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## EXISTENCE OF REGULAR SOLUTIONS FOR A ONE-DIMENSIONAL SIMPLIFIED PERFECT-PLASTIC PROBLEM (\*)

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*Abstract — This work is devoted to the regularity of solutions for a one-dimensional simplified model in Perfect Plasticity*

*In a first step, we are interested in the stress problem, which is the dual problem to the displacement one. First, we establish the Inf Sup equality. We give then an explicit computation of the limit load.*

*In a second step, we produce a necessary and sufficient condition, using the stress problem, for the existence of a regular solution ( $\varepsilon$  in  $W^{1,1}$ ) of the displacement problem. We can then introduce the notion of regular limit loads,  $\underline{\lambda}_{reg}, \bar{\lambda}_{reg}$ , which are so that if  $\lambda$  belongs to  $]\underline{\lambda}_{reg}, \bar{\lambda}_{reg}[$ , the displacement problem admits a solution  $u$  in  $W^{1,1}$  which satisfies the limit conditions. In a third step, we apply the theory to some classical examples.*

*Résumé — Ce travail est consacré à l'étude de l'existence de solutions régulières pour un modèle unidimensionnel simplifié en Plasticité Parfaite*

*Dans un premier temps, on s'intéresse au problème en contrainte, qui se trouve être le problème dual de celui en déplacement. On commence par établir l'égalité Inf-Sup. On donne ensuite un calcul explicite de la charge limite.*

*Dans un deuxième temps, on produit une condition nécessaire et suffisante, à partir du problème en contrainte, pour l'existence d'une solution « régulière » ( $\varepsilon$  dans  $W^{1,1}$ ) au problème en déplacement. On introduit la notion de charges limites régulières, en deçà desquelles on est assuré de l'existence de solutions régulières. On applique cette théorie à des exemples classiques de Plasticité.*

### 1. INTRODUCTION

This paper is devoted to the study of the regularity of the solutions of perfect-plastic problems, for a one-dimensional model. For previous works on plasticity, the reader can consult Suquet [11], Temam [12], Strang-Temam [13], Kohn-Temam [8] and Giaquinta-Modica [6].

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In these works, the authors have proved that when the load is not « too large », there exists a solution to the displacement problem in some weak sense. In particular,  $u$  belongs to  $BD = \{u \in L^1(\Omega, \mathbb{R}^N), \epsilon_y(u) \in M^1\} - M^1$  is the space of bounded measures on  $\Omega$ , and  $\epsilon(u)$  is the symmetric part of the gradient. In the one dimensional case, it reduces to  $BV(]0, 1[)$ . Until now, only very few things are known about regularity or uniqueness.

The first original result in that direction concerns the two dimensional case with  $\lambda = 0$ , and was proved by P. Sternberg, G. Williams and W. Ziemer [10]. Let us note that Kohn and Strang [7] have previously been able to construct a « regular » solution for the antiplane shear, in the two dimensional case, and for particular shape of  $\Omega$  and particular boundary data. Moreover, the load  $\lambda$  was zero.

Here we consider the following one-dimensional problem

$$\text{Inf}(P_\lambda) = \text{Inf}_{\substack{u \in W^{1,1}(]0, 1[) \\ u(0) = \alpha \\ u(1) = \beta}} \left\{ \int_0^1 \Psi(u'(t)) dt - \lambda \int_0^1 f(t) u(t) dt \right\} \quad (1)$$

where  $\Psi$  is a convex continuous function which is at most linear at infinity and coercive on  $L^1$ ,  $f$  is the load,  $\lambda$  is a parameter.

It is known [12], that there exists a convex subset  $\mathcal{A}$  of  $\mathbb{R}$  such that for  $\lambda \in \mathcal{A}$ ,  $\text{Inf}(P_\lambda) > -\infty$ . Moreover, one can prove the existence of solutions to  $\text{Inf}(P_\lambda)$  in a very weak sense, we must introduce a relaxed form of  $\text{Inf}(P_\lambda)$

$$\text{Inf}(P_{\lambda R}) = \text{Inf}_{u \in BV(]0, 1[)} \left\{ \int_0^1 \Psi(u') + \Psi_\infty(u(1^+) - \beta) + \right. \\ \left. + \Psi_\infty(\alpha - u(0^-)) - \lambda \int_0^1 fu \right\},$$

and note that  $\text{Inf}(P_{\lambda R}) = \text{Inf}(P_\lambda)$  ( $\Psi_\infty$  is the asymptotic function of  $\Psi$ , which is defined as  $\Psi_\infty(x) = \lim_{t \rightarrow +\infty} \frac{\Psi(tx)}{t}$ )

The main result of this paper concerns the existence of regular solutions for (1), for  $\lambda$  small enough, more precisely, we prove the existence of regular limit loads  $\underline{\lambda}_r, \bar{\lambda}_r$  such that for every  $\lambda$  in the convex set  $\mathcal{A}_r = ]-\underline{\lambda}_r, \bar{\lambda}_r[$ ,  $\text{Inf}(P_\lambda)$  possesses a solution in  $W^{1,1}(]0, 1[)$ . Let us point out that there may or may not exist a regular solution for  $\lambda = \bar{\lambda}_r$ , and that  $\underline{\lambda}_r$  may be different from  $\underline{\lambda}$ .

The plan of this paper is as follows :

— in section 2, we recall the expression of the displacement problem, the assumptions on  $\Psi$ , and the relations which link the displacement and stress problems. We give the explicit expression of the limit load  $\bar{\lambda}$  ;

— in section 3, we look for necessary and sufficient conditions to have a solution  $u$  in  $W^{1,1}([0, 1[)$ . For that aim, we use a change of variable in the stress problem. We give in Proposition 2.2 a sufficient condition on  $\Psi^*$  and  $f$  to have a « regular » solution  $u$  for  $(P_\lambda)$  (by « regular », we mean a solution which belongs to  $W^{1,1}([0, 1[))$  ,

— in section 4, we introduce the notion of Convex Set of Regularity and Regular Limit Loads, and we give some examples.

**2. PRELIMINARY RESULTS**

**2.1. Definition of the problem**

Let us consider the problem

$$\text{Inf } (P_\lambda) = \text{Inf}_{\substack{u \in W^{1,1}([0, 1[) \\ u(0) = \alpha \\ u(1) = \beta}} \left\{ \int_0^1 \Psi(u'(t)) dt - \lambda \int_0^1 f(t) u(t) dt \right\}, \quad (2)$$

where

$$\alpha \neq \beta,$$

$$f \in L^2([0, 1[),$$

$$\lambda \in \mathbb{R},$$

and  $\Psi$  is convex, lower semi continuous, proper and such that

$$\Psi(x) \sim \begin{matrix} x \\ +\infty \end{matrix}$$

$$\Psi(x) \sim \begin{matrix} -x \\ -\infty \end{matrix}.$$

We suppose here that  $\Psi$  has only a finite number of discontinuity points, but it is not difficult to extend the following results to a countable number of points

Let  $(a_n)_{n \in \mathbb{N}_n}$  be a finite, non decreasing, bounded sequence such that

$$\Psi = \begin{cases} \Psi(x) = \Psi_0(x) & \text{if } x \in ]-\infty, a_1] \\ \Psi(x) = \Psi_i(x) & \text{if } x \in [a_i, a_{i+1}] \\ \Psi(x) = \Psi_n(x) & \text{if } x \in [a_n, +\infty[ , \end{cases}$$

where  $\Psi_i \in C^1([a_{i-1}, a_i])$ , and  $\Psi'_i$  is either constant or strictly decreasing and continuous. Let us notice that  $\Psi'_i(a_i) = \Psi'_{i-1}(a_i)$  since  $\Psi$  is a convex and semi-continuous function.

*Remark 2.1 :*  $\Psi$  being convex,  $\Psi'$  is non decreasing and we have

$$\text{Sup}_{]a_i, a_{i+1}[} \Psi'_i \leq \text{Inf}_{]a_{i+1}, a_{i+2}[} \Psi'_{i+1} .$$

We define then

$$K = \{ \sigma \in L^\infty(]0, 1[), |\sigma|_\infty \leq 1 \} ,$$

$$S_{ad}(\lambda) = \{ \sigma \in L^\infty(]0, 1[), \sigma' + \lambda f = 0 \text{ a.e.} \} .$$

*Remark 2.2 :*  $\sigma \in S_{ad}(\lambda) \Rightarrow \sigma \in W^{1,\infty}(]0, 1[)$ .

Let

$$\text{Sup}(P_\lambda^*) = \text{Sup}_{\sigma \in S_{ad}(\lambda) \cap K} \left\{ - \int_0^1 \Psi^*(\sigma(t)) dt + \beta\sigma(1) - \alpha\sigma(0) \right\} . \quad (3)$$

PROPOSITION 2.1 :

$$\text{Inf}(P_\lambda) = \text{Sup}(P_\lambda^*) . \quad (4)$$

*Proof :* We apply the theory of Convex Analysis developed in the book of I. Ekeland and R. Temam (see [5]) :

Let  $V$  and  $V^*$ ,  $Y$  and  $Y^*$  be two couples of dual spaces,  $A : V \rightarrow Y$  be a linear continuous operator,  $F$  and  $G$  be two convex functions

$$F : V \rightarrow \mathbb{R}$$

$$G : Y \rightarrow \mathbb{R}$$

and  $F^*$  and  $G^*$  be their conjugate functions in the sense of Fenchel (i.e.  $\forall p^* \in V^*$ ,  $F^*(p^*) = \text{Sup}_{p \in V} \{ \langle p^*, p \rangle - F(p) \}$ ); let  $(P)$  and  $(P^*)$  the two following conjugate problems :

$$(P) = \text{Inf}_{v \in V} \{ F(v) + G(A(v)) \}$$

and

$$(P^*) = \text{Sup}_{p^* \in Y^*} \{-F^*(A^* p^*) - G^*(-p^*)\}.$$

then

THEOREM 2.2 :

$$-\infty \leq \text{Inf}(P) \leq \text{Sup}(P^*) < +\infty. \tag{5}$$

Furthermore, if there exists  $u_0 \in V$  such that  $F(u_0) < +\infty$ , and if  $G$  is continuous at the point  $A(u_0)$ , then

$$\text{Inf}(P) = \text{Sup}(P^*).$$

Here we take  $V = W^{1,1}(]0, 1[)$ ,  $Y = L^1(]0, 1[)$ ,  $Y^* = L^\infty(]0, 1[)$ , and the operator

$$\begin{aligned} A : V &\rightarrow Y \\ u &\mapsto u'. \end{aligned}$$

The functionals  $F$  and  $G$  are

$$\begin{aligned} G(u) &= \int_0^1 \Psi(u(t)) dt \\ F(u) &= \begin{cases} -\lambda \int_0^1 f(t) u(t) dt & \text{if } u \in C \\ +\infty & \text{elsewhere,} \end{cases} \end{aligned}$$

where  $C = \{u \in V \mid u(0) = \alpha \text{ and } u(1) = \beta\}$  is a convex subset of  $W^{1,1}(]0, 1[)$ .  $F$  and  $G$  are convex. Using the Theorem of Krasnoselskii (see [9]), we obtain

$$G^*(\sigma) = \int_0^1 \Psi^*(\sigma(t)) dt.$$

Here,

$$F^*(A^* \sigma) = \begin{cases} \beta\sigma(1) - \alpha\sigma(0) & \text{if } (-\sigma) \in S_{ad}(\lambda) \\ +\infty & \text{elsewhere.} \end{cases}$$

Using Theorem I.4.2 in [5], we get

$$-\infty \leq \text{Sup}(P_\lambda^*) \leq \text{Inf}(P_\lambda) < +\infty.$$

Furthermore, having the existence of  $u_0 \in V$ , such that  $F(u_0) < +\infty$ , and  $G$  being continuous at  $u'_0$ ,

$$\text{Inf}(P_\lambda) = \text{Sup}(P_\lambda^*),$$

and  $\text{Sup}(P_\lambda^*)$  has at least one solution □

## 2.2. An explicit expression of the limit load

Let  $\mathcal{A} = \{\lambda \in \mathbb{R}/K \cap S_{ad}(\lambda) \neq \emptyset\}$ . It follows immediately from the definitions of  $K$  and  $S_{ad}(\lambda)$  that  $\mathcal{A}$  is a bounded convex set containing 0 (but we should notice that it is exactly the real line  $\mathbb{R}$  if  $f \equiv 0$ ), thus there exists  $(\bar{\lambda}, \underline{\lambda}) \in \mathbb{R}^2$  such that  $\mathcal{A} = ]-\underline{\lambda}, \bar{\lambda}[$ . Here below, we give an explicit expression of  $\bar{\lambda}$  and  $\underline{\lambda}$ .

PROPOSITION 2.3

$$\underline{\lambda} = \bar{\lambda} = \frac{2}{\text{Sup}_{(a,x) \in [0,1]^2} \left| \int_a^x f(t) dt \right|}, \quad (6)$$

*Proof* Let  $\lambda \in \mathcal{A}$ , and  $\sigma \in S_{ad}(\lambda) \cap K$

We have  $\sigma' + \lambda f = 0$ , thus  $\forall (a, x) \in [0, 1]^2$

$$\sigma(x) = \sigma(a) - \lambda \int_a^x f(t) dt, \quad (7)$$

and  $\sigma \in K$ , thus

$$-1 - \sigma(a) \leq -\lambda \int_a^x f(t) dt \leq 1 - \sigma(a)$$

$|\sigma(a)| \leq 1$  implies that

$$|\lambda| \int_a^x f(t) dt \leq 2, \quad (8)$$

$$|\lambda| \text{Sup}_{(a,x) \in [0,1]^2} \left| \int_a^x f(t) dt \right| \leq 2, \quad (9)$$

$$|\lambda| \leq \frac{2}{\text{Sup}_{(a,x) \in [0,1]^2} \left| \int_a^x f(t) dt \right|} \quad (10)$$

Now, we show the reverse assertion ; if we denote by  $G$  the antiderivative of  $f$  which takes value 0 at zero,

$$G : [0, 1] \rightarrow \mathbb{R}$$

$$x \mapsto \int_0^x f(t) dt .$$

We can define

$$B_m = \{x \in [0, 1] / \forall y \in [0, 1], G(y) \geq G(x)\}$$

$$B_M = \{x \in [0, 1] / \forall y \in [0, 1], G(y) \leq G(x)\} .$$

$G$  being continuous, Heine's theorem ensures us that neither  $B_m$  nor  $B_M$  is void.

Now, let  $\lambda \in \mathbb{R}$  be such that

$$|\lambda| \leq \frac{2}{\sup_{(a, x) \in [0, 1]^2} \left| \int_a^x f(t) dt \right|} .$$

— Taking  $\lambda > 0$ ,  $a \in B_M$ , and  $x$  in  $[0, 1]$ , we get

$$\int_0^x f(t) dt \leq \int_0^a f(t) dt$$

$$\int_a^x f(t) dt \leq 0$$

thus  $\forall x \in [0, 1]$ ,

$$-2 \leq \lambda \int_a^x f(t) dt \leq 0 .$$

Let

$$\sigma : [0, 1] \rightarrow \mathbb{R}$$

$$x \mapsto -1 + \lambda \int_a^x f(t) dt .$$

$$\sigma \in S_{ad}(\lambda) \cap K \Rightarrow \lambda \in \mathcal{A} .$$



— If  $\lambda < 0$ , taking  $a$  in  $B_m$ , and using an analogous process, we obtain the whole conclusion of Proposition 1.2.  $\square$

### 2.3. The extremality relation

From now on, we will denote by  $\partial$  the subdifferential.

PROPOSITION 2.4: *Let  $u$  be a solution of  $\text{Inf}(P_\lambda)$  and  $\sigma$  a solution of  $\text{Sup}(P_\lambda^*)$ . Then the three following assertions (11), (12), (13) hold for almost all  $t \in [0, 1]$ ,*

$$\Psi^*(\sigma(t)) + \Psi(u'(t)) - u'(t)\sigma(t) = 0. \quad (11)$$

$$\sigma(t) \in \partial\Psi(u'(t)). \quad (12)$$

$$u'(t) \in \partial\Psi^*(\sigma(t)). \quad (13)$$

*Proof:* Theorem 1.2 implies that  $\text{Inf}(P_\lambda) = \text{Sup}(P_\lambda^*)$ , from which we can write

$$\int_0^1 \Psi(u'(t)) dt - \lambda \int_0^1 f(t) u(t) dt = \beta\sigma(1) - \alpha\sigma(0) - \int_0^1 \Psi^*(\sigma(t)) dt.$$

Since  $\sigma \in S_{ad}(\lambda)$ ,  $\sigma'(t) = -\lambda f(t)$  and  $\forall t \in [0, 1]$

$$\int_0^1 (\Psi(u'(t)) + \Psi^*(\sigma(t))) dt + \int_0^1 \sigma'(t) u(t) dt = \beta\sigma(1) - \alpha\sigma(0).$$

Hence, integrating by parts,

$$\int_0^1 \sigma'(t) u(t) dt = - \int_0^1 \sigma(t) u'(t) dt + \beta\sigma(1) - \alpha\sigma(0).$$

Therefore we obtain

$$\int_0^1 [\Psi(u'(t)) + \Psi^*(\sigma(t)) - \sigma(t)u'(t)] dt = 0. \quad (14)$$

Using the definition of  $\Psi^*$ ,

$$\Psi^*(\sigma(t)) = \text{Sup}_{X \in \mathbb{R}} \{X\sigma(t) - \Psi(X)\}$$

$$\geq u'(t)\sigma(t) - \Psi(u'(t)).$$

We derive that  $\Psi(u'(t)) + \Psi^*(\sigma(t)) - u'(t)\sigma(t)$  has a constant sign for almost all  $t \in ]0, 1[$ . Therefore, (14) implies that

$$\Psi(u'(t)) + \Psi^*(\sigma(t)) - u'(t)\sigma(t) = 0,$$

for almost every  $t \in ]0, 1[$ .

The assertions (11), (12) and (13) are equivalent since  $\Psi$  is convex, lower semi continuous and proper (see [5]). □

**2.4. Computation of  $\Psi^*$**

*Remark 2.3 :*  $\Psi'_i$  being non decreasing, it admits a limit on the right of  $a_{i+1}$  and a limit on the left of  $a_i$ , that we denote respectively by  $\Psi'_{d'}(a_{i+1})$  and  $\Psi'_{g'}(a_i)$ .

PROPOSITION 2.5 :

$\Psi^*(Y) =$

$$\begin{cases} +\infty & \text{if } |Y| > 1 \\ Y\Psi_0'^{-1}(Y) - \Psi_0(\Psi_0'^{-1}(Y)) & \text{if } Y \in ]-1, \Psi'_g(a_1)] \\ Y\Psi_i'^{-1}(Y) - \Psi_i(\Psi_i'^{-1}(Y)) & \text{if } Y \in ]\Psi'_{d'}(a_i), \Psi'_g(a_{i+1})] \\ a_{i+1}Y - \Psi_i(a_{i+1}) & \text{if } Y \in ]\Psi'_{d'}(a_{i+1}), \Psi'_{d'}(a_{i+1})] \\ Y\Psi_n'^{-1}(Y) - \Psi_n(\Psi_n'^{-1}(Y)) & \text{if } Y \in ]\Psi'_{d'}(a_n), 1[. \end{cases}$$

*Remark 2.4 :*  $\Psi^*$  being continuous since it is convex, *lsc* and proper, we can deduce the value of  $\Psi^*$  at the points  $\Psi'_{d'}(a_i)$  and  $\Psi'_g(a_{i+1})$  from Proposition (2.5).

*Proof :* The proof of this Proposition is technical and may be found in [1].□

**3. A SUFFICIENT CONDITION ON  $\lambda$  TO HAVE A SOLUTION  $u$  IN  $W^{1,1}(]0, 1[)$**

In this section, we assume that  $\lambda$  belongs to  $] -\bar{\lambda}, \bar{\lambda}[$ .

**3.1. A change of variable for the stress function**

Let  $\lambda \in A, \sigma \in S_{ad}(\lambda) \cap K$  and  $a \in B_m$ . We can write :

$$\sigma(x) = X - \lambda \int_a^x f(t) dt$$

where  $X$  equals  $\sigma(a)$  Therefore if  $b$  belongs to  $B_M$ , we get for all  $x$  in  $[0, 1]$

$$0 \leq \int_a^x f(t) dt \leq \int_a^b f(t) dt, \quad (15)$$

and we must consider two different cases

— If  $\lambda \leq 0, \forall x \in [0, 1]$

$$X \leq \sigma(x) \leq X - \lambda \int_a^b f(t) dt, \quad (16)$$

— If  $\lambda > 0$

$$X - \lambda \int_a^b f(t) dt \leq \sigma(x) \leq X, \quad (17)$$

and then  $\sigma$  achieves its extrema in  $a$  and  $b$

We shall denote by  $\sigma_\lambda(X, t)$  the function  $\sigma_\lambda(X, t) = X - \lambda \int_a^t f(v) dv$   
We now define the functional

$$G_\lambda(X) = - \int_0^1 \Psi^*(\sigma_\lambda(X, t)) dt + \sigma_\lambda(X, 1) \beta - \sigma_\lambda(X, 0) \alpha,$$

and we remark that looking for  $\text{Sup}(P_\lambda^*)$  is equivalent to maximizing  $G_\lambda(X)$  We will then use the point  $X \in [-1, 1]$  as a new variable

The definition sets of  $G_\lambda$  depends on the sign of  $\lambda$

$$\lambda \geq 0, D_G = ] - 1 + \lambda \int_a^b f(s) ds, 1[$$

$$\lambda \leq 0, D_G = ] - 1, 1 + \lambda \int_a^b f(s) ds[$$

since  $\sigma_\lambda \in K$

Let us suppose that  $\lambda > 0$  (if  $\lambda < 0$ , the proofs and the results are analogous)

$G_\lambda$  is continuous on  $D_G$ , and a.e differentiable

PROPOSITION 3.1  $G_\lambda$  is right and left differentiable on  $D_G$  and  $\forall x \in D_G$

$$G'_{\lambda d}(X) = \beta - \alpha - \int_0^1 \Psi_d^*(\sigma_\lambda(X, t)) dt \quad (18)$$

$$G'_{\lambda g}(X) = \beta - \alpha - \int_0^1 \Psi_g^*(\sigma_\lambda(X, t)) dt \quad (19)$$

*Proof:* This result is a consequence of Lebesgue's dominated convergence Theorem. Let us introduce  $X_0 \in D_G$  and  $I$ , a neighbourhood of  $X_0$  such that  $\bar{I} \subset D_G$ . Let  $(h_n)_{n \in \mathbb{N}}$  be a sequence of non negative reals, such that :

$$\begin{cases} h_n + X_0 \in I, \quad \forall n \in \mathbb{N}, \\ \lim_{n \rightarrow \infty} h_n = 0. \end{cases}$$

Let  $\varphi_n(X_0, t) = \frac{1}{h_n} [\Psi^*(\sigma_\lambda(X_0 + h_n, t)) - \Psi^*(\sigma(X_0, t))]$ .

Since  $\Psi^* \circ \sigma_\lambda$  is convex, it is right and left differentiable on  $I$  and for every  $t$  in  $[0, 1]$ ,

$$\Psi_g^{*'}(\sigma_\lambda(X_0, t)) \leq \Psi_d^{*'}(\sigma_\lambda(X_0, t)) \leq \varphi_n(X_0, t) \tag{20}$$

$$\varphi_n(X_0, t) \leq \Psi_g^{*'}(\sigma_\lambda(X_0 + h_n, t)) \leq \Psi_d^{*'}(\sigma_\lambda(X_0 + h_n, t)) \tag{21}$$

From (17), we have  $\forall t \in [0, 1], \forall X \in I$ :

$$X - \lambda \int_a^b f(s) ds \leq \sigma_\lambda(X, t) \leq X.$$

Since  $\Psi^*$  is convex, its derivatives are non decreasing, and for all  $X \in I$

$$\text{Inf}_I \{ \Psi_d^{*'} \} \leq \Psi_d^{*'}(\sigma_\lambda(X, t)) \leq \text{Sup}_I \{ \Psi_d^{*'} \} \tag{22}$$

$$\text{Inf}_I \{ \Psi_g^{*'} \} \leq \Psi_g^{*'}(\sigma_\lambda(X, t)) \leq \text{Sup}_I \{ \Psi_g^{*'} \}. \tag{23}$$

From these, we obtain a uniform bound on  $\varphi_n(X_0, t)$

$$|\varphi_n(X_0, t)| \leq A. \tag{24}$$

The hypothesis of Lebesgue's dominated convergence Theorem being verified, we get

$$\lim_{n \rightarrow +\infty} \int_0^1 \varphi_n(X_0, t) dt = \int_0^1 \left( \lim_{n \rightarrow +\infty} \varphi_n(X_0, t) \right) dt.$$

But  $\lim_{n \rightarrow +\infty} \varphi_n(X_0, t) = \Psi_d^{*'}(\sigma_\lambda(X_0, t))$  and

$$\int_0^1 \varphi_n(X_0, t) dt = \frac{1}{h_n} [G_\lambda(X_0 + h_n) - G_\lambda(X_0)].$$

Since  $(h_n)_{n \in \mathbb{N}}$  is arbitrary,  $G_\lambda$  is right differentiable, and for all  $X_0 \in D_G$ ,

$$G'_{\lambda, d}(X_0) = \beta - \alpha - \int_0^1 \Psi_d^{*'}(\sigma_\lambda(X_0, t)) dt. \tag{25}$$

We obtain the same result for the left derivative, taking  $(h_n)_{n \in \mathbb{N}}$  a sequence of negative reals, and obtain that  $G_\lambda$  is left differentiable and

$$G'_{\lambda, g}(X_0) = \beta - \alpha - \int_0^1 \Psi_g^*(\sigma_\lambda(X_0, t)) dt. \quad (26)$$

□

### 3.2. A key result

PROPOSITION 3.2 : Let  $\lambda \geq 0$ , if

$$\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) \leq 0 \quad \text{and} \quad \lim_{X \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) \geq 0, \quad (27)$$

then  $\text{Inf}(P_\lambda)$  has at least one solution.

We need the following Lemma for the proof of this Proposition :

LEMMA 3.3 : Let  $f$  be a continuous function on  $[a, b]$ , right and left differentiable on  $]a, b[$ , such that

$$\begin{cases} f'_d \text{ and } f'_g \text{ are non decreasing} \\ \lim_{x \rightarrow a^+} f'_d(x) < 0 \\ \lim_{x \rightarrow b^-} f'_g(x) > 0. \end{cases}$$

Then there exists  $x_0$  in  $]a, b[$  such that

$$\begin{cases} f'_g(x_0) \leq 0 \\ f'_d(x_0) \geq 0. \end{cases}$$

A proof of this lemma can be found in [1], and is based upon a generalized form of the fundamental calculus Theorem.

*Proof* (of Proposition 2.2) :

The supremum of  $G_\lambda$ , under the assumptions above, can be achieved on  $D_G$  or on its boundary.

1.

$$\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) < 0 \quad \text{and} \quad \lim_{X \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) > 0.$$

Using lemma 2.3, we know that there exists  $X_0$  in  $D_G$  such that

$$\begin{cases} G'_{\lambda, d}(X_0) \leq 0 \\ G'_{\lambda, g}(X_0) \geq 0. \end{cases}$$

Since  $X_0$  is in  $D_G$ ,  $|\sigma_\lambda(X_0, \cdot)|$  never takes the value 1 and then  $\Psi^*(\sigma_\lambda(X_0, \cdot))$  is well defined.

Let  $(c, d) \in \mathbb{R}^2$  be such that

$$\begin{cases} c + d = 1 \\ cG'_{\lambda, g}(X_0) + dG'_{\lambda, d}(X_0) = 0 \end{cases}$$

and define

$$v : [0, 1] \rightarrow \mathbb{R}$$

$$t \mapsto c\Psi_g^*(\sigma_\lambda(X_0, t)) + d\Psi_d^*(\sigma_\lambda(X_0, t)).$$

*Remark 3.1:* In fact, the previous definition means that  $v(t) \in \partial\Psi^*(\sigma_\lambda(X_0, t))$ , for almost every  $t$  in  $[0, 1]$ .

Let now  $u$  be the antiderivative of  $v$  which equals  $\alpha$  at 0.

$$u : [0, 1] \rightarrow \mathbb{R}$$

$$t \mapsto \alpha + \int_0^t v(s) ds.$$

$$u(1) = \alpha + c \int_0^1 \Psi_g^*(\sigma_\lambda(X_0, t)) dt + d \int_0^1 \Psi_d^*(\sigma_\lambda(X_0, t)) dt$$

$$= \alpha + \beta - \alpha$$

$$= \beta.$$

Since  $v(t) \in \partial\Psi^*(\sigma_\lambda(X_0, t))$ ,  $u$  and  $\sigma_\lambda$  verify the extremality relation. Thus,  $u$  and  $\sigma_\lambda$  are solutions of  $\text{Inf}(P_\lambda)$  and  $\text{Sup}(P_\lambda^*)$ .

2.

$$\lim_{x \rightarrow 1^-} G'_{\lambda, g}(X) = 0 \quad \text{or} \quad \lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) = 0.$$

Then,  $G_j$  has its supremum at 1 or at  $(-1 + \lambda \int_a^b f(s) ds)$ . Let us assume that  $\lim_{X \rightarrow 1} G'_{\lambda, g}(X) = 0$ , since  $G'_{\lambda, g}$  and  $G'_{\lambda, d}$  are decreasing,

$$\lim_{X \rightarrow 1} G'_{\lambda, g}(X) = 0 \Rightarrow \lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) \geq 0$$

As a first step, we shall prove that

$$\lim_{X \rightarrow 1} \int_0^1 \Psi_g^*(\sigma_\lambda(X, t)) dt = \int_0^1 \lim_{X \rightarrow 1} \Psi_g^*(\sigma_\lambda(X, t)) dt \quad (28)$$

Since we assume  $\lim_{X \rightarrow 1} G'_{\lambda, g}(X) = 0$ , we have

$$\beta - \alpha = \lim_{X \rightarrow 1} \int_0^1 \Psi_g^*(\sigma_\lambda(X, t)) dt$$

which implies that the left hand side of (28) is finite

Let  $(X_n)_{n \in \mathbb{N}}$  be an arbitrary increasing sequence of  $[0, 1]$  such that

$$\sigma_\lambda(X_0, t) > -1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} X_n = 1$$

Then, by Fatou's Lemma

$$\int_0^1 \lim_{X \rightarrow +\infty} \Psi_g^*(\sigma_j(X_n, t)) dt \leq \beta - \alpha < +\infty$$

Furthermore,  $\Psi_g^*(\sigma_j(\cdot, t))$  is non decreasing and continuous with respect to  $X$  in a neighbourhood of 1. We have then for  $(n, t) \in \mathbb{N} \times ]0, 1[$

$$\Psi_g^*(\sigma_\lambda(X_n, t)) \leq \lim_{n \rightarrow +\infty} \Psi_g^*(\sigma_\lambda(X_n, t))$$

We now apply Lebesgue's theorem, to obtain

$$\lim_{n \rightarrow +\infty} \int_0^1 \Psi_g^*(\sigma_\lambda(X_n, t)) dt = \int_0^1 \lim_{n \rightarrow +\infty} \Psi_g^*(\sigma_\lambda(X_n, t)) dt$$

Since  $X_n$  is increasing, for every  $X_0 < X < 1$ , there exists  $n_0 \in \mathbb{N}$  such that  $X_{n_0-1} \leq X \leq X_{n_0}$ , and by the non decreasing behaviour of  $\Psi_g^*$ , we get

$$\begin{aligned} \Psi_g^*(\sigma_\lambda(X, t)) &\leq \Psi_g^*(\sigma_\lambda(X_{n_0}, t)) \\ &\leq \lim_{n \rightarrow +\infty} \Psi_g^*(\sigma_\lambda(X_n, t)) \end{aligned}$$

Lebesgue's Theorem ensures that

$$\lim_{X \rightarrow 1} \int_0^1 \Psi_g^*(\sigma_\lambda(X, t)) dt = \int_0^1 \lim_{X \rightarrow 1^-} \Psi_g^*(\sigma_\lambda(X, t)) dt. \tag{29}$$

We now construct a solution of  $\text{Inf}(P_\lambda)$ .

$$\begin{aligned} \text{Let } v : [0, 1] &\mapsto \bar{\mathbb{R}} \\ t &\mapsto \lim_{X \rightarrow 1^-} \Psi_g^*(\sigma_\lambda(X, t)). \end{aligned}$$

From above, we know that  $v \in L^1(]0, 1[)$

$$\begin{aligned} \text{Let } u : [0, 1] &\mapsto \mathbb{R} \\ t &\mapsto \int_0^t v(s) ds + \alpha. \end{aligned}$$

We can see that  $u \in L^\infty([0, 1])$  and

$$\begin{aligned} u(1) &= \alpha + \int_0^1 \lim_{n \rightarrow +\infty} \Psi_g^*(\sigma_\lambda(X_n, t)) dt \\ &= \alpha + \lim_{n \rightarrow +\infty} \int_0^1 \Psi_g^*(\sigma_\lambda(X_n, t)) dt \\ &= \alpha + \beta - \alpha \\ &= \beta. \end{aligned}$$

And, almost everywhere,

$$\lim_{X \rightarrow 1} \Psi_g^*(\sigma_\lambda(X, t)) = \Psi_g^*(\sigma_\lambda(1, t)),$$



since  $\Psi_g^{*'}$  is continuous on the left of every point of discontinuity of  $\Psi^*$ , and since  $\Psi_g^{*'} \circ \sigma_\lambda$  is non decreasing in  $X$ .

Hence,  $v(t) \in \partial \Psi^*(\sigma_\lambda(1, t))$  a.e. Thus  $u$  and  $\sigma_\lambda$  are respectively solutions of  $(P_\lambda)$  and  $(P_\lambda^*)$ .  $\square$

Now, if  $\lambda < 0$ , we have a similar result :

PROPOSITION 3.4 : *If  $\lambda < 0$  and if*

$$\lim_{X \rightarrow (1 + \lambda \int_a^b f(t) dt)^-} G'_{\lambda, g}(X) \leq 0 \quad \text{and} \quad \lim_{X \rightarrow (-1)^+} G'_{\lambda, d}(X) \geq 0$$

then  $\text{Inf}(P_\lambda)$  has at least one solution.

### 3.3. Example

We illustrate the previous result by treating a famous example, which appears in the calculus of variations :

Let  $\Psi \cdot \mathbb{R} \rightarrow \mathbb{R}$

$$X \mapsto \sqrt{1 + X^2}$$

Let  $f \cdot [0, 1] \rightarrow \mathbb{R}$

$$X \mapsto \begin{cases} 0 & \text{if } X \in [0, \frac{1}{2}] \\ 1 & \text{if } X \in [\frac{1}{2}, 1] \end{cases}.$$

It is not difficult to see that

$$\bar{\lambda} = 4$$

We assume that  $\lambda = 2, \beta = \alpha$ .

The computation of  $\Psi^*$  gives

$\Psi^* \cdot [-1, 1] \rightarrow \mathbb{R}$

$$Y \mapsto -\sqrt{1 - Y^2}$$

$$\text{Inf}(P_\lambda) = \inf_{\substack{u \in W^{1,1}([0,1]) \\ u(0) = \alpha \\ u(1) = \beta}} \left\{ \int_0^1 \sqrt{1 + (u'(t))^2} dt - \lambda \int_{\frac{1}{2}}^1 u(t) dt \right\} \quad (30)$$

$$\text{Sup } (P_\lambda^*) = \text{Sup}_{\sigma \in S_{ad}(\lambda) \cap K} \left\{ - \int_0^1 \sqrt{1 - \sigma^2(t)} dt + \alpha(\sigma(1) - \sigma(0)) \right\}. \tag{31}$$

Here  $\sigma_2(t) = X - 2 \int_0^t f(s) ds$

i.e.  $\sigma_2(t) = \begin{cases} X & \text{if } t \in [0, \frac{1}{2}] \\ X - 2(t - \frac{1}{2}) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$

$$\begin{aligned} G_2(X) &= \int_0^1 \sqrt{1 - \sigma_2^2(t)} dt + \alpha(\sigma_2(1) - \sigma_2(0)) \\ &= \int_0^{\frac{1}{2}} \sqrt{1 - X^2} dt + \int_{\frac{1}{2}}^1 \sqrt{1 - (X + 1 - 2t)^2} dt + \\ &\quad + \alpha(\sigma_2(1) - \sigma_2(0)) \end{aligned}$$

$$\begin{aligned} G_2'(X) &= - \frac{X}{\sqrt{1 - X^2}} - \int_{\frac{1}{2}}^1 \frac{X + 1 - 2t}{\sqrt{1 - (X + 1 - 2t)^2}} dt \\ &= - \frac{X}{\sqrt{1 - X^2}} - \frac{\sqrt{1 - (X - \frac{1}{2})^2}}{2} + \frac{\sqrt{1 - X^2}}{2}. \end{aligned}$$

$$\lim_{x \rightarrow 1^-} G_2'(X) = -\infty.$$

$$\lim_{x \rightarrow 0^+} G_2'(X) = \frac{1}{2}.$$

The assumptions of Proposition (2.2) are verified. The computation of  $X_0$ ,  $X_0$  being such that  $G_2'(X_0) = 0$  gives us a solution of  $\text{Inf } (P_\lambda)$ . In this example, this computation requires the resolution of a sixth order equation.

In fact,  $\Psi^*$  being  $C^1(-1, 1[)$ ,  $u$  is unique, and we have

$$\begin{aligned} t \in [0, \frac{1}{2}] \quad u(t) &= \alpha + t \frac{X_0}{\sqrt{1 - X_0^2}} \\ t \in [\frac{1}{2}, 1] \quad u(t) &= \alpha + \frac{\sqrt{1 - (X_0 + 1 - 2t)^2}}{2} - \frac{\sqrt{2X_0 - X_0^2}}{2}. \end{aligned}$$

**3.4. The necessity viewpoint for condition (27)**

PROPOSITION 3.5 : *Let us suppose that  $\lambda > 0$ .*

1. If  $\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) > 0$  and

$$(a) \quad \text{meas}(B_m) = 0$$

$\text{Inf}(P_\lambda)$  possesses at least one solution if and only if

$$\int_0^1 \text{Inf}(\partial \Psi^*(\sigma_\lambda(1, t))) dt \leq \beta - \alpha \quad (32)$$

$$\beta - \alpha \leq \int_0^1 \text{Sup}(\partial \Psi^*(\sigma_\lambda(1, t))) dt. \quad (33)$$

$$(b) \quad \text{meas}(B_m) \neq 0$$

$\text{Inf}(P_\lambda)$  possesses at least one solution if and only if  $\Psi_n$  is linear.

2. If  $\lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, g}(X) < 0$  and

$$(a) \quad \text{meas}(B_M) = 0$$

$\text{Inf}(P_\lambda)$  possesses at least one solution if and only if:

$$\int_0^1 \text{Inf}(\partial \Psi^*(\sigma_\lambda(-1 + \lambda \int_a^b f(s) ds, t))) dt \leq \beta - \alpha \quad (34)$$

$$\beta - \alpha \leq \int_0^1 \text{Sup}(\partial \Psi^*(\sigma_\lambda(-1 + \lambda \int_a^b f(s) ds, t))) dt. \quad (35)$$

$$(b) \quad \text{meas}(B_M) \neq 0$$

$\text{Inf}(P_\lambda)$  possesses at least one solution if and only if  $\Psi_0$  is linear.

*Proof:* We give the details for case (1), the proof of case (2) may be done in a similar way. Let us suppose that  $\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) > 0$  and  $\text{meas}(B_m) \neq 0$ . We will denote  $\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X)$  by  $\varepsilon$ . Two cases may occur:

— Either  $\Psi_n$  is strictly convex.

Thus  $\partial \Psi^*(1) = \{+\infty\}$ . The extremality relation implies that if  $u$  is a solution,  $u'(t)$  belongs to  $\partial \Psi^*(\sigma_\lambda(1, t))$ , for almost every  $t$  in  $[0, 1]$ .

We would have  $\int_0^1 u'(t) dt = +\infty$ , from which we derive a contradiction.

— Either  $\Psi_n$  is linear, hence  $\partial \Psi^*(1) = [a_n, +\infty]$ .

Let us denote by  $(X_i)_{1 \leq i \leq k}$  the increasing sequence of the points of discontinuity of  $\Psi^*$ , and define the following sets and reals

$$T_i = \{t \in [0, 1] / \sigma_\lambda(1, t) = X_i\}$$

$$T = \bigcup_{1 \leq i \leq k} T_i$$

$$\lambda_i = \text{meas}(T_i)$$

$$b_i^- = \Psi_g^{**}(X_i)$$

$$b_i^+ = \Psi_d^{**}(X_i)$$

$$\lambda_n = \text{meas}(B_m).$$

Let us point out that we will need the following equality

$$\lim_{X \rightarrow 1^-} \Psi_g^{**}(\sigma_\lambda(X, t)) = \Psi_g^{**}(\sigma_\lambda(1, t))$$

for every  $t \notin B_m$ , proved in section 2.2. At this stage, we are once more led to distinguish two cases (i and ii).

$$i) \quad \text{If } \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) \geq \varepsilon.$$

Let  $\mu$  be in  $[0, 1]$  such that  $\mu = \frac{\varepsilon}{\sum_{i=1}^k \lambda_i (b_i^+ - b_i^-)}$ ,

$$v(t) = \begin{cases} \Psi^{**}(\sigma_\lambda(1, t)) & \forall t \in [0, 1] \setminus \bigcup B_m \\ \mu b_i^+ + (1 - \mu) b_i^- & \forall t \in T_i \\ a_n & \forall t \in B_m \end{cases}$$

and

$$u : [0, 1] \rightarrow \mathbb{R}$$

$$t \mapsto \alpha + \int_0^t v(s) ds.$$

It is easy to check that  $u$  is a solution of  $\text{Inf}(P_\lambda)$ :

$$u' \in \partial \Psi^*(\sigma_\lambda(1, t)), \quad \forall t \in [0, 1]$$

$$u \in W^{1,1}([0, 1[)$$

$$u(0) = \alpha$$

and

$$u(1) = \alpha + \lambda_n a_n + \mu \lambda_i (b_i^+ - b_i^-) + \sum_{i=1}^k \lambda_i b_i^- + \int_{[0,1] \setminus \bigcup B_m} \Psi_g^*(\sigma_\lambda(1, t)) dt.$$

But

$$\begin{aligned} \int_{[0,1] \setminus \bigcup B_m} \Psi_g^*(\sigma_\lambda(1, t)) dt &= \lim_{x \rightarrow 1} \int_0^1 \Psi_g^*(\sigma_\lambda(X, t)) dt \\ &\quad - \lim_{x \rightarrow 1^-} \int_{B_m \cup T} \Psi_g^*(\sigma_\lambda(X, t)) dt \\ &= \beta - \alpha - \varepsilon - \lambda_n a_n - \sum_{i=1}^k \lambda_i b_i^-. \end{aligned}$$

Therefore

$$\begin{aligned} u(1) &= \alpha + \lambda_n a_n + \varepsilon + \sum_{i=1}^k \lambda_i b_i^- + \beta - \alpha - \varepsilon - \lambda_n a_n - \sum_{i=1}^k \lambda_i b_i^- \\ &= \beta. \end{aligned}$$

$$\text{ii) If } \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) < \varepsilon.$$

Let  $\mu$  be such that  $\mu = \varepsilon - \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-)$ , and  $(v, u)$  be defined as

$$v(t) = \begin{cases} \Psi_g^*(\sigma_\lambda(1, t)) & \forall t \in [0, 1] \setminus \bigcup B_m \\ b_i^+ & \forall t \in T_i \\ a_n + \frac{\mu}{\lambda_n} & \forall t \in B_m. \end{cases}$$

$$u : [0, 1] \rightarrow \mathbb{R}$$

$$t \rightarrow \alpha + \int_0^t v(s) ds.$$

$u$  is a solution of  $\text{Inf}(P_\lambda)$ , since

$$u' \in \partial \Psi^*(\sigma_\lambda(1, t)), \quad \forall t \in [0, 1]$$

$$u \in W^{1,1}([0, 1[)$$

$$u(0) = \alpha$$

and

$$\begin{aligned}
 u(1) &= \alpha + \lambda_n a_n - \varepsilon - \sum_{i=1}^k \lambda_i b_i^- + \beta - \alpha + \sum_{i=1}^k \lambda_i b_i^+ + \mu \\
 &= \beta.
 \end{aligned}$$

Let us assume that we are in case 1.a), i.e. that  $\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) = \varepsilon > 0$  and  $\text{meas}(B_m) = 0$ . On the complementary set of  $T$ ,  $\Psi_g^*(\sigma_\lambda(1, t)) = \Psi_d^*(\sigma_\lambda(1, t))$ , and then

$$\begin{aligned}
 \int_{[0, 1] \setminus \text{TV}B_m} \Psi_g^*(\sigma_\lambda(1, t)) dt &= \int_{[0, 1] \setminus \text{TV}B_m} \Psi_d^*(\sigma_\lambda(1, t)) dt \\
 &= \beta - \alpha - \varepsilon - \sum_{i=1}^k \lambda_i b_i^-.
 \end{aligned}$$

If  $u$  is a solution for  $\text{Inf}(P_\lambda)$ ,  $u$  and  $\sigma$  verify the extremality relation, and

$$\begin{aligned}
 \forall t \in [0, 1] \setminus \text{TV}B_m, \quad u'(t) &= \Psi_g^*(\sigma_\lambda(1, t)) \\
 \forall t \in T, \quad b_i^- &\leq u'(t) \leq b_i^+ \\
 \sum_{i=1}^k \lambda_i b_i^- &\leq \int_T u'(t) dt \leq \sum_{i=1}^k \lambda_i b_i^+.
 \end{aligned}$$

Then, by adding the two inequalities,

$$\beta - \alpha - \varepsilon \leq \int_0^1 u'(t) dt \leq \beta - \alpha - \varepsilon + \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) \tag{36}$$

$$-\varepsilon \leq 0 \leq \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) - \varepsilon. \tag{37}$$

These last inequalities provide a necessary condition for  $\text{Inf}(P_\lambda)$  to have a regular solution. In fact, this condition is also sufficient. Indeed, let  $\mu \in ]0, 1[$  such that

$$\mu \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) = \varepsilon$$

and  $v \in L^1(]0, 1[)$  defined by

$$v(t) = \begin{cases} \Psi_g^{*'}(\sigma_\lambda(1, t)) & \forall t \in [0, 1] \setminus \bigcup B_m \\ \mu b_i^+ + (1 - \mu) b_i^- & \forall t \in T_i \\ +\infty & \forall t \in B_m. \end{cases}$$

Let then  $u$  be defined by

$$u(t) = \alpha + \int_0^t v(s) ds \quad \forall t \in [0, 1]. \quad (38)$$

$u \in W^{1,1}([0, 1[)$ ,  $u(0) = \alpha$ , and

$$\begin{aligned} u(1) &= \alpha + \int_{[0,1] \setminus \bigcup B_m} v(s) ds + \sum_{i=1}^k \int_{T_i} v(s) ds \\ &= \alpha + \beta - \alpha - \varepsilon - \sum_{i=1}^k \lambda_i b_i^- + \sum_{i=1}^k \lambda_i (\mu b_i^+ + \\ &\quad + (1 - \mu) b_i^-) \\ &= \beta - \varepsilon + \mu \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-) \\ &= \beta. \end{aligned}$$

Once more,  $u$  is a solution of  $\text{Inf}(P_\lambda)$ .

One can remark that

$$\forall t \in T_i \quad \text{Inf} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} = b_i^-$$

$$\text{Sup} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} = b_i^+$$

hence

$$\int_{[0,1]} \text{Inf} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} dt \leq \beta - \alpha - \varepsilon$$

$$\int_{[0,1]} \text{Sup} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} dt \leq \beta - \alpha - \varepsilon + \sum_{i=1}^k \lambda_i (b_i^+ - b_i^-).$$

Then, condition (37) is equivalent to :

$$\int_{[0, 1]} \text{Inf} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} dt \leq \beta - \alpha \leq \int_{[0, 1]} \text{Sup} \{ \partial \Psi^*(\sigma_\lambda(1, t)) \} dt .$$

□

4. THE CONVEX SET OF REGULARITY AND THE REGULAR LIMIT LOADS

4.1.  $\lambda = 0$  .

We want to prove here that there always exists a solution for  $(P_0)$ .

PROPOSITION 4.1 : *Inf  $(P_0)$  has at least one regular solution.*

*Proof :*

$$\text{Inf} (P_0) = \text{Inf}_{\substack{u \in W^{1,1}([0, 1]) \\ u(0) = \alpha \\ u(1) = \beta}} \left\{ \int_0^1 \Psi(u'(t)) dt \right\} \tag{40}$$

$$\text{Sup} (P_0^*) = \text{Sup}_{X \in [-1, 1]} \{ (\beta - \alpha) X - \Psi^*(X) \} . \tag{41}$$

The extremality relation is here

$$\Psi^*(X) - \Psi(u'(t)) - Xu'(t) = 0 \tag{42}$$

or equivalently

$$u'(t) \in \partial \Psi^*(X) \tag{43}$$

We must distinguish two cases :

1. If  $\lim_{Y \rightarrow (-1)^+} \Psi_d^{**}(Y) \leq \beta - \alpha$  and  $\lim_{Y \rightarrow 1} \Psi_g^{**}(Y) \geq \beta - \alpha$ , then we can construct a regular solution of  $\text{Inf} (P_0)$  as we did in Proposition (2.2).

2. If  $\lim_{Y \rightarrow 1^-} \Psi_g^{**}(Y) < \beta - \alpha$  or  $\lim_{Y \rightarrow (-1)^+} \Psi_d^{**}(Y) > \beta - \alpha$ .

Assume that we are in the first case, then  $G'_{0,g}$  and  $G'_{0,d}$  are non negative, and  $G_0$  takes its supremum at 1.

Since  $\sigma$  is constant, the extremality relation becomes

$$u'(t) \in \partial \Psi^*(1) . \tag{44}$$



We must consider separately the situations (a) and (b) below

(a)  $\Psi_n$  is strictly convex. Then

$$\Psi^*(Y) = Y\Psi_n'^{-1}(Y) - \Psi_n(\Psi_n'^{-1}(Y)) \quad \text{on } ]\Psi_n'(a_n), 1[$$

and  $\lim_{Y \rightarrow 1^-} \Psi_g^{**}(Y) = +\infty$  which is in contradiction with the assumption above.

(b)  $\Psi_n'$  is constant, we must distinguish two cases .

$$i. \quad \lim_{X \rightarrow a_n^-} \Psi'(X) \neq 1 .$$

$$\text{Then } \Psi^*(Y) = a_n Y - \Psi_n(a_n) \quad \text{on } ]\Psi_{n-1}'(a_n), 1[$$

$$\text{and } \lim_{Y \rightarrow 1^-} \Psi_g^{**}(Y) = a_n \quad \text{with } \Psi^*(1) = a_n - \Psi_n(a_n) .$$

$$ii. \quad \lim_{X \rightarrow a_n} \Psi(X) = 1 .$$

$\Psi_{n-1}'$  is strictly convex.

Then

$$\Psi^*(Y) = Y\Psi_{n-1}'^{-1}(Y) - \Psi_{n-1}(\Psi_{n-1}'^{-1}(Y)) \quad \text{on } ]\Psi_{n-1}'(a_{n-1}), 1[$$

$$\lim_{Y \rightarrow 1^-} \Psi_g^{**}(Y) = a_n \quad \text{with } \Psi^*(1) = a_n - \Psi_n(a_n)$$

We finally get

$$\lim_{Y \rightarrow 1^-} \Psi_g^{**}(Y) < \beta - \alpha \Leftrightarrow \begin{cases} a_n < \beta - \alpha \\ \Psi_n' \text{ is constant} . \end{cases} \quad (45)$$

In this case,  $[a_n, +\infty) \subset \partial\Psi^*(1)$ , thus  $\beta - \alpha \in \partial\Psi^*(1)$ .

We can take  $u$  linear with value  $\alpha$  and  $\beta$  at 0 and 1 respectively. Then  $u \in W^{1,1}(]0, 1[)$ , and  $u$  is a solution for  $(P_0)$

In the other case, i.e.  $\lim_{Y \rightarrow (-1)^+} \Psi_d^{**}(Y) > \beta - \alpha$ , one can verify that  $a_1 > \beta - \alpha$ , and  $\Psi_0'$  is constant.  $\square$

## 4.2. The convex set of Regularity

Let

$$A_c^+ = \{\lambda \in ]0, \bar{\lambda}[ / G_\lambda \text{ verifies conditions (27)}\} \cup \{0\}$$

$$A_c^+ = \{\lambda \in ]-\bar{\lambda}, 0[ / G_\lambda \text{ verifies conditions (27)}\} \cup \{0\} .$$

PROPOSITION 4.2 : *There exists  $(\bar{\lambda}_r, \underline{\lambda}_r) \in (\mathbb{R}^+)^2$  such that*

- 1  $\forall \lambda \in A_r = ]-\underline{\lambda}_r, \bar{\lambda}_r[$ ,  $\text{Inf}(P_\lambda)$  possesses at least a solution.
2.  $\{\lambda \in ]-\bar{\lambda}, \bar{\lambda}[ \setminus A_r, \text{Inf}(P_\lambda) \text{ possesses a solution}\}$  is at most countable.

For the proof of this Proposition, we will need the following Lemma :

LEMMA 4.3 :

$A_c^+$  is an interval of  $\mathbb{R}$ .

Remark 4.1 :  $\text{Inf}(P_0)$  possesses at least a solution ; as a consequence, we may assume that 0 is in  $A_c^+$ .

Proof (of Lemma 4.3) :

Let  $\lambda > 0, \lambda \in A_c^-$ , and  $\lambda_1 \in ]0, \lambda[$ .

1. Then  $\lambda_1 \int_a^t f(s) ds < \lambda \int_a^t f(s) ds$  and then

$$\sigma_{\lambda_1}(X, t) > \sigma_{\lambda}(X, t) \quad \forall X \in D_G, \quad \forall t \in [0, 1].$$

Since  $\Psi^*$  is convex, we get

$$\int_0^1 \Psi_g^*(\sigma_{\lambda_1}(X, t)) dt > \int_0^1 \Psi_g^*(\sigma_{\lambda}(X, t)) dt \quad \text{and then} \quad G'_{\lambda_1, g}(X) < G'_{\lambda, g}(X)$$

$\forall X \in D_G$ ; hence

$$\lim_{X \rightarrow 1^-} G'_{\lambda_1, g}(X) \leq \lim_{X \rightarrow 1^-} G'_{\lambda, g}(X).$$

Finally

$$\lim_{X \rightarrow 1^-} G'_{\lambda_1, g}(X) \leq 0.$$

2. Here we have

$$\lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) = \lim_{x \rightarrow (-1)^+} G'_{\lambda, d}(X + \lambda \int_a^b f(s) ds).$$

$$\lambda_1 \int_t^b f(s) ds < \lambda \int_t^b f(s) ds \quad \text{and then}$$

$$\sigma_{\lambda_1}(X + \lambda_1 \int_a^b f(s) ds, t) < \sigma_{\lambda}(X + \lambda \int_a^b f(s) ds, t)$$

for all  $X$  in  $D_G$ , for every  $t$  in  $[0, 1]$ .

$$G'_{\lambda_1, g}(X + \lambda_1 \int_a^b f(s) ds) > G'_{\lambda, g}(X + \lambda \int_a^b f(s) ds)$$

$$\lim_{x \rightarrow (-1 + \lambda_1 \int_a^b f(s) ds)^+} G'_{\lambda, d}(X) \geq \lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d}(X).$$

Finally

$$\lim_{x \rightarrow (-1 + \lambda_1 \int_a^b f(s) ds)^+} G'_{\lambda_1 d}(X) \geq 0$$

One can see, using Proposition 2.2, that  $\text{Inf}(P_{\lambda_1})$  possesses at least a solution  $\lambda_1$  is in  $A_c^+$ . Moreover, since  $A_c^+$  is bounded in  $\mathbb{R}$ , it possesses an upper bound, that we will denote by  $\bar{\lambda}_c$ .

Finally,

$$A_c^- = [0, \bar{\lambda}_c[$$

$$A_c^+ = ] - \underline{\lambda}, 0]$$

□

*Remark 4.2*  $\bar{\lambda}_c$  may or may not belong to  $A_c^+$

*Proof* (of Proposition 4.2) Let us take  $\lambda$  in  $[0, \bar{\lambda}[$ , and let us assume that  $\lambda \notin A_c^+$ , i.e.

$$\lim_{x \rightarrow 1} G'_{\lambda g}(X) > 0 \quad \text{or} \quad \lim_{x \rightarrow (-1 + \lambda \int_a^b f(t) dt)^+} G'_{\lambda d}(X) < 0$$

To fix the ideas, we assume that  $\lim_{x \rightarrow 1} G'_{\lambda g}(X) > 0$ . We must distinguish two cases

1 First, let us assume that  $\text{meas}(B_m) \neq 0$ , and

(a)  $\Psi_n$  is strictly convex

$\forall \lambda > \bar{\lambda}_c$ ,  $\text{Inf}(P_\lambda)$  has no solution (cf Proposition 2.5, case 1 b)

$$A_r^+ = A_c^+$$

(b)  $\Psi_n$  is linear

$\forall \lambda > \bar{\lambda}_c$ ,  $\text{Inf}(P_\lambda)$  possesses at least a solution,

$$A_r^+ = [0, \bar{\lambda}[$$

2 Let us suppose now that  $\text{meas}(B_m) = 0$

Proposition 2.5 tells us that  $\text{meas}(T) \neq 0$

We define  $T_{\lambda,t}$  as  $T_{\lambda,t} = \{t \in [0, 1] / \sigma_\lambda(1, t) = X_t\}$ .  
 Let us point out that

$$i \neq j \Leftrightarrow T_{\lambda,i} \cap T_{\lambda,j} = \emptyset$$

$$T_{\lambda,i} \cap T_{\lambda',j} \neq \emptyset \Leftrightarrow T_{\lambda,i} = T_{\lambda',j}.$$

Let  $\Delta = \inf_{i \in \mathbb{N}_{k-1}^*} \{X_{i+1} - X_i\}$  and let  $\delta > 0$  be such that

$$\delta < \frac{\Delta}{4 \int_a^b f(s) ds}. \tag{46}$$

Let  $\lambda_0 \in [\bar{\lambda}_c, \bar{\lambda}]$ ,  $(\lambda, \lambda') \in [\lambda_0 - \delta, \lambda_0 + \delta]^2$  and let  $(i, j) \in (\mathbb{N})^2$ ,  $0 < i \leq j < k$ . We argue by contradiction, and assume that  $T_{\lambda,i} \cap T_{\lambda',j} \neq \emptyset$ . We have for

$$t \in T_{\lambda,i} \cap T_{\lambda',j}$$

$$\begin{aligned} X_j - X_i &= 1 - \lambda' \int_a^t f(s) ds - 1 + \lambda \int_a^t f(s) ds \\ &= (\lambda - \lambda') \int_a^t f(s) ds. \end{aligned}$$

Since  $j \geq i$ ,  $\lambda \geq \lambda'$ , and  $\lambda - \lambda' \leq 2\delta$ ,

$$\begin{aligned} X_j - X_i &\leq (\lambda - \lambda') \int_a^b f(s) ds \\ &\leq 2\delta \int_a^b f(s) ds \\ &< \Delta \end{aligned}$$

from which we derive a contradiction.

Hence

$$\forall \lambda_0 \in [\bar{\lambda}_c, \bar{\lambda}], \quad \forall (\lambda, \lambda') \in [\lambda_0 - \delta, \lambda_0 + \delta], \quad \forall (i, j) \in (\mathbb{N})^2,$$

$$0 < i \leq j < k,$$

$$T_{\lambda,i} \cap T_{\lambda',j} = \emptyset. \tag{47}$$

Let then  $(\lambda_j)_{1 \leq j \leq N}$  be a non decreasing sequence in  $[\bar{\lambda}_c, \bar{\lambda}]$ , such that

$$\begin{cases} \lambda_1 - \delta = \bar{\lambda}_c \\ \lambda_i + \delta = \lambda_{i+1} - \delta \\ \bar{\lambda} \in [\lambda_n + \delta, \lambda_N + 3\delta[. \end{cases}$$

We have then  $[\bar{\lambda}_c, \bar{\lambda}] = I \bigcup_{j=1}^N [\lambda_j - \delta, \lambda_j + \delta[$

Let  $A_j = \{\lambda \in [\lambda_j - \delta, \lambda_j + \delta[ / \text{Inf}(P_\lambda) \text{ possesses at least a solution}\}$ , and let

$$T_s = \bigcup_{\lambda \in A_j} T_\lambda.$$

$T_s$  is a subset of  $[0, 1]$ , hence  $\text{meas}(T_s) \leq 1, \forall \lambda \in A_j, \text{meas}(T_\lambda) \neq 0$ .

Let  $L_p$  be  $\left\{ \lambda \in A_j / \frac{1}{p+1} \leq \text{meas}(T_\lambda) < \frac{1}{p} \right\}$ .

$L_p$  is finite, since otherwise  $\text{meas}(T_s) \geq \text{meas}\left(\bigcup_{\lambda \in L_p} T_\lambda\right) \geq +\infty$ .

Let  $L_0 = \{\lambda \in A_j, \text{meas}(T_\lambda) \geq 1\}$ ;

$$T_s = \bigcup_{\lambda \in \mathbb{N}} \bigcup_{\lambda \in L_k} T_\lambda$$

$$A_j = \bigcup_{\lambda \in \mathbb{N}} L_k$$

Then  $A_j$  is a countable union of at most countable sets, i.e  $A_j$  is at most countable.

$\{\lambda \in [0, \bar{\lambda}] \setminus M\}$  is at most countable. Indeed, this set is the union of the  $A_j$ .

The other case, i.e.  $\lim_{x \rightarrow (-1 + \lambda \int_a^b f(t) dt)^+} G'_{\lambda, d}(X) < 0$  is dealt with in the

same fashion, using  $\text{meas}(B_M)$  instead of  $\text{meas}(B_m)$

One can deal with the negative  $\bar{\lambda}_c$  in the same fashion □

### 4.3. Some famous examples

In this subsection, we present three examples :

- 1 the rigid-plastic model,
2. a strictly convex function,
- 3 the elasto-plastic model.

#### 4 3 1. The rigid-plastic model

For the rigid-plastic model, the functional  $\Psi$  is

$$\Psi(X) = |X|. \tag{48}$$

The computation of  $\Psi^*$  gives

$$\Psi^*(Y) = \begin{cases} +\infty & \text{if } |Y| > 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{49}$$

Then

$$\begin{aligned} \partial\Psi^*(X) &= \{0\} & \text{if } |X| < 1 \\ \partial\Psi^*(1) &= [0, +\infty] \\ \partial\Psi^*(-1) &= [-\infty, 0], \end{aligned}$$

and

$$\begin{aligned} \Psi_{d'}^*(Y) &= \begin{cases} +\infty & \text{if } Y = 1 \\ 0 & \text{elsewhere} \end{cases} \\ \Psi_{g'}^*(Y) &= \begin{cases} -\infty & \text{if } Y = 1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

One can then compute the left and right derivative of  $G_\lambda$ . If  $\lambda$  is non negative,

a)  $\text{meas}(B_m) \neq 0$

$$G'_{\lambda,d}(X) = \begin{cases} \beta - \alpha & \text{if } -1 + \lambda \int_a^b f(s) ds \leq X < 1 \\ -\infty & \text{elsewhere.} \end{cases}$$

a')  $\text{meas}(B_m) = 0$

$$G'_{\lambda,d}(X) = \begin{cases} \beta - \alpha & \text{if } -1 + \lambda \int_a^b f(s) ds \leq X \leq 1 \\ -\infty & \text{elsewhere.} \end{cases}$$

b)  $\text{meas}(B_M) \neq 0$

$$G'_{\lambda,d}(X) = \begin{cases} \beta - \alpha & \text{if } -1 + \lambda \int_a^b f(s) ds < X \leq 1 \\ +\infty & \text{elsewhere.} \end{cases}$$

b')  $\text{meas}(B_M) = 0$

$$G'_{\lambda,d}(X) = \begin{cases} \beta - \alpha & \text{if } -1 + \lambda \int_a^b f(s) ds \leq X \leq 1 \\ +\infty & \text{elsewhere.} \end{cases}$$

The existence of regular solutions depends then on  $\beta - \alpha$ . In fact, Propositions 2.2 and 2.4 do not provide any solution if  $\beta \neq \alpha$ .

— If  $\beta = \alpha$ ,  $u \equiv \beta$  is solution of  $\text{Inf}(P_\lambda)$ , for all  $\lambda$  in  $[0, \bar{\lambda}[$ .

— If  $\beta \neq \alpha$ , the assumption of Proposition 2.2 are never satisfied. The only possible solutions are provided by Proposition 2.5.

(a)  $\beta > \alpha$

Then  $\lim_{x \rightarrow 1^-} G'_{\lambda, g} > 0$ ,

i.  $\text{meas}(B_m) \neq 0$

$$\bar{\lambda}_c = \bar{\lambda}$$

ii.  $\text{meas}(B_m) = 0$

$$\bar{\lambda}_c = 0.$$

(b)  $\beta < \alpha$

Then  $\lim_{x \rightarrow (-1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, d} > 0$ ,

i.  $\text{meas}(B_M) \neq 0$

$$\bar{\lambda}_c = \bar{\lambda}$$

ii.  $\text{meas}(B_M) = 0$

$$\bar{\lambda}_c = 0.$$

The negative part of  $\mathcal{A}$  can be treated in the same fashion.

#### 4.3.2. A strictly convex function

Let us consider once more the following example, given in Section 2.3

$$\Psi(X) = \sqrt{1 + X^2}. \tag{50}$$

The computation of  $\Psi^*$  gives

$$\Psi^*(Y) = \begin{cases} +\infty & \text{if } |Y| > 1 \\ -\sqrt{1 - Y^2} & \text{else.} \end{cases}$$

Then

$$\Psi^*(Y) = \frac{Y}{\sqrt{1 - Y^2}} \quad \text{if } |Y| < 1$$

$$\partial \Psi^*(1) = \{ +\infty \}$$

$$\partial \Psi^*(-1) = \{ -\infty \}.$$

Four cases could happen :

1.  $\text{meas} (B_m) \neq 0$  and  $\text{meas} (B_M) \neq 0$ .

In this case, we have

$$\lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) = -\infty < 0$$

$$\lim_{X \rightarrow (-1 + \lambda \int_a^b f(t) dt)^+} G'_{\lambda, d}(X) = +\infty > 0.$$

Proposition 2.2 tells then that in  $\text{inf} (P_\lambda)$  possesses at least a regular solution for all  $\lambda$  in  $[0, \bar{\lambda}[$ . Finally

$$A_r = ] - \bar{\lambda}, \bar{\lambda}[ .$$

2.  $\text{meas} (B_m) = 0$  and  $\text{meas} (B_M) \neq 0$ .

In this case, we have

$$\lim_{X \rightarrow (-1 + \lambda \int_a^b f(t) dt)^+} G'_{\lambda, d}(X) = +\infty > 0.$$

Let us call  $A = \int_a^b f(t) dt$ .

From  $X - \lambda A \leq \sigma_\lambda(X, T) \leq X$ , one can find

$$\beta - \alpha - \frac{X}{\sqrt{1 - X^2}} \leq G'_{\lambda, g}(X) \leq \beta - \alpha - \frac{X - \lambda A}{\sqrt{1 - (X - \lambda A)^2}}. \tag{51}$$

At last,

$$-\infty \leq \lim_{X \rightarrow 1^-} G'_{\lambda, g}(X) \leq \beta - \alpha - \frac{1 - \lambda A}{\sqrt{1 - (1 - \lambda A)^2}}. \tag{52}$$

The right hand side of these inequalities is a continuous function w.r.t.  $\lambda$ , and its limits when  $\lambda$  goes to  $0^+$  is  $-\infty$ . Then, for  $\lambda$  small enough, this expression is negative. In fact, the bound is

$$\bar{\lambda}_l = \frac{1 - \frac{\beta - \alpha}{\sqrt{1 + (\beta - \alpha)^2}}}{A}.$$



Let now  $\lambda$  be greater than  $\bar{\lambda}_l$ . Then,

$$\lim_{x \rightarrow 1^-} G'_{\lambda g} > 0.$$

But  $\Psi^d$  is differentiable on  $] - 1, 1[$ , and  $\text{meas}(B_m) = 0$ . Hence,

$$\begin{aligned} \int_0^1 \text{Inf}(\partial \Psi^*(\sigma_\lambda(1, t))) dt &\leq \beta - \alpha = \int_0^1 \text{Sup}(\partial \Psi^*(\sigma_\lambda(1, t))) dt \\ &= \int_0^1 \Psi^{*'}(\sigma_\lambda(1, t)) dt. \end{aligned}$$

In any case, Proposition 2.5 does not provide any solution to  $\text{Inf}(P_r)$   $A_r^+ = [0, \bar{\lambda}_l[$ .

3. The two other cases, and the case  $\lambda < 0$  can be treated in the same way

*Remark 4.3* · The solutions are all provided by Proposition 2.2

### 4.3.3 The elasto-plastic model

The functional  $\Psi$  for the elasto-plastic case is given by

$$\Psi(X) \begin{cases} \frac{1}{2} |X|^2 & \text{if } |X| \leq 1 \\ |X| - \frac{1}{2} & \text{either} \end{cases}$$

The computation of  $\Psi^{*c}$  gives

$$\Psi^{*c}(Y) = \begin{cases} \frac{1}{2} |Y|^2 & \text{if } |Y| \leq 1 \\ +\infty & \text{either} \end{cases} \tag{54}$$

Then

$$\partial \Psi^*(X) = \{X\} \quad \text{if } |X| < 1$$

$$\partial \Psi^*(1) = [1, +\infty]$$

$$\partial \Psi^*(-1) = [-\infty, -1],$$

and

$$\Psi^{*'}_d(Y) = \begin{cases} +\infty & \text{if } Y = 1 \\ Y & \text{else,} \end{cases}$$

$$\Psi^{*'}_g(Y) = \begin{cases} -\infty & \text{if } Y = -1 \\ Y & \text{else,} \end{cases}$$

$$\lim_{x \rightarrow 1} G'_{j, g}(X) = \beta - \alpha - 1 + \lambda \int_0^1 \int_a^1 f(s) ds dt,$$

$$\lim_{x \rightarrow (1 + \lambda \int_a^b f(s) ds)^+} G'_{\lambda, a}(X) = \beta - \alpha + 1 - \lambda \int_a^b f(s) ds + \lambda \int_0^1 \int_a^t f(s) ds dt$$

1 If  $|\beta - \alpha| > 1$ , the derivatives of  $G_\lambda$  do not take the value 0, and the existence of solutions is a consequence of Proposition (2.5), cases 1 b and 2 b, since  $\Psi_0$  and  $\Psi_2$  are linear, it gives the same result as the rigid plastic case

2 If  $|\beta - \alpha| \leq 1$ , being small enough, the two derivatives can have opposite sign. Let  $\lambda > 0$ ,

$$(a) \text{ if } \beta - \alpha \leq 1 - 2 \frac{\int_0^1 \int_a^t f(s) ds dt}{\int_a^b f(s) ds}, \text{ let } \bar{\lambda}_1 = \bar{\lambda}$$

$$(b) \text{ else let } \bar{\lambda}_1 = \frac{1 - (\beta - \alpha)}{\int_0^1 \int_a^t f(s) ds dt}$$

In the second case, the limits of the derivatives have opposite signs if and only if  $\lambda$  is smaller than  $\bar{\lambda}_1$ , and then  $(P_\lambda)$  could have a solution. One can deal with the second limit in a same way

$$\text{If } \beta - \alpha \geq -1 + 2 \frac{\int_0^1 \int_t^b f(s) ds dt}{\int_a^b f(s) ds}, \text{ then } \bar{\lambda}_2 = \bar{\lambda},$$

$$\text{else } \bar{\lambda}_2 = \frac{\beta - \alpha}{\int_0^1 \int_t^b f(s) ds}$$

Let  $\bar{\lambda}_l = \min \{ \bar{\lambda}_1, \bar{\lambda}_2 \}$

One can define  $\lambda_l$  in a same way,  $\lambda$  being negative. Then

- 1 If  $\text{meas}(B_m) = 0$  and  $\text{meas}(B_M) = 0$  then  $A_r = ] - \lambda_l, \bar{\lambda}_l [$
- 2 If  $\text{meas}(B_m) = 0$  and  $\text{meas}(B_M) = 0$  then  $A_r = ] - \lambda, \bar{\lambda}_l [$

3. If  $\text{meas}(B_m) \neq 0$  and  $\text{meas}(B_M) = 0$  then  $\mathcal{A}_r = ] - \underline{\lambda}, \bar{\lambda}[$ .
4. If  $\text{meas}(B_m) \neq 0$  and  $\text{meas}(B_M) \neq 0$  then  $\mathcal{A}_r = ] - \underline{\lambda}, \bar{\lambda}[$ .

*Remark 4.4* : The elasto-plastic bar is a mix of the two previous cases.

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