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TRANSVERSALS OF CIRCUITS IN THE LEXICOGRAPHIC PRODUCT OF DIRECTED GRAPHS

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A directed graph G consists of a set V(G) of vertices and a set E(G) consisting of ordered pairs (x,y) (called edges) of distinct vertices x,y. The directed circuit of length n ($n \geq 2$) is the directed graph consisting of the vertices $x_1, x_2, \ldots, x_n$ and the edges (x_i, x_{i+1}), $1 \leq i \leq n-1$, and (x_n,x_1). Let S_q denote the graph with q vertices and no edges.
The lexicographic product $G \otimes H$ of the directed graphs G, H can be defined as follows :
For each vertex x of G, let H_x be a copy of H such that $x,y \in V(G)$, $x \neq y$ implies $H_x \cap H_y = \emptyset$. Then add to the graph $\bigcup_{x \in V(G)} H_x$ all edges (x_1,y_1) such that $x_1 \in H_x$, $y_1 \in H_y$ and $(x,y) \in E(G)$.
G being a directed graph, we define $\tau(G) = \min \{m | \exists A \subseteq E(G) : |A|=m, G-A$ contains no directed circuit$\}$.
We also define $\tau_k(G) = \min \{m | \exists A \subseteq E(G) : |A|=m, G-A$ contains no directed circuit of length less than $k\}$. J.C.Bermond showed that $\tau(G \otimes H) \leq |V(G)| \tau(H) + |V(H)|^2 \tau(G)$ and conjectured [2, Conjecture 2] that equality holds. (The weaker conjecture for tournaments was made in [1]). We shall here prove this conjecture.

THEOREM 1

Let G and H be directed graphs with p and q vertices, respectively. Then $\tau(G \otimes H) = p\tau(H) + q^2\tau(G)$.

Clearly $\tau(G) = \tau_k(G)$ when $k > |V(G)|$, so Theorem 1 follows from Theorem 2 below.

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THEOREM 2

Let G and H be directed graphs with p and q vertices, respectively. Then for every $k \geq 2$, $\tau_k(G \otimes H) = p\tau_k(H) + q^2\tau_k(G)$.

Proof : Let $A \subseteq E(G)$, $B \subseteq E(H)$ such that $|A| = \tau_k(G)$, $|B| = \tau_k(H)$ and $G - A$, respectively $H - B$, contains no circuit of length less than k. From each of the graphs $H_x (x \in V(G))$ we delete the edges of B and for each edge of A we delete the corresponding q^2 edges of $G \otimes H$. We have then deleted $p\tau_k(H) + q^2\tau_k(G)$ edges from $G \otimes H$ and the resulting directed graph contains no directed circuit of length less than k.
This proves the inequality $\tau_k(G \otimes H) \leq p\tau_k(H) + q^2\tau_k(G)$. The reverse inequality follows from Theorem 3 below.
If $\mathfrak{F}$ is a family of directed graphs and G is a directed graph then an $\mathfrak{F}$-tranversal of G is a subset A of E(G) such that every subgraph of G which is isomorphic to one of the graphs of $\mathfrak{F}$ contains an edge of A. We define $f(G, \mathfrak{F})$ as the minimum number of elements in an $\mathfrak{F}$-transversal of G. If, in particular, $\mathfrak{F}$ consists of the directed circuits of length less than k then clearly $f(G, \mathfrak{F}) = \tau_k(G)$.

THEOREM 3

Let G and H be directed graphs with p and q vertices, respectively, and let $\mathfrak{F}$ be a family of directed graphs. Then $f(G \otimes H, \mathfrak{F}) \geq pf(H, \mathfrak{F}) + q^2 f(G, \mathfrak{F})$.

Proof : Since $G \otimes H$ is the edge-disjoint union of $G \otimes S_q$ and the graphs H_x, $x \in V(G)$, we have $f(G \otimes H, \mathfrak{F}) \geq pf(H, \mathfrak{F}) + f(G \otimes S_q, \mathfrak{F})$, so it is sufficient to show that $f(G \otimes S_q, \mathfrak{F}) \geq q^2 f(G, \mathfrak{F})$. Let A be any $\mathfrak{F}$-tranversal of $G \otimes S_q$. Select one vertex from each of the graphs $(S_q)_x$, $x \in V(G)$, and consider the subgraph of $G \otimes S_q$ spanned by these vertices. This subgraph is isomorphic to G, and there are q^p such subgraphs, say $G_1, G_2, \ldots, G_{q^p}$. For each i, $1 \leq i \leq q^p$, $A \cap E(G_i)$ is an $\mathfrak{F}$-tranversal of G_i so

$$(1) \quad |A \cap E(G_i)| \geq f(G_i, \mathfrak{F}) = f(G, \mathfrak{F}) \quad \text{for each } i, \ 1 \leq i \leq q^p.$$

Moreover, each edge of A is contained in precisely q^{p-2} of the graphs G_i, $1 \leq i \leq q^p$. Hence

$$(2) \quad q^{p-2}|A| = q^{p-2} \left| \bigcup_{i=1}^{q^p} (A \cap E(G_i)) \right| = \sum_{i=1}^{q^p} |A \cap E(G_i)|.$$

Combining (1) and (2) we obtain the inequality $q^{p-2}|A| \geq q^p f(G,\mathfrak{F})$ i.e. $|A| \geq q^2 f(G,\mathfrak{F})$. Since A is any $\mathfrak{F}$-transversal, $f(G \otimes S_q) \geq q^2 f(G,\mathfrak{F})$.

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