ (T_h^\gamma d(\,.\,, h) - d(\,.\,, 0) T_h^\gamma) T_h^\delta \mathcal{R}_h^\beta$$
$$+ d(\,.\,, 0) \mathcal{R}_h^{(m)-\alpha} - \mathcal{R}_h^{(m)-\alpha} d(\,.\,, 0) .$$

It holds that

$$\| T_h^{\gamma+\delta} \mathcal{R}_h^{\beta} - \mathcal{R}_h^{(m)-\alpha} \|_{0 \leftarrow \theta} \leqslant$$

$$\leqslant \| T_h^{\gamma+\delta} - \mathcal{I}_h \|_{0 \leftarrow \theta} \, \| \mathcal{R}_h^{\beta} \|_{\theta \leftarrow \theta} + \| \mathcal{R}_h^{\beta} - \mathcal{R}_h^{(m)-\alpha} \|_{0 \leftarrow \theta} \leqslant ch^{\theta}$$

by lemma 3.12 (4) and interpolation,

$$\| T_h^{\gamma} d(\,.\,,h) - d(\,.\,,0)\, T_h^{\gamma} \|_{0 \leftarrow 0} \leqslant$$

$$\leqslant \| T_h^{\gamma} d(\,.\,,h) - d(\,.\,,h)\, T_h^{\gamma} \|_{0 \leftarrow 0} + \| (d(\,.\,,h) - d(\,.\,,0))\, T_h^{\gamma} \|_{0 \leftarrow 0} \leqslant ch^{\chi}$$

because of $d = c_{\alpha\beta\gamma\delta} \in C^{0,\chi}(\overline{\mathbb{R}^d \times (0, h_0)})$ $(|\alpha| = m)$, and finally

$$\| d(\,.\,,0)\, \mathcal{R}_h^{(m)-\alpha} - \mathcal{R}^{(m)-\alpha} d(\,.\,,0) \|_{0 \leftarrow 0} \leqslant ch^{\chi}$$

by lemma 3.12 (5) and again the smoothness of d.

All the above estimates together show (c'), with which the proof is completed. □

3.4. Regularity of L_h, the general difference operator

So far we have discussed the regularity of $L_h'(+\lambda I_h)$. In our definition of L_h', we have assumed that the coefficients $c_{\alpha\beta\gamma\delta}$ are bounded. To obtain regularity results, we had to impose some additional conditions concerning the smoothness of the $c_{\alpha\beta\gamma\delta}$. Because of all these restrictions on the $c_{\alpha\beta\gamma\delta}$, the regularity theorems concerning L_h' are for instance not applicable to difference operators arising from discretisations which, at points near the boundary, depend on certain distances between these points and the boundary (see examples 3.28). Since such operators are quite familiar, we will introduce in this subsection a class of more general difference operators L_h and formulate sufficient conditions for the regularity of such L_h, which make less severe demands upon the underlying discretisations at points near the boundary.

DEFINITION 3.16: *We consider $L_h : \mathbb{G}(\Omega_h) \to \mathbb{G}(\Omega_h)$ of the following general form :*

$$L_h = h^{-2m} \sum_{\mu \in \mathbb{Z}^d} b_{\mu}(\,.\,,h)\, T_h^{\mu}$$

with $b_{\mu}(x,h) = 0$, if $x \notin \Omega_h$ or $x + \mu h \notin \Omega_h$, and with an $M \in \mathbb{N}$, independent of h, such that $b_{\mu}(\,.\,,.\,) = 0$ if $|\mu| > M$ and $(I_h - \gamma_h(M))(L_h - L_h') = 0$, for some L_h' of the form given in definitions 3.5 *(e).*

The two propositions below give sufficient conditions for the $m + \theta$ regularity of L_h.

PROPOSITION 3.17 : *Let $h\mathbb{Z}^d \backslash \Omega_h$ have the d.c.p. and let*

$$\| (L_h + \lambda I_h)^{-1} \|_{m,0 \leftarrow -m,0} \leqslant c, \quad \| (L_h' + \lambda I_h)^{-1} \|_{m+\theta,0 \leftarrow -m+\theta,0} \leqslant c$$

(L_h' induced by L_h as indicated in definition 3.16).
Then $\| (L_h + \lambda I_h)^{-1} \|_{m+\theta,0 \leftarrow -m+\theta,0} \leqslant c$ (7).

Proof: As in the proof of theorem 3.15, we can take $\theta \geqslant 0$ and $\lambda = 0$.

Since $L_h(I_h - \gamma_h(2M)) = L_h'(I_h - \gamma_h(2M))$, we have

$$\begin{aligned} L_h^{-1} &= L_h'^{-1} + L_h^{-1}\{L_h' - L_h(I_h - \gamma_h(2M))\} L_h'^{-1} - \gamma_h(2M) L_h'^{-1} \\ &= L_h'^{-1} + I_h(L_h^{-1} L_h' - I_h)\gamma_h(2M) L_h'^{-1}. \end{aligned}$$

The assumptions concerning L_h^{-1} and $L_h'^{-1}$ together with the estimates $\| I_h \|_{m+\theta,0 \leftarrow -m,0} \leqslant ch^{-\theta}$, $\| \gamma_h(2M) \|_{m,0 \leftarrow m+\theta,0} \leqslant ch^{\theta}$ give the result. □

PROPOSITION 3.18 : *If* $\| (L_h + \lambda I_h)^{-1} \|_{m+\theta,0 \leftarrow -m+\theta,0} \leqslant c$ *for a certain* λ *and* $\| L_h^{-1} \|_{0,0 \leftarrow 0,0} \leqslant c$ *(stability), then* $\| L_h^{-1} \|_{m+\theta,0 \leftarrow -m+\theta,0} \leqslant c$.

Proof: From the identity

$$L_h^{-1} = (L_h + \lambda I_h)^{-1} + \lambda L_h^{-1}(L_h + \lambda I_h)^{-1},$$

it follows easily that $\| L_h^{-1} \|_{0,0 \leftarrow -m+\theta,0} \leqslant c$. Hence

$$L_h^{-1} = (L_h + \lambda I_h)^{-1} + \lambda (L_h + \lambda I_h)^{-1} L_h^{-1}$$

gives $\| L_h^{-1} \|_{m+\theta,0 \leftarrow -m+\theta,0} \leqslant c$. □

Remark 3.19 : For the demonstration of the m regularity of $L_h + \lambda I_h$ (*cf.* proposition 3.17) in some situations, we refer to [7] theorem 2.4 step 1, [9] Lemma 9.2.7, Satz 9.2.8, 9.2.9 or remark 3.23 below in combination with the examples 3.28.

Remarks 3.20 : We now discuss the condition of stability in proposition 3.18. Only in some special cases where $L_h = L_h^*$ can the eigenvalues of L_h be computed and therefore $\| L_h^{-1} \|_{0,0 \leftarrow 0,0}$. In many other cases, where

(7) In [5] section 2.5, which corresponds to this lemma, it is not used that the difference between L_h and L_h' is located at the boundary. As a result much stronger conditions concerning L_h are needed there. (In the notation of [5], it suffices for our lemma that $(I + \ell_h L_h^{-1})^{-1} : \mathscr{H}_0^{\theta-m} \to \mathscr{H}_0^{\theta-m}$ is bounded for $\theta = 0$ only.) Moreover in contrast to criterion 2.1 in [5], we achieve $m + \theta$ regularity of L_h for the same θ as is assumed for L_h'.

with respect to the canonical basis of $\mathbb{G}(\Omega_h)$ we have diag $(L_h) \geq 0$ and $R_h = \text{diag}\,(L_h) - L_h \geq 0$ (i.e. all entries ≥ 0), the boundedness of $\|L_h^{-1}\|_{\infty \leftarrow \infty}$ (*cf.* definition 2.5) and $\|L_h^{*-1}\|_{\infty \leftarrow \infty}$ can be shown using the concepts of « irreducibility » and « M-matrix » (see e.g. [9] §§ 4.3, 4.4, 4.6, 4.8, 5.1.4). Then using $\|L_h^{-1}\|^2_{0,0 \leftarrow 0,0} \leq \|L_h^{-1}\|_{\infty \leftarrow \infty} \|L_h^{*-1}\|_{\infty \leftarrow \infty}$, the desired stability follows.

Conversely, if Ω is bounded and $m \geq d/2$, then $\|L_h^{-1}\|_{m+\theta,0 \leftarrow -m+\theta,0}$ is bounded for every θ in some neighbourhood of 0 only if $\|L_h^{-1}\|_{\infty \leftarrow \infty}$ and $\|L_h^{*-1}\|_{\infty \leftarrow \infty}$ are bounded. Indeed, the boundedness of Ω implies $\|I_h\|_{0,0 \leftarrow \infty} \leq c$, lemma 2.6 gives $\|I_h\|_{\infty \leftarrow m+\theta,0} \leq c$ if $\theta > 0$, and so

$$\|L_h^{-1}\|_{\infty \leftarrow \infty} \leq \|I_h\|_{\infty \leftarrow m+\theta,0} \|L_h^{-1}\|_{m+\theta,0 \leftarrow -m+\theta,0} \|I_h\|_{-m+\theta,0 \leftarrow \infty} \leq c\,.$$

The result $\|L_h^{*-1}\|_{\infty \leftarrow \infty} \leq c$ is obtained by using the $m + \theta$ regularity for a $\theta < 0$.

3.5. Regularity of scaled difference operators

There are (general) difference operators L_h, with

$$\|L_h^{-1}\|_{m+\theta,0 \leftarrow -m+\theta,0} \leq c(=c(\theta))$$

for θ small enough, while $\|L_h\|_{-m+\theta,0 \leftarrow m+\theta,0}$ does not have a bound that is uniform in $h \in (0, h_0]$: The coefficients of the « difference star » of L_h at the boundary multiplied by h^{2m} are unbounded (see the examples in this subsection). The unboundedness of $\|L_h\|_{-m+\theta,0 \leftarrow m+\theta,0}$ is undesirable, since it hinders a number of applications of the regularity result (see e.g. [4] example on p. 431).

In this subsection, we scale such an L_h using an operator D_h in order to obtain $\|D_h L_h\|_{-m+\theta,0 \leftarrow m+\theta,0} \leq c$. Moreover, for some rather usual situations we will show that this can be done whilst the $m + \theta$ regularity of $D_h L_h$ is retained. Although, in view of a consistent discretisation, the construction of $D_h L_h$ seems to be unnatural, there are useful applications. For instance in [9] § 9.2, it appears that regularity of $D_h L_h$ can be used to prove an « optimal » error estimate for the solution u_h of the discretized boundary value problem $L_h u_h = f_h$.

Notations 3.21 : Write $\mathscr{L}_h = \text{diag}\,(\mathscr{L}_h) - \mathscr{R}_h$, $L'_h = \text{diag}\,(L'_h) - R'_h$ and $L_h = \text{diag}\,(L_h) - R_h$, with respect to the canonical bases on $\mathbb{G}(h\mathbb{Z}^d)$ or $\mathbb{G}(\Omega_h)$ respectively. We assume that $(\text{diag}\,(L_h))^{-1}$ exists and define $D_h = (\text{diag}\,(L_h))^{-1}\,\text{diag}\,(L'_h)$.

Remark 3.22 : If $h\mathbb{Z}^d \backslash \Omega_h$ has the d.c.p., $\|D_h L_h\|_{0,0 \leftarrow 0,0} \leqslant ch^{-2m}$ and $\|L'_h\|_{-m+\theta,0 \leftarrow m+\theta,0} \leqslant c$ (*cf.* proposition 3.14) (L'_h induced by L_h), then $\|D_h L_h\|_{-m+\theta,0 \leftarrow m+\theta,0} \leqslant c$ (use lemma 2.10).

Remark 3.23 : If $h\mathbb{Z}^d \backslash \Omega_h$ has the d.c.p. and $\|D_h\|_{0,0 \leftarrow 0,0} \leqslant c$, then $m+\theta$ regularity of $D_h L_h$ implies $m+\theta$ regularity of L_h. Indeed, write

$$L_h^{-1} = (D_h L_h)^{-1} D_h = (D_h L_h)^{-1} \{I_h + \gamma_h(2M)(D_h - I_h)\} ,$$

then

$$\|\gamma_h(2M)(D_h - I_h)\|_{-m+\theta,0 \leftarrow -m+\theta,0} \leqslant$$

$$\leqslant \|\gamma_h(2M)\|_{-m+\theta,0 \leftarrow 0,0} (\|D_h\|_{0,0 \leftarrow 0,0} + 1) \|I_h\|_{0,0 \leftarrow -m+\theta,0} \leqslant c$$

gives the result.

$D_h L_h$ can be considered as a special difference operator of the form given in definition 3.16. Therefore, to prove $m+\theta$ regularity of $D_h L_h$ for all θ small enough, the propositions 3.17 and 3.18 can be applied to $D_h L_h$. In order to check the desired stability of $D_h L_h$ (*cf.* prop. 3.18), the reader is referred to the discussion following that proposition and to footnote (8). The two following propositions give sufficient conditions for the m regularity of $D_h L_h + \lambda I_h$ (*cf.* proposition 3.17).

PROPOSITION 3.24 : *Let* L'_h, *induced by* $D_h L_h$, *be* $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-*coercive* (*cf. theorem* 3.6), $c_{\alpha\beta\gamma\delta}$ *constant if* $\alpha = 0$ *or (exclusive)* $\beta = 0$, $\sum_{\gamma,\delta} c_{00\gamma\delta} \geqslant 0$, $\operatorname{diag}(L'_h) \geqslant 0$, $\mathcal{R}_h \geqslant 0$ and $-\eta R'_h \leqslant D_h R_h \leqslant R'_h$ *for some* $\eta < 1$, *independent of* $h \in (0, h_0]$.

Then $D_h L_h$ *is* $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-*coercive. In particular, there exist* $\lambda_0 \geqslant 0$, $c > 0$ *such that*

$$\forall \lambda \geqslant \lambda_0 \quad \|(D_h L_h + \lambda I_h)^{-1}\|_{m,0 \leftarrow -m,0} \leqslant 1/c .$$

Proof :

$$\operatorname{Re} \langle\!\langle D_h L_h u_h, u_h \rangle\!\rangle = \frac{1-\eta}{2} \operatorname{Re} \langle\!\langle L'_h u_h, u_h \rangle\!\rangle +$$

$$+ \operatorname{Re} \left\langle\!\!\left\langle \left(\frac{1+\eta}{2} \operatorname{diag}(L'_h) + \frac{1-\eta}{2} R'_h - D_h R_h \right) u_h, u_h \right\rangle\!\!\right\rangle .$$

Since $\frac{1-\eta}{2} > 0$, it suffices to show that the real part of the second inner product on the right is non-negative. This non-negativity follows from Gershgorin's circle theorem applied to the Hermitian part of the matrix in

combination with the following observations :
Because of the assumptions concerning the coefficients $c_{\alpha\beta\gamma\delta}$, the row- and column-sums of $\mathscr{L}_h$ are non-negative and so by $\mathscr{R}_h \geqslant 0$, the same holds for L'_h ; diag $(L'_h) \geqslant 0$; $R'_h \geqslant 0$; $\left| \frac{1-\eta}{2} R'_h - D_h R_h \right| \leqslant \frac{1+\eta}{2} R'_h$ [8]. □

Remark 3.25 : Since $D_h L_h$ is coercive is equivalent to $\frac{1}{2}((D_h L_h)^* + D_h L_h)$ is coercive, the conditions of proposition 3.24 can be relaxed to corresponding conditions on $\frac{1}{2}((D_h L_h)^* + D_h L_h)$.

For general $\mathscr{L}_h$, i.e ; without the assumptions of proposition 3.24 concerning the $c_{\alpha\beta\gamma\delta}$, we use the following notations :

Notations 3.26 : Write $\mathscr{L}_h = \mathscr{L}_h^{(1)} + \mathscr{L}_h^{(2)}$, where $\mathscr{L}_h^{(2)}$ is of the form given in definitions 3.5 with « $c_{\alpha\beta\gamma\delta}$ » $= 0$ if $|\alpha| + |\beta| = 2m$. Similarly, write $L'_h = L'^{(1)}_h + L'^{(2)}_h$ and $L_h = L^{(1)}_h + L^{(2)}_h$. If $(\text{diag}\,(L^{(1)}_h))^{-1}$ exists, define $D^{(1)}_h = (\text{diag}\,(L^{(1)}_h))^{-1}\,\text{diag}\,(L'^{(1)}_h)$.

PROPOSITION 3.27 : Let $h\mathbb{Z}^d \backslash \Omega_h$ have the d.c.p.,

$$\| D_h L_h - D^{(1)}_h L^{(1)}_h \|_{0,0 \leftarrow 0,0} \leqslant c h^{-2m+\eta}$$

for some $\eta > 0$ and $D^{(1)}_h L^{(1)}_h$ $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercive (*cf.* proposition 3.24).
Then $D_h L_h$ is $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercive.

Proof : We can assume $\eta \leqslant 1$. Write $L'^{(2)}_h = L'^{(3)}_h + L'^{(4)}_h$, where $L'^{(3)}_h (L'^{(4)}_h)$ does not contain coefficients « $c_{\alpha\beta\gamma\delta}$ » if $|\alpha| = m(|\beta| = m)$. Since

$$D_h L_h - D^{(1)}_h L^{(1)}_h = \\ = \gamma_h(M) \left\{ D_h L_h - D^{(1)}_h L^{(1)}_h - L'^{(2)}_h \right\} \gamma_h(2M) + L'^{(3)}_h + L'^{(4)}_h ,$$

$$\| \gamma_h(M) \left\{ D_h L_h - D^{(1)}_h L^{(1)}_h - L^{(2)}_h \right\} \gamma_h(2M) \|_{-m+\eta,0 \leftarrow m,0} \leqslant c$$

(use lemma 2.10),

$\| L'^{(3)}_h \|_{-m+1,0 \leftarrow m,0} \leqslant c$ and $\| L'^{(4)}_h \|_{-m,0 \leftarrow m-1,0} \leqslant c$, the result follows by applying lemma 2.4. □

[8] The proof shows that, under the conditions of the proposition, strong $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercivity of L'_h, i.e. Re $\langle\!\langle L'_h u_h, u_h \rangle\!\rangle \geqslant c \| u_h \|^2_{m,0}$, implies strong $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercivity of L_h and so $\| L_h^{-1} \|_{m,0 \leftarrow -m,0} \leqslant 1/c$ (thus certainly $\| L_h^{-1} \|_{0,0 \leftarrow 0,0} \leqslant 1/c$, *cf.* proposition 3.18).

Examples 3.28 : $d = 2$, Ω bounded, $\Omega_h = \Omega \cap h\mathbb{Z}^d$,

$$L = -\frac{\partial}{\partial x_1} a_1(x) \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} a_2(x) \frac{\partial}{\partial x_2},$$

$m = 1$, $a_i \geqslant \varepsilon > 0$ on Ω $(i = 1, 2)$,

$$\mathscr{L}_h = \partial^*_{h,1} a_1\left(x - \frac{1}{2} he_1\right) \partial_{h,1} + \partial^*_{h,2} a_2\left(x - \frac{1}{2} he_2\right) \partial_{h,2}$$

$(« c_{10\,10\,10\,00}(x, h) » = a_1\left(x - \frac{1}{2} he_1\right),\ « c_{01\,01\,01\,00}(x, h) » = a_2\left(x - \frac{1}{2} he_2\right))$
is elliptic (and has consistency order 2 if the a_i's are sufficiently smooth).

From theorem 3.6, it follows that L_h' is $(\mathbb{G}(\Omega_h), \|.\|_{m,0})$-coercive if $a_i \in C^0(\overline{\mathbb{R}^2})$. Because of the boundedness of Ω and lemma 2.7, direct calculations show that even without the assumption $a_i \in C^0(\overline{\mathbb{R}^2})$

$$\text{Re } \langle\!\langle L_h' u_h, u_h \rangle\!\rangle \geqslant c \|u_h\|^2_{m,0} \text{ (strong } (\mathbb{G}(\Omega_h), \|.\|_{m,0})\text{-coercivity)}.$$

We will now investigate the m regularity of the scaled versions of two difference operators, which both coincide with L_h' in the interior of Ω_h, but differ from L_h' at the boundary.

(I) In the (generalized) *Collatz* discretisation, for an $x \in \Omega_h$, for which $x + s_e he_1$, $x - s_s he_2$, $x - s_w he_1$, $x + s_n he_2 \in \Omega \cup \Gamma$ for some s_e, s_s, s_w, $s_n \in (0, 1]$, with $s_e . s_s . s_w . s_n < 1$, the equation
$\mathscr{L}_h u_h(x)$

$$\begin{aligned}(= h^{-2}\{&(a_1(x + 1/2\, he_1) + a_1(x - 1/2\, he_1) + a_2(x + 1/2\, he_2)\\ &+ a_2(x - 1/2\, he_2))\, u_h(x) - a_1(x - 1/2\, he_1)\, u_h(x - he_1)\\ &- a_1(x + 1/2\, he_1)\, u_h(x + he_1) - a_2(x - 1/2\, he_2)\, u_h(x - he_2)\\ &- a_2(x + 1/2\, he_2)\, u_h(x + he_2)\}) = « f_h(x) » \quad \text{(the right-hand side)}\end{aligned}$$

is replaced by the sum of zero order interpolation formulas in both directions. These formulas are chosen such that the resulting L_h is symmetric.

If $s_e = 1$, $s_w \leqslant 1$ (similarly if $s_e \leqslant 1$, $s_w = 1$), the formula in the x_1-direction is chosen as

$$\begin{aligned}h^{-2}\Big\{ &\frac{1 + s_w}{2\, s_w} (a_1(x + 1/2\, he_1) + a_1(x - 1/2\, he_1))\, u_h(x) - \\ &\qquad - a_1(x + 1/2\, he_1)\, u_h(x + he_1)\\ &- \left(\frac{1 - s_w}{2\, s_w} a_1(x + 1/2\, he_1) + \frac{1 + s_w}{2\, s_w} a_1(x - 1/2\, he_1)\right) u_h(x - s_w\, he_1)\Big\} = 0\end{aligned}$$

and if $s_e, s_w < 1$, we take

$$h^{-2}((a_1(x + 1/2\, he_1) + a_1(x - 1/2\, he_1))/2) \times$$
$$\times \left\{ \frac{s_e + s_w}{s_e\, s_w} u_h(x) - \frac{1}{s_w} u_h(x - s_w\, he_1) - \frac{1}{s_e} u_h(x + s_e\, he_1) \right\} = 0 .$$

Note that in the last case, the formula is a linear (first order) interpolation formula. In the first case, this is only true if a_1 is a constant (that is why we called the method *generalized* Collatz discretisation). However, if $a_1 \in C^{0,1}(\bar{\Omega})$ the extra error as a consequence of non-constant a_1 is of the same order as the interpolation error if a_1 is constant.

In the x_2-direction we use similar formulas. The ultimate equation for $u_h(x)$ is the sum of the appropriate formulas in both directions (*cf.* [9] § 4.8.2 (4.8.16)). After eliminating the boundary values in the obtained equations, we get a symmetric L_h, which meets the conditions of proposition 3.24. From footnote (8), it appears then that $D_h L_h$ is strongly $(\mathbb{G}(\Omega_h), \|\cdot\|_{m,0})$-coercive and thus

$$\| (D_h L_h)^{-1} \|_{m,0 \leftarrow -m,0} \leqslant 1/c .$$

(II) As a second example, we discuss the *Shortley-Weller* discretisation. In this discretisation $\partial_{h,1}^{*} a_1(x - 1/2\, he_1)\, \partial_{h,1} u_h(x)$ is replaced by

$$\{- a_1(x + 1/2\, s_e\, he_1)\, (u_h(x + s_e\, he_1) - u_h(x))/s_e\, h$$
$$+ a_1(x - 1/2\, s_w\, he_1)\, (u_h(x) - u_h(x - s_w\, he_1))/s_w\, h\} \Big/ \frac{(s_e + s_w)\, h}{2}$$
$$= h^{-2}\Bigg\{ \left(\frac{2\, a_1(x + 1/2\, s_e\, he_1)}{s_e(s_e + s_w)} + \frac{2\, a_1(x - 1/2\, s_w\, he_1)}{s_w(s_e + s_w)} \right) u_h(x)$$
$$- \frac{2\, a_1(x - 1/2\, s_w\, he_1)}{s_w(s_e + s_w)} u_h(x - s_w\, he_1) - \frac{2\, a_1(x + 1/2\, s_e\, he_1)}{s_e(s_e + s_w)} u_h(x + s_e\, he_1)\Bigg\}$$

(if $s_e \cdot s_w < 1$) and the same procedure is used in the x_2-direction.

For $a_1 = a_2 = 1$ (Poisson equation), it holds that $0 \leqslant D_h R_h \leqslant R_h'$ (*cf.* proposition 3.24) and so $\| (D_h L_h)^{-1} \|_{m,0 \leftarrow -m,0} \leqslant 1/c$.

For $a_1/a_2 = 1$ and $a_1 (= a_2) \in C^0(\overline{\mathbb{R}^2})$, $0 \leqslant D_h R_h \leqslant R_h' + o(1)\, h^{-2m}$, whereas in the interior as always $D_h R_h = R_h'$. From this, it follows that

$$\text{Re}\, \langle\!\langle D_h L_h u_h, u_h \rangle\!\rangle \geqslant c \| u_h \|_{m,0}^2 - o(1)\, h^{-2m} \| \gamma_h(2\, M)\, u_h \|_{0,0}^2$$

and thus $\| (D_h L_h)^{-1} \|_{m,0 \leftarrow -m,0} \leqslant 1/\tilde{c}$ (h_0 small enough), if $h\mathbb{Z}^d \backslash \Omega_h$ has the d.c.p. (*cf.* lemma 2.10).

For arbitrary a_1/a_2, $D_h R_h \leq R'_h$ can not be expected even if a_1 and a_2 are constants. However, if we neglect a possible part of $\mathbb{R}^d \backslash \Omega$ between neighbouring points of Ω_h while setting up the discretisations, $\operatorname{Re} \langle\langle D_h L_h u_h, u_h \rangle\rangle \geq c \| u_h \|^2_{m,0}$ and thus the boundedness of $\| (D_h L_h)^{-1} \|_{m,0 \leftarrow -m,0}$ can be proved in the same way as in [7] (theorem 2.4 step 1) if $a_i \in C^0(\overline{\mathbb{R}^2})$, $\frac{1}{\mu} < a_1/a_2 < \mu$, with

$$\mu = \left(2\sqrt{3} + 3 + (1 + \sqrt{3})\sqrt{2 + 2\sqrt{3}}\right)^2 \approx 165 ,$$

and if $h\mathbb{Z}^d \backslash \Omega_h$ has the d.c.p.

We finally note that using the same sort of analysis, it can be proved that $\langle\langle L_h u_h, u_h \rangle\rangle \geq c \| u_h \|^2_{m,0}$ without conditions concerning a_1/a_2.

4. PROOF OF THEOREM 2.12

Before giving the proof we state three lemmas. First we formulate an interpolation lemma, which is in fact a special case of the general interpolation theorem for Hilbert scales, which we used earlier. The proof of this lemma can be found in, for instance, [4] lemma 4.

LEMMA 4.1 : *Let H_1 and H_2 be two complex Hilbert spaces. Let $A : H_1 \to H_2$ linear, bounded, Λ_i, $\Lambda_i^{-1} : H_i \to H_i$ linear, bounded and positive definite $(i = 1, 2)$.*

Then for all α, β, $\gamma \in \mathbb{R}$ with $\alpha \leq \beta \leq \gamma$, it holds that

$$\| \Lambda_2^{\beta} A \Lambda_1^{-\beta} \|_{H_2 \leftarrow H_1} \leq \| \Lambda_2^{\alpha} A \Lambda_1^{-\alpha} \|_{H_2 \leftarrow H_1}^{\frac{\gamma - \beta}{\gamma - \alpha}} \| \Lambda_2^{\gamma} A \Lambda_1^{-\gamma} \|_{H_2 \leftarrow H_1}^{\frac{\beta - \alpha}{\gamma - \alpha}} .$$

LEMMA 4.2 : *If Ω has the s.l.L.p., then there is a linear operator E that maps functions on Ω onto functions on $\mathbb{R}^d$ with the properties*

(*a*) *$E(f)|_{\Omega} = f$ for all $f \in C^{\infty}(\bar{\Omega})$; that is, E is an extension operator.*

(*b*) *$\forall k \in \mathbb{N}_0$, the map $E : H^k(\Omega) \to H^k(\mathbb{R}^d)$ is bounded, the bound depending only on k, d, « L » and « R » (cf. definition* 2.14).

Proof [11] chapter VI § 3 theorem 5. An inspection of the proof there shows the assertion about the bound in (*b*). □

LEMMA 4.3 : *For all $k \in \mathbb{N}_0$, there are linear maps*

$$\mathscr{P}_h : \mathbb{G}(h\mathbb{Z}^d) \to H^k(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$$

and $\mathscr{R}_h : L^2(\mathbb{R}^d) \to \mathbb{G}(h\mathbb{Z}^d)$ with the properties :

(*a*) $\| \mathscr{P}_h \|_{\ell \leftarrow \ell} \leq c$ $(\ell \in \{0, k\})$ (*b*) *$\mathscr{P}_h$ is local (cf. lemma* 3.12 (7))

(c) $\mathscr{R}_h \mathscr{P}_h = \mathscr{I}_h$ (d) $\|\mathscr{R}_h\|_{\ell \leftarrow \ell} \leq c \ (\ell \in \{0, k\})$

(e) $\mathscr{R}_h$ *is* pseudo local *i.e.* $\exists c,\ d_1 > 0$ *such that for all* $u \in C_0^\infty(\mathbb{R}^d)$, $\|v_h\|_0 \leq ch^k \|u\|_k$, *where* $v_h \in \mathbb{G}(h\mathbb{Z}^d)$ *is defined by*

$$\begin{aligned} v_h &= \mathscr{R}_h u \quad \textit{on } \{x \in h\mathbb{Z}^d : \text{dist}\ (x, \text{supp}\ (u)) \geq d_1 h\} \\ &= 0 \qquad \textit{elsewhere}. \end{aligned}$$

Proof: $\mathscr{P}_h := \mathscr{P}_h^{(k)}$ (see definitions 3.13 and 3.11) has the properties (*a*) and (*b*) (see lemma 3.12 (8) (7)). [2] § 4.2.5 shows the existence of an $\mathscr{R}_h$ with property (*d*) and $\mathscr{R}_h \mathscr{P}_h = \mathscr{I}_h$ ((*c*)). Using [2] § 5.1.3, one can verify that there is a local $\tilde{\mathscr{R}}_h$: $L^2(\mathbb{R}^d) \to \mathbb{G}(h\mathbb{Z}^d)$ with $\|\mathscr{I}_h - \mathscr{P}_h \tilde{\mathscr{R}}_h\|_{0 \leftarrow k} \leq ch^k$. The estimate

$$\|\mathscr{R}_h - \tilde{\mathscr{R}}_h\|_{0 \leftarrow k} = \|\mathscr{R}_h(\mathscr{I}_h - \mathscr{P}_h \tilde{\mathscr{R}}_h)\|_{0 \leftarrow k} \leq ch^k$$

in combination with the locality of $\tilde{\mathscr{R}}_h$ shows that $\mathscr{R}_h$ has property (*e*) (9). □

Proof of theorem 2.12 : By the definition of $\|.\|_{-s,0}$ for $s > 0$, it suffices to give the proof for $s \in [0, k]$.

(α) At first we prove the inequality $\|u_h\|_{s,0} \leq \|B_h^{s/2k} u_h\|_{0,0} (s \in [0, k]$, $u_h \in \mathbb{G}(\Omega_h))$, or equivalently

$$\|\mathscr{B}_h^{s/2k} \omega_h B_h^{-s/2k}\|_{0 \leftarrow 0,0} \leq 1 \quad (s \in [0, k])\ (cf.\ (2.5)).$$

For $s = 0$, we have that $\|\omega_h\|_{0 \leftarrow 0,0} = 1$ and for $s = k$

$$\begin{aligned} \|\mathscr{B}_h^{1/2} \omega_h B_h^{-1/2}\|_{0 \leftarrow 0,0} &= \|B_h^{-1/2} \omega_h^* \mathscr{B}_h^{1/2} \mathscr{B}_h^{1/2} \omega_h B_h^{-1/2}\|_{0,0 \leftarrow 0,0}^{1/2} \\ &= \|B_h^{-1/2} B_h B_h^{-1/2}\|_{0,0 \leftarrow 0,0}^{1/2} \\ &= 1. \end{aligned}$$

---

(9) In [5] proof of lemma 2.1, the existence of $\mathscr{P}_h, \mathscr{R}_h$ is assumed as in lemma 4.3, but with in addition

$$\|\mathscr{P}_h \mathscr{R}_h u - u\|_0 \leq \|\mathscr{P}_h w_h - u\|_0 \quad \text{for all} \quad u \in L^2(\mathbb{R}^d), \quad w_h \in \mathbb{G}(h\mathbb{Z}^d)$$

and $\mathscr{R}_h$ local. It is easily seen that then necessarily holds that $\mathscr{R}_h = (\mathscr{P}_h^* \mathscr{P}_h)^{-1} \mathscr{P}_h^*$. From the locality of $\mathscr{R}_h$, it follows that $\mathscr{R}_h \mathscr{R}_h^* = (\mathscr{P}_h^* \mathscr{P}_h)^{-1}$ is local. If we now assume that $\mathscr{P}_h$ is of the usual finite element form $\mathscr{P}_h u_h(x) = \sum_{j \in \mathbb{Z}^d} u_h(jh)\, \mu\left(\frac{x}{h} - j\right)$ with $\mu \in H^k(\mathbb{R}^d)$ having a compact support, it appears by means of Fourier analysis that $\{\mu(. - j) : j \in \mathbb{Z}^d\}$ is an orthogonal set in $L^2(\mathbb{R}^d)$. However, at least in the usual case of positive μ, for $k > 0$ this conflicts with the requirement $\mathscr{P}(\mathbb{G}(h\mathbb{Z}^d)) \subset H^k(\mathbb{R}^d)$ unless $\mu \in H_0^k(]0, 1[^d + c)$. However, in the latter case the corresponding $\mathscr{P}_h$, if not 0, does not satisfy $\|\mathscr{P}_h\|_{k \leftarrow k} \leq c$.

Lemma 4.1 now gives the result $\| \mathscr{B}_h^{s/2k} \omega_h B_h^{-s/2k} \|_{0 \leftarrow 0,0} \leq 1$ $(s \in [0, k])$.

(β) For the proof of the inequality $\| B_h^{s/2k} u_h \|_{0,0} \leq C \| u_h \|_{s,0}$ $(s \in [0, k]$, $u_h \in \mathbb{G}(\Omega_h))$, we will construct an $N_h : \mathbb{G}(h\mathbb{Z}^d) \to \mathbb{G}(\Omega_h)$ with $N_h \omega_h = I_h$ and $\| B_k^{s/2k} N_h v_h \|_{0,0} \leq C \| v_h \|_s$ $(s \in [0, k]$, $v_h \in \mathbb{G}(h\mathbb{Z}^d))$, which is equivalent to

$$\| B_h^{s/2k} N_h \mathscr{B}_h^{-s/2k} \|_{0,0 \leftarrow 0} \leq c \quad (s \in [0, k]) .$$

From the existence of such an N_h the desired result follows directly by substituting $v_h = \omega_h u_h$ $(u_h \in \mathbb{G}(\Omega_h))$.

Because of lemma 4.1, it suffices to show that $\| B_h^{s/2k} N_h \mathscr{B}_h^{-s/2k} \|_{0,0 \leftarrow 0} \leq c$ for $s \in \{0, k\}$. This is equivalent to $\| N_h \|_{s,0 \leftarrow s} \leq c$ for $s \in \{0, k\}$, which for $s = k$ we see by writing

$$\| N_h \|_{k,0 \leftarrow k} =$$

$$= \| \mathscr{B}_h^{1/2} \omega_h N_h \mathscr{B}_h^{-1/2} \|_{0 \leftarrow 0} = \| \mathscr{B}_h^{-1/2} N_h^* \omega_h^* \mathscr{B}_h^{1/2} \mathscr{B}_h^{1/2} \omega_h N_h \mathscr{B}_h^{-1/2} \|_{0 \leftarrow 0}^{1/2}$$

$$= \| \mathscr{B}_h^{-1/2} N_h^* B_h^{1/2} B_h^{1/2} N_h \mathscr{B}_h^{-1/2} \|_{0 \leftarrow 0}^{1/2} = \| B_h^{1/2} N_h \mathscr{B}_h^{-1/2} \|_{0,0 \leftarrow 0} .$$

We define $N_h = \omega_h^* \mathscr{R}_h F(h) \mathscr{P}_h$ as follows.

We take $\mathscr{P}_h : \mathbb{G}(h\mathbb{Z}^d) \to H^k(\mathbb{R}^d)$, $\mathscr{R}_h : L^2(\mathbb{R}^d) \to \mathbb{G}(h\mathbb{Z}^d)$ as in lemma 4.3. By the locality of $\mathscr{P}_h$, there exists a $d_2 > 0$ such that for all $u_h \in \mathbb{G}(\Omega_h)$, supp $(\mathscr{P}_h \omega_h u_h) \subset \overline{\Omega(h(d_2 + D))}$ (for D, *cf.* notation 1.1).

Remark 2.16 shows that for all $h \in (0, h_0]$, where h_0 is « small enough », $\Omega(h(d_2 + D))$ and thus $\mathbb{R}^d \backslash \overline{\Omega(h(d_2 + D))}$ has the s.l.L.p. with the « R » and « L » from Ω (*cf.* definition 2.14). Thus by lemma 4.2, for all $h \in (0, h_0]$ there exists an extension

$$E(h) : H^\ell(\mathbb{R}^d \backslash \overline{\Omega(h(d_2 + D))}) \to H^\ell(\mathbb{R}^d) \quad (\ell \in \mathbb{N}_0) ,$$

which is bounded uniformly in $\ell \in \{0, k\}$, $h \in (0, h_0]$.

Hence, if we define $R(h)$ as the restriction of functions in $\mathbb{R}^d$ to $\mathbb{R}^d \backslash \overline{\Omega(h(d_2 + D))}$ and $F(h) : H^\ell(\mathbb{R}^d) \to H^\ell(\mathbb{R}^d)$ $(\ell \in \mathbb{N}_0)$ by

$$F(h) = \mathscr{I} - E(h) R(h) ,$$

then $F(h)$ is bounded uniformly in $\ell \in \{0, k\}$, $h \in (0, h_0]$ too.

With the above definitions, we have

$$N_h \omega_h = \omega_h^* \mathscr{R}_h F(h) \mathscr{P}_h \omega_h = \omega_h^* \mathscr{R}_h \mathscr{P}_h \omega_h = I_h$$

and $\| N_h \|_{0,0 \leftarrow 0} \leq c$.

Since for $k > 0$, $\|\omega_h^*\|_{k,0\leftarrow k}$ is unbounded with respect to h, we have to do some more work in order to get the remaining estimate $\|N_h\|_{k,0\leftarrow k}(=\|\omega_h N_h\|_{k\leftarrow k}) \leq c$. Because of

$$\|\mathscr{P}_h\|_{k\leftarrow k} \leq c, \quad \|\mathscr{R}_h F(h)\|_{k\leftarrow k} \leq c \quad \text{and} \quad \|\mathscr{I}_h\|_{k\leftarrow 0} \leq ch^{-k},$$

it suffices to show

$$\|(\mathscr{I}_h - \omega_h \omega_h^*)\, \mathscr{R}_h F(h)\|_{0\leftarrow k} \leq ch^k .$$

Define for $u \in H^k(\mathbb{R}^d)$, $v_h \in \mathbb{G}(h\mathbb{Z}^d)$ by

$$\begin{aligned} v_h &= \mathscr{R}_h F(h)\, u \ \text{ on } (\mathbb{R}^d \backslash \Omega((d_1 + d_2 + D)\, h)) \cap h\mathbb{Z}^d \ (d_1 \text{ from lemma 4.3}) \\ &= 0 \qquad \text{elsewhere on } h\mathbb{Z}^d . \end{aligned}$$

Since for all $u \in H^k(\mathbb{R}^d)$, $\text{supp}\,(F(h)\, u) \subset \overline{\Omega(h(d_2 + D))}$, the pseudo locality of $\mathscr{R}_h$ (lemma 4.3) implies the existence of a c, independent of u (and h), with

$$\|v_h\|_0 \leq ch^k \|F(h)\, u\|_k .$$

The function $(\mathscr{I}_h - \omega_h \omega_h^*)\, \mathscr{R}_h F(h)\, u - v_h \in \mathbb{G}(h\mathbb{Z}^d)$ is identically zero outside $\Omega_h' \backslash \Omega_h$, where $\Omega_h' = \Omega((d_1 + d_2 + D)\, h) \cap h\mathbb{Z}^d$. By remark 2.11 and lemma 2.10, with « ℓ » $= d_1 + d_2 + 2\, D$, applied to Ω_h', the existence is shown of a c', independent of u (and h), with

$$\begin{aligned} \|(\mathscr{I}_h - \omega_h \omega_h^*)\, \mathscr{R}_h F(h)\, u - v_h\|_0 &\leq ch^k \|\mathscr{R}_h F(h)\, u - v_h\|_k \leq \\ &\leq c' h^k (\|\mathscr{R}_h\|_{k\leftarrow k} + \|\mathscr{I}_h\|_{k\leftarrow 0} \cdot ch^k)\, \|F(h)\, u\|_k . \end{aligned}$$

By $\|F(h)\|_{k\leftarrow k} \leq c$ and $\|\mathscr{R}_h\|_{k\leftarrow k} \leq c$, the proof is completed ([10]). □

([10]) In [5] proof of lemma 2.1, Calderón's extension theorem ([1] theorem 4.32) is used for the construction of $F(h)$ instead of Stein's extension theorem (lemma 4.2). Calderon's extension operator needs less smoothness of the boundary than the s.l.L.p., in fact a kind of « uniform c.p. » suffices. However, this extension operator (and thus « N_h ») depends on the degree (ℓ) of the underlying Sobolev space, which implies that interpolation (lemma 4.1) cannot be used.

ACKNOWLEDGMENTS

The author would like to thank Prof. dr. A. van der Sluis and Dr. G. L. G. Sleijpen for helpful discussions. He is grateful to Prof. dr. W. Hackbusch for giving his comments on an earlier version of this paper, which led to the introduction of the discrete cone property.

REFERENCES

[1] R. A. ADAMS, *Sobolev spaces,* Academic Press, New York, 1975.

[2] J. P. AUBIN, *Approximation of elliptic boundary-value problems,* Wiley-Interscience, New York, 1972.

[3] K. P. BUBE and J. C. STRIKWERDA, *Interior regularity estimates for elliptic systems of difference equations,* SIAM J. Numer. Anal. 20 (1983), 653-670.

[4] W. HACKBUSCH, *Convergence of Multi-Grid Iterations Applied to Difference Equations,* Math. Comp. 34 (1980), 425-440.

[5] W. HACKBUSCH, *On the regularity of difference schemes,* Ark. Mat. 19 (1981), 71-95.

[6] W. HACKBUSCH, *Analysis of discretizations by the concept of discrete regularity.* In : The Mathematics of Finite Elements and Applications IV-MAFELAP 1981 (J. R. Whiteman, ed.) 369-376, Academic Press, London, 1982.

[7] W. HACKBUSCH, *On the regularity of difference schemes, Part II. Regularity estimates for linear and nonlinear problems,* Ark. Mat. 21 (1983), 3-28.

[8] W. HACKBUSCH, *Multi-Grid Methods and Applications,* Springer-Verlag, Berlin, 1985.

[9] W. HACKBUSCH, *Theorie und Numerik elliptischer Differentialgleichungen,* B. G. Teubner, Stuttgart, 1986.

[10] J. NEČAS, *Sur la coercivité des formes sesquilinéaires, elliptiques,* Rev. Roumaine Math. Pures Appl. 9 (1964), 47-69.

[11] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, New Jersey, 1970.

[12] F. STUMMEL, *Elliptische Differenzenoperatoren unter Dirichletrandbedingungen,* Math. Z. 97 (1967), 169-211.

[13] V. THOMÉE, *Elliptic Difference Operators and Dirichlet's Problem*, Contributions to Differential Equations 3 (1964), 301-324.

[14] V. THOMÉE and B. WESTERGREN, *Elliptic Difference Equations and Interior Regularity,* Numer. Math. 11 (1968), 196-210.

[15] J. WLOKA, *Partielle Differentialgleichungen*, B. G. Teubner, Stuttgart, 1982.