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AN OBJECTIVE AND PRACTICAL METHOD FOR DESCRIBING AND UNDERSTANDING RATIOS

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RÉSUMÉ — Méthode objective et pratique pour décrire et comprendre les rapports
Cet article explore l'utilisation de l'algorithme euclidien comme moyen très utile pour manipuler les rapports, surtout dans les cas où de bonnes approximations rationnelles sont nécessaires. Cette discussion est illustrée à partir d'une analyse de l'architecture grecque par J.J. Coulton. Cet article veut être un compte-rendu pratique, mais il comprend aussi une discussion de certains aspects théoriques, ainsi que de la relation de cette procédure à une nouvelle interprétation des mathématiques grecques de l'époque de Platon.

SUMMARY — *This article explores the use of the Euclidian algorithm as a most useful way of handling ratios, especially when good rational approximations are required. Illustrations are taken from a discussion of the analysis of Greek architecture by J.J Coulton. Although this is intended as a practical account, some discussions of theoretical aspects are included, and also of the relationship of this procedure to a new interpretation of early Greek mathematics.*

This note[2] was prompted by a reading of J.J. Coulton, 'Towards Understanding Greek Temple Design: General Considerations'[3], and I shall preface each section with a quotation from his article[4].

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[2] This article was written in 1982 and circulated in limited way, but never published. Since then, Louis Frey has made great use of some of the techniques described here in a series of increasingly detailed and penetrating analyses; see, for example, 'Médiétés et approximations chez Vitruve', 'Pour un modèle du chapiteau ionique vitruvien', and 'La transmission d'un canon: les temples ioniques'. (Detailed references are given in the bibliography.) It is with his encouragement, and because I think the techniques described here have more general interest and scope, that I am now publishing it here.

[3] I would like to thank Dr. Coulton for his comments on an early draft of this article, which led to significant extensions and clarifications of my proposals.

[4] In this first quotation, I have alternated the ratios so that they appear as 2:1 etc., rather than 1:2 etc. This avoids a distracting preliminary step in the procedure, to be explained in Section 3, below. Thereafter the ratios are given as they occur in Coulton's article.

1. THE METHOD

"If two parts of a building are found to be related to each other as $2\frac{1}{2}:1$, the architect might have visualised this ratio simply as $2\frac{1}{2}:1$; but he might also have visualised it as 15:6 or as $13\frac{1}{3}:5\frac{1}{3}$ — or as an approximation to $\sqrt{6}:1$ or $3\pi:4$; but it is at least highly unlikely that he visualised it in all of these ways" (p.61).

The basic problem is to compare two lengths in a way that is independent of any measurement system, or choice of unit, or even to carry out the comparison in a purely geometrical way that does not introduce any intermediate system of units; to find those ratios of small numbers which approximate the number particularly closely; and to identify at what point the approximation procedure becomes more accurate than the tolerances in the original measurements.

The method is based on the Euclidean algorithm. Suppose we wish to compare two lines or numbers a_0 and a_1, that is to describe the ratio $a_0:a_1$. (In the first example above $a_0 = 2\frac{1}{2}$, $a_1 = 1$, etc.) Subtract a_1 from a_0 as many times as possible to leave a remainder a_2,

$$a_0 - n_0 a_1 = a_2 \quad \text{with} \quad 0 \leq a_2 \leq a_1 .$$

(I recommend that the reader to whom these operations are unfamiliar carries out the procedure on two strips of paper, tearing them into successively smaller and smaller pieces.) Record the number n_0, which gives approximate information about $a_0:a_1$, and retain only a_1 and a_2; this involves no loss of information. If a_2 is zero, then $a_0:a_1 = n_0:1$; otherwise $a_0:a_1$ will be *greater* than $n_0:1$, and more precise information will depend on knowing something about that first remainder a_2. So repeat the operation with a_1 and a_2 :

$$a_1 - n_1 a_2 = a_3 \quad \text{with} \quad 0 \leq a_3 \leq a_2;$$

record n_1, and retain only a_2 and a_3. If a_3 is zero, then we can calculate that $a_1 = n_1 a_2$ and $a_0 = n_0 n_1 a_2 + a_2$, so $a_0:a_1 = (n_0 n_1 + 1):n_1$; otherwise $a_0:a_1$ will be *less* than $(n_0 n_1 + 1):n_1$. Yet closer information can be obtained by continuing the procedure, and decoding the results. I shall describe a simple and systematic way of evaluating these approximations $n_0:1$, $(n_0 n_1 + 1):n_1$, ... in Section 3, below.

The sequence of numbers $n_0, n_1, n_2,$... will then completely and objectively describe the ratio $a_0:a_1$ and, for convenience, I shall write $a_0:a_1 = [n_0, n_1, n_2, ...]$. Here are Coulton's examples.

(a) $2\frac{1}{2}:1$

$$2\tfrac{1}{2} - 2 \times 1 = \tfrac{1}{2},$$

$1 - 2 \times \frac{1}{2} = 0$, and the procedure terminates.

So $2\frac{1}{2}:1 = [2, 2]$. This same pattern will arise from 15:6, or $13\frac{1}{3}:5\frac{1}{3}$.

(b) $\sqrt{6}:1$. These calculations can be performed in a variety of ways. To begin with, let us use decimal expansions, writing $\sqrt{6}:1$ as 2·4495... :1.

$$2{\cdot}4495... - 2 \times 1 = 0{\cdot}4495... ,$$

$$1 - 2 \times 0{\cdot}4495... = 0{\cdot}1010... ,$$

$$0{\cdot}4495... - 4 \times 0{\cdot}1010... = 0{\cdot}0454... , \text{ etc.}$$

So $\sqrt{6}:1 = [2, 2, 4, ...]$.

This particular evaluation becomes more suggestive if we rescale line by line: in line 2, instead of comparing 1 and 0·4495..., compare $(0{\cdot}4495...)^{-1} = 2{\cdot}2247...$ and 1; and then do a similar rescaling on each subsequent line. We then get

$$\begin{aligned} && 2{\cdot}4495.. &= 2 \times 1 + 0{\cdot}4495... ;\\ (0{\cdot}4495...)^{-1} &=& 2{\cdot}2247.. &= 2 \times 1 + 0{\cdot}2247... ,\\ (0{\cdot}2247...)^{-1} &=& 4{\cdot}4495.. &= 4 \times 1 = 0{\cdot}4495... ,\\ (0{\cdot}4495...)^{-1} &=& 2{\cdot}2247.. &= \text{etc} \end{aligned}$$

and now the numbers in the second and third lines seem to repeat indefinitely, so that we seem to get $\sqrt{6}:1 = [2, 2, 4, 2, 4, 2, 4, ...]$. We can prove that this sequence is indeed periodic by performing this same calculation in the following form :

$$\begin{aligned} \sqrt{6} &= 2 + (\sqrt{6} - 2)\\ (\sqrt{6} - 2)^{-1} = \tfrac{1}{2}(\sqrt{6} + 2) &= 2 + \tfrac{1}{2}(\sqrt{6} - 2)\\ (\tfrac{1}{2}(\sqrt{6} - 2))^{-1} = (\sqrt{6} + 2) &= 4 + (\sqrt{6} - 2), \end{aligned}$$

and now the last two steps will repeat indefinitely.

In the rescaled form, the procedure is well adapted for the simplest of pocket calculating machines, especially programmable machines (but further refinements, to be described below, are more easily set up as spreadsheets on personal computers). Perform the following steps: enter a number; read out its integer part and reciprocate its fractional part; read out the integer part of this new number and reciprocate its fractional part; etc. If this operation is performed on $x = \frac{3\pi}{4}$ we get

(c) $$\frac{3\pi}{4}:1 = [2, 2, 1, 4, 5, ...] .$$

2. HANDLING APPROXIMATION DATA

> "We can, however, suppose that the dimensions of each part of a building were governed by a system of proportion, but defined by being rounded out to the nearest convenient dimension" (p.64).
> "In so far as Greek temple design is indeed based on proportion and dimension, then within the limits of accuracy current for the building concerned, every member of a Greek building must have been defined *either* as a convenient number of feet or parts of a foot, *or* as a convenient and/or desirable proportion of a part of the building which had itself already been defined" (p.65).
> "A question which continually recurs is what degree of accuracy are we to expect in the design" (p.89).

For a variety of reasons, we must treat our data as only approximate, and so significant only to within tolerances that are either built in as part of the process of design and construction, or arise from the state of the remains of the building or the inaccuracies in the measurement process. There is no point, therefore, in pursuing our exploration beyond this degree of accuracy. The simplest way of doing this is to stop as soon as one of the remainders a_2, a_3, a_4, ... becomes less than the tolerance in the data, if this is known, or, at the least, to take note of the size of these remainders. Note that this information is lost if we use the rescaled process of Section 1(b); the original form of the algorithm is the most satisfactory for serious use. Another more elaborate way of handling tolerances is given in the next section.

3. FINDING SIMPLE APPROXIMATIONS

> "The difficulty is that some quite simple vulgar fractions are hard to recognise when expressed decimally; thus 0·426 may not be recognised as approximately $\frac{3}{7}$, nor 0·0775 as approximately $\frac{1}{13}$" (p.75).

A small remainder at any step in this procedure will immediately manifest itself in a large suceeding integer n_k; then ignoring such a small remainder should give rise to a particularly good approximation, and this will correspond to ignoring the large n_k and all subsequent terms. Most of the n_k, in a randomly chosen ratio, will be 1 or 2; in practical contexts, 4 or above in the third coefficient or beyond (i.e. in n_2, n_3, n_4, etc.) can be considered as large. Take, for example the case of $\sqrt{6}:1 = [2, 2, 4, 2, 4, ...]$. By this argument, we would expect this to be close to $[2, 2] = 5:2 = 2\frac{1}{2}:1$, and this explains one of the examples in the opening quotation of Section 1. Moreover we see that the last example in this first quotation, $\frac{3\pi}{4}:1 = [2, 2, 1, 4, 5, ...]$, is less close to $[2, 2] = 2\frac{1}{2}:1$, and is in fact much closer to $[2, 2, 1] = 2\frac{1}{3}:1$ [5]; while the first approximation in the passage above, again taken the greater to the lesser, $1: 0{\cdot}426 = [2, 2, 1, 7, ...]$, is also very close to $[2, 2, 1] = 7:3$, and therefore also close to $\frac{3\pi}{4}:1$. The second example above gives $0{\cdot}0775:1 = [0, 12, 1, 9, 3]$, which is close to $[0, 12, 1] = 1:13$.

All of these examples have been simple to evaluate, but it is clear that we shall need a quick and systematic way of working out these expressions $[n_0, n_1, n_2, ... n_k] = p_k:q_k$ if they get any more complicated; try, for example, to evaluate the approximation $[2, 2, 4, 2, 4]$ to $\sqrt{6}:1$. Here is such a method; I shall first describe it, and then give a brief explanation of what is going on.

First write down $\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}$; then, starting above the next column write down the values of $n_0, n_1, n_2, ...$, and calculate p and q, using at each stage their values in the previous two columns, as in the following example of $\sqrt{6}:1$:

			n_0	n_1	n_2	n_3	n_4	...
$\sqrt{6}:1 =$			2	2	4	2	4	...
p	0	1	2	5	22	49	218	...
q	1	0	1	2	9	20	89	...
type			under	over	under	over	under	...

Here, in the n_0 column, $p_0 = 2 \times 1 + 0 = 2$, $q_0 = 2 \times 0 + 1 = 1$; then $p_1 = 2 \times 2 + 1 = 5$, and so on up to $p_4 = 4 \times 49 + 22 = 218$, $q_4 = 4 \times 20 + 9 = 81$. This process generates the ratios $[2] = 2:1$; $[2, 2] = 5:2$; $[2, 2, 4] = 22:9$; $[2, 2, 4, 2] = 49:20$; etc., and we find that they form a rapidly improving sequence of approximations to the ratio $\sqrt{6}:1$ by ratios of integers, and, as indicated in the bottom row, these approximations alternate round the given ratio :

$$2:1 < 22:9 < 218:89 < ... < \sqrt{6}:1 < ... < 49:20 < 5:2.$$

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[5] Louis Frey has pointed out to me that $\frac{4\pi}{5}:1 = [2, 1, 1, 18, 2, 1, 366, 3, ...]$ is much closer to $[2, 1, 1] = [2, 2] = 2\frac{1}{2}:1$, as we now can clearly see, and is very close to $[2, 1, 1, 18, 2, 1] = 284:113$, an approximation that derives from the excellent approximation 355:113 to $\pi:1$. Perhaps Coulton originally intended to write $4\pi:5$.

These approximations $p{:}q$ are, in fact, the best possible using numbers of up to that size. For example $\sqrt{6}{:}1 = 2{\cdot}449489...$ while $[2, 2, 4, 2, 4] = 49{:}20 = 2{\cdot}45$, and no ratio involving denominators up to 20 (and, in fact, considerably larger than that) will give a closer approximation.

These examples and this procedure illustrate two features worth noting. First, the effect of taking ratios the lesser to the greater, rather than the greater to the lesser, is to introduce an initial term of 0, and so to reverse the roles of a_0 and a_1 and to replace $p{:}q$ by its reciprocal; evaluate the approximations to $1{:}\sqrt{6}$ to see what I mean. Secondly, there is an ambiguous representation of terminating ratios, that we can sometimes use to simplify their evaluation. For example, $[0, 2, 2, 1] = [0, 2, 3]$, which is the reciprocal of $[2, 3] = 2\frac{1}{3}{:}1$; or, in general, $[n_0, n_1, ..., n_k] = [n_0, n_1, ..., n_k - 1, 1]$ if $n_k \geq 2$. We see this as follows: in the last stages of the subtraction process, we can have either something like

$$12 - 3 \times 4 = 0, \quad \text{that is} \quad a_k - n_k a_{k+1} = 0 \quad \text{with} \quad n_k \geq 2, \text{ and the process terminates,}$$

or, alternatively,

$$12 - 2 \times 4 = 4, \quad \text{that is} \quad a_k - (n_k - 1) \times a_{k+1} = a_{k+1}, \quad \text{and then}$$
$$4 - 1 \times 4 = 0, \quad \text{that is} \quad a_{k+1} - 1 \times a_{k+1} = 0, \text{ and the process terminates one step later.}$$

In addition to the very good approximations that arise by truncating the expansion, there are other good approximations that arise by decreasing a final large term. An example that arose in an actual analysis of some data will illustrate and explain this: $5{\cdot}943 : 2{\cdot}409 = [2, 2, 7, 13, ...]$. Our theory so far tells us that if we neglect the large term 7 and beyond, this will generate a good approximation $[2, 2] = 5{:}2$, an overestimate; and neglecting the next and even bigger term 12 and beyond will generate an excellent approximation, $[2, 2, 7] = 37{:}15$, an underestimate. But also, in addition, the terms $[2, 2, 6] = 32{:}13$, $[2, 2, 5] = 27{:}11$, and $[2, 2, 4] = 22{:}9$ will be good approximations; and so indeed will be $[2, 2, 3] = 17{:}7$, $[2, 2, 2] = 12{:}5$, and $[2, 2, 1] = [2, 3] = 7{:}3$, though these will be poorer absolute underestimates that $[2, 2]$ was an overestimate. The situation is illustrated in Figure 1; the change-over occurs at the half-way point, here between 4 (greater than half of 7) and 3 (less than half); and when the final term is even, the half-way value can be either better or worse[6]. Finally there is the issue of the accuracy of the data, which was here given to 3 decimal places. In the absence of any other information, we can only interpret the number 5.943, for example, as denoting anything in the interval from $5{\cdot}9425$ to $5{\cdot}9435$; so this ratio denotes anything in the interval from $5{\cdot}925 : 2{\cdot}4095 = 2{\cdot}4663... {:}1 = [2, 2, 6, 1, 10,...]$ to $5{\cdot}9435 : 2{\cdot}4085 = 2{\cdot}4677... {:}1 = [2, 2, 7, 4, 10, ...]$. Hence all points inside the interval from $[2, 2, 6, 1]$ ($= [2,2, 7]$) to $[2, 2, 7, 4]$ are, in effect, indistinguishable from each other and from the given ratio. Figure 1 gives some impression of just how rapidly these approximations converge to this ratio.

These two kinds of approximations — now generally called the 'convergents' and the 'intermediate convergents' of the given ratio — are the best approximations using numbers of a specified size, where the details of this statement, like the precise meanings of 'best approximation', etc., can be precisely defined in a variety of different ways[7].

6 There is an arcane test to determine if $[n_0, n_1, ... \frac{1}{2}n_k]$ will be a better absolute approximation than $[n_0, n_1, n_2, ... n_{k-1}]$: Is $[n_{k-1}, n_{k-2}, ... n_1]$ less than $[n_{k+1}, n_{k+2}, ...]$?

7 For more details, see my book *The Mathematics of Plato's Academy: A New Reconstruction*, Chapter 9.

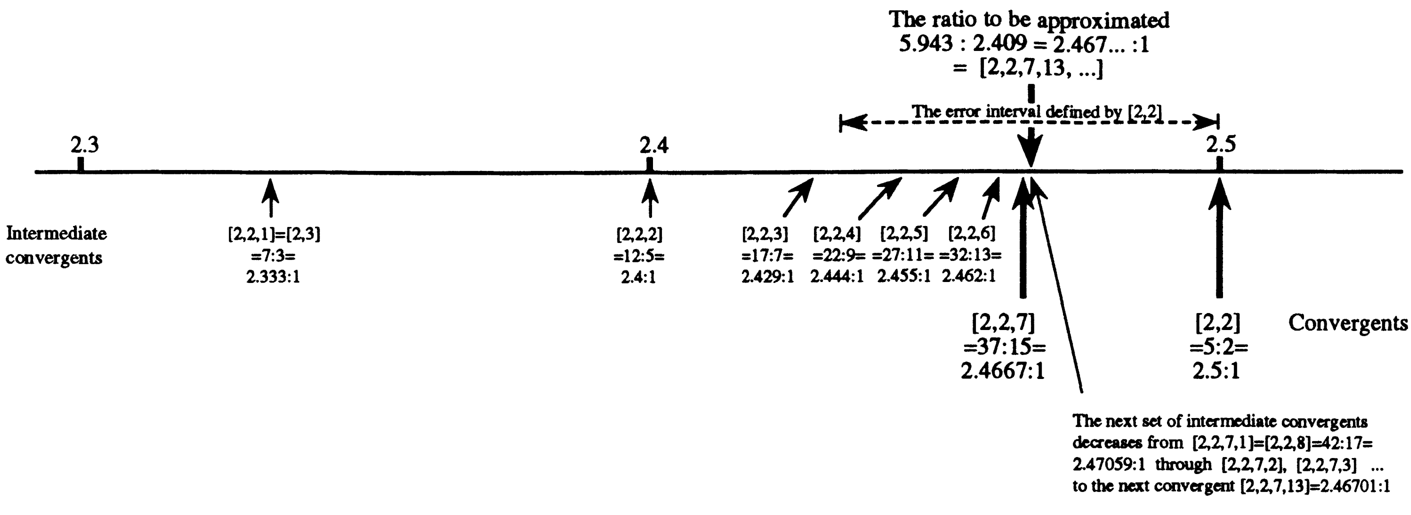

Figure 1. Approximations to 5.943 : 2.409 = [2, 2, 7, 13, ...]

Here is a brief explanation of what is happening[8]. We have

$$a_0 - n_0 a_1 = a_2 \qquad \text{i.e.} \qquad \frac{a_0}{a_1} = n_0 + \frac{a_2}{a_1}$$

$$a_1 - n_1 a_2 = a_3 \qquad \text{i.e.} \qquad \frac{a_1}{a_2} = n_1 + \frac{a_3}{a_2}$$

etc.

Now substitute each expression in the expression above it, starting at the top:

$$\frac{a_0}{a_1} = n_0 + \frac{a_2}{a_1} = n_0 + \cfrac{1}{n_1 + \cfrac{a_3}{a_2}} = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{a_4}{a_3}}} = \text{etc.},$$

that is

$$\frac{a_0}{a_1} = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots}},$$

a so-called continued fraction, for which we again employ the abbreviation $[n_0, n_1, n_2, \ldots]$. If we ignore a remainder in this process, this will correspond to truncating this expression, which will yield the corresponding convergent $[n_0, n_1, \ldots, n_k] = p_k{:}q_k$. If we then think carefully how these fractions behave (see the next section), we see that

$$p_0{:}q_0 < p_2{:}q_2 < \ldots < a_0{:}a_1 < \ldots < p_3{:}q_3 < p_1{:}q_1$$

and that

$$p_{k+1} = n_k p_k + p_{k-1}$$
$$q_{k+1} = n_k q_k + q_{k-1}$$

where $p_0 = q_{-1} = 1$, $p_{-1} = q_0 = 0$. Finally, the intermediate convergents fit between the convergents in these inequalities.

[8] Readers who do not like mathematics or its history should skip the rest of this section!

It is worth describing a completely different and general procedure which generates all the apparatus of this kind of expansion and its associated approximations, both the convergents and the intermediate convergents. We shall use the following propositions :

$$\text{If} \quad p{:}q < r{:}s \quad \text{then} \quad p{:}q < (p+r){:}(q+s) < r{:}s.$$

(This has the homely interpretation that the average speed of a day's journey will always lie between the averages of morning and afternoon; or that a cricketer's seasonal average will lie between the half-seasonal averages). So, to approximate a given ratio $a{:}b$, start with initial under- and over-estimates $p{:}q$ and $r{:}s$; generate the new estimate $(p+r){:}(q+s)$ and check whether it is less than, equal to, or greater than $a{:}b$; replace the appropriate initial estimate by it; and then continue. The process can be started with the universal estimates $0{:}1 < a{:}b < 1{:}0$. Here is an illustration of its operation on the ratio $\sqrt{6}{:}1$, where we use the test that $p{:}q$ is less than, equal to, or greater than $\sqrt{6}{:}1$ according as p^2 is less than, equal to, or greater than $6q^2$.

under-estimate	**over-estimate**	**new estimate**	p^2	$6q^2$	**under/ over**
0:1	1:0	1:1	1	6	under
1:1		<u>**2:1**</u>	4	6	<u>**under**</u>
2:1		3:1	9	6	over
	3:1	<u>**5:2**</u>	25	24	<u>**over**</u>
	5:2	7:3	49	54	under
7:3		12:5	144	150	under
12:5		17:7	289	294	under
17:7		<u>**22:9**</u>	484	486	<u>**under**</u>
22:9		27:11	729	726	over
etc.					

The approximations in column 3 occur in improving runs of under- and overestimates and the pattern of these runs gives the expansion: here 2 under-estimates, 2 over-estimates, 4 under-under estimates, ... corresponds to the expansion $\sqrt{6}{:}1 = [2, 2, 4, ...]$. The run-end approximations just before the estimate changes side will be the convergents, the best at any stage; these here are the underlined ratios 2:1, 5:2, 22:9. The other approximations will be intermediate convergents that I have just described. (This is also illustrated by Figure 1.) Observe also that the third column of new estimates can be built up from a knowledge of only the last column ; and this is precisely what the calculation of the approximations in Section 1 is doing. Many of the other properties of these expansions can also be described in terms of this algorithm.

This method can be used to generate good (indeed the best possible) approximations to a wide variety of ratios (roots, logarithms, etc.) in a way that does not involve having recourse to calculating machines or mathematical tables[9].

4. INEQUALITIES AND ARITHMETIC

> "It is hard to say at once in which order such terms as $\frac{5}{17}$, $\frac{12}{37}$ and $\frac{13}{42}$ should be placed, let alone whether the difference between the largest and the middle value is more or less than the difference between the middle and the smallest value" (p.74).

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[9] A very clear discussion of the technique, with copious examples, is given in Fletcher, 'Approximating by vectors'.

Look now at the effect of varying the successive terms n_k. As n_0 increases, so surely does the corresponding ratio $a_0{:}a_1$. Then n_1 measures the size of the first remainder a_2 against a_1; so as n_1 *increases*, a_2 descreases, so the ratio $a_0{:}a_1$ *decreases*, since a_2 is a subinterval of a_0. Next, n_2 measures the size of the second remainder a_3, a subinterval of a_1, against a_2; so as n_2 increases, a_3 decreases, so a_1 decreases, and $a_0{:}a_1$ increases. This pattern continues: increasing an even-indexed term n_0, n_2, n_4 ... will increase the corresponding ratio $a_0{:}a_1$, while *increasing* an *odd*-indexed term $n_1, n_3, n_5, \ldots$ will *decrease* $a_0{:}a_1$.

The expansions of the three examples above are as follows:

$$\begin{aligned} 5{:}17 &= [0, 3, 2, 2] \\ 12{:}37 &= [0, 3, 12], \text{ and} \\ 13{:}42 &= [0, 3, 4, 3]. \end{aligned}$$

These ratios differ first in their even-indexed terms n_2, and increasing this term will correspond to increasing the ratio. Hence 5:17 is the smallest ratio, 13:42 is the middle, and 12:37 the largest.

This rule needs to be extended to deal with the case of terminating ratios, as follows: express the ratios using the alternative that does *not* end with a term equal to 1 and then adjoin a nominal term of ∞ at the end (corresponding to performing the subtraction process with a zero remainder). Then, to compare two ratios, look at the first place at which they differ; if it is even-indexed, the ratios will be in standard 'dictionary' order; if odd-indexed, in reversed dictionary order. It was not necessary to invoke this rule in the examples above, but it explains why $[2,1] = [3] < [3, 2]$, neither of which relationship seems to satisfy the rule as stated.

5. ARITHMETIC WITH RATIOS

> "The [so-called] Egyptian system of fractions is normally used throughout these works [Hero's *Stereometrica* and *De Mensuris*], and the handling of this cumbersome method is most impressive to a novice in the art (e.g. $104\frac{1}{2}\frac{1}{7}\frac{1}{14}\frac{1}{21} - 21\frac{1}{2}\frac{1}{3}\frac{1}{12} = 82\frac{1}{2}\frac{1}{3}\frac{1}{84}$" (pp.81-2).

It is worth noting that descriptions of arithmetic with ratios of numbers (corresponding to our fractional arithmetic) are notoriously absent from Euclid's *Elements* and from Greek mathematics in general, while even everyday calculations, and perhaps even the arithmetic of mathematicians, was performed in a way that seems strange and complicated to us. In fact, the evidence for a Greek use of anything corresponding to our way of manipulating fractions is very slight, almost non-existent, despite what is confidently asserted in many histories of Greek mathematics; instead, Greek calculations seem to have been carried out Egyptian-style, using sums of unit fractions[10]. In this note I am describing practical applications of a method of describing ratios that may have been in use by pre-Euclidean mathematicians, though I think that this historical aspect is incidental to my proposal here. More details of these historical matters will be given in Section 8, below.

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[10] See, for example, Heath, *History of Greek Mathematics* i, pp41-45; the section called 'The ordinary Greek form, variously written' is, at best, highly misleading. An excellent summary of some of the evidence is given by Coulton on pp74-84, and I discuss the topic in detail in Chapter 7 of my *The Mathematics of Plato's Academy,* where I argue that early Greek fractional calculations (excluding some astronomical calculations, as noted in Section 7, below) may have been carried out using unit fractions. A brief summary of this proposal is given in my article, 'Logistic and fractions', in the general survey *Histoire de Fractions, Fractions d'Histoire.*

These continued fraction expansions have the curious feature that the procedures for performing arithmetic on them are very difficult to find: common informed mathematical opinion today is that it is just about possible but is computationally unwieldy. In fact, a simple procedure has, been described by R.W. Gosper[11]. Even with this, however, it is not easy to see quickly whether the difference, in the example of Section 4, above, between 13:42 and 5:17 is greater or less than the difference between 12:37 and 13:42. One might guess at first from the variation in the n_2 terms, from 2 to 4 to 12, that it is less; in fact it is greater. In general, a small change in a small value of some index k has a much greater effect than quite a large change in a large value of this same index or a small change in a higher index; here, the change from 2 to 4 in n_2 has a greater effect than the change from 4 to 12. (Once again, Figure 1 illustrates what is happening). A similar phenomenon can be seen in the behaviour of the denominators of simple fractions: the difference between $\frac{1}{2}$ and $\frac{1}{4}$ is greater than the difference between $\frac{1}{4}$ and $\frac{1}{12}$.

6. INCOMMENSURABLE RATIOS

> "The assumption has been ... that the proportional systems used in Greek architecture were based on commensurable ratios, ... [but] many of the general remarks above apply equally to systems based on incommensurable ratios" (p.73).

The theory of these expansions (known now to mathematicians under the name of the theory of continued fractions) is supremely well adapted to two topics: finding information about good rational approximations, and finding certain kinds of properties of the solutions of quadratic equations. (The two solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, numbers of the form $\frac{p+\sqrt{q}}{r}$, which I shall hereafter call 'quadratic surds'; and I will pass freely between these numbers and the corresponding ratios, $\frac{p+\sqrt{q}}{r}:1$ or $(p+\sqrt{q}):r$.) Consider first the expansion of the simplest kind of such an incommensurable ratio, $\sqrt{p}:\sqrt{q}$, corresponding to the positive solution of $qx^2 - p = 0$, where $\frac{p}{q}$ is not a perfect square. The astonishing result here is that the continued fraction expansion of these ratios will *always* exhibit the striking form

$$\sqrt{p}:\sqrt{q} = [n_0, n_1, n_2, ..., n_2, n_1, 2n_0, n_1, n_2, ..., n_2, n_1, 2n_0, ...]$$

in which the first term n_0 is followed by an indefinitely repeating period comprising a palindromic block terminated by $2n_0$. (The palindromic block may be reduced to a single term, or be missing from certain simple ratios.) Table 2 sets out all ratios $\sqrt{p}:\sqrt{q}$ for $1 \le q < p \le 10$; in the first column, two repetitions of the period are given and the commensurable ratios $\sqrt{4}:1$ and $\sqrt{9}:1$ are not expanded; elsewhere only one period is given, or the ratio is identified as a ratio that has already been calculated. It will be evident to anybody who explores this table that some general patterns appear to be emerging for some kinds of ratios:

$$\begin{aligned}
\sqrt{(n^2+1)}:1 &= [n, 2n, 2n, ...] , \\
\sqrt{(n^2+2)}:1 &= [n, n, 2n, n, 2n, ...] , \\
\sqrt{(n+1)}:\sqrt{n} &= [1, 2n, 2, 2n, 2, ...] , \\
\sqrt{(n+2)}:\sqrt{n} &= [1, n, 2, n, 2, ...] .
\end{aligned}$$

[11] This algorithm is described in my *Mathematics of Plato's Academy*, pp354-60.

Table 2: The continued fraction expansions of $\sqrt{p} : \sqrt{q}$, $1 \le q < p \le 10$.

q=	1	2	3	4	5	6	7	8	9
p=									
2	1,2,2,...								
3	1,1,2,1,2,...	1,4,2,...							
4	2:1	√2:1	1,6,2,...						
5	2,4,4,..	1,1,1,2,...	1,3,2,...	1,8,2,...					
6	2,2,4,2,4,...	√3:1	√2:1	√3:√2	1,10,2,...				
7	2,1,1,1,4,1,1,1,4,...	1,1,6,1,2,...	1,1,1,8,1,1,2,...	1,3,10,3,2,...	1,5,2,...	1,12,2,...			
8	2,1,4,1,4,...	2:1	1,1,1,1,2,...	√2:1	1,3,1,3,2,...	2:√3	1,14,2,...		
9	3:1	2,8,4,...	√3:1	3:2	1,2,1,12,1,2,2,...	√3:√2	1,7,2,...	1,16,2,...	
10	3,6,6,...	√5:1	1,1,4,1,2,...	√5:√2	√2:1	√5:√3	1,5,8,5,2,...	√5:2	1,18,2,...

The characteristic form of the ratios $\sqrt{p}:\sqrt{q}$ makes them instantly recognisable. Conversely, given any expansion of this periodic palindromic form $[n_0, n_1, n_2, ..., n_2, n_1, 2n_0, n_1, n_2, ..., n_2, n_1, 2n_0, ...]$, the ratio of which this is the expansion can be calculated as follows:

In general, if $x = [n_0, n_1, n_2, ...]$, this means that $x = n_0 + \theta_0$ where $0 \leq \theta_0 \leq 1$; $\theta_0^{-1} = x_1 = n_1 + \theta_1$, etc. Substituting each expression into the one preceding it, we get

$$\frac{a_0}{a_1} = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots}},$$

as was described in Section 3, above. In the case when x has the palindromic periodic form, we then write

$$x + n_0 = [2n_0, n_1, n_2, ..., n_2, n_1, 2n_0, n_1, ...n_1, ...];$$

the period now starts with the first term, and the expansion got by omitting the first complete period is again identical to $x + n_0$. Hence

$$x + n_0 = 2n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_2 + \cfrac{1}{n_1 + \cfrac{1}{x + n_0}}}}}}.$$

When this expansion is simplified, it gives rise to a quadratic equation of the special simple form $qx^2 - p = 0$.

The expansion of a general quadratic surd, $\frac{p+\sqrt{q}}{r}:1$, will eventually be periodic, of the form, $[m_0, m_1, ..., m_k, n_1, n_2, ..., n\ , n_1, n_2, ..., n\ , ...]$, but little more can be said in advance. (This behaviour is analogous to this displayed by the decimal expansion of a fraction; for example $\frac{4996}{3770} = 1{\cdot}36027027027...$.) One particular quadratic surd has come to fascinate those who seek interesting ratios, for reasons not unconnected with its expansion: the ratio $x:1$ whose expansion is $[1, 1, 1, 1, ...]$ will satisfy $x = 1 + \frac{1}{x}$, so $x^2 - x - 1 = 0$, so $x = \frac{1}{2}(1 \pm \sqrt{5})$. The positive root $x = \frac{1}{2}(1 + \sqrt{5}) = 1{\cdot}6180...$ gives the ratio of the notorious 'golden section'; and, from what I have described here, it can be seen that, since it has *no* terms larger than 1 in its expansion, it cannot be specially well-approximated by any ratio of integers. Now calculate the approximations:

n_k			1	1	1	1	1	1	...
p_k	0	1	1	2	3	3	8	13	...
q_k	1	0	1	1	2	3	5	8	...
			under	over	under	over	under	over ,	

and we see that they are, as is well known, the ratios of the successive Fibonacci numbers. We can read off from Table 1 other kinds of approximation to the golden ratio, for example, $\sqrt{8}:\sqrt{3} = [1, 1, 1, 1, 2, ...]$; this ratio $\sqrt{8}:\sqrt{3}$ will give a slightly poorer overestimate than the ratio $[1, 1, 1, 1, 1, 1] = 13:8$. (In decimal numbers $\frac{1}{2}(1 + \sqrt{5}) = 1{\cdot}6180...$, $13:8 = 1{\cdot}625...$, and $\sqrt{8}:\sqrt{3} = 1{\cdot}6330...$).

Vast and unfounded claims are often made for the golden section[12]. At the very least, one virtue of this theory is that, by providing an explanation of many of the mathematical properties of this ratio, it provides a procedure which enables us to set most of these remarkable properties in a more revealing context; and perhaps this objective test for the presence of this ratio may help to put some of the more extravagant claims where they belong.

Among other incommensurable ratios, it is worth knowing that $\pi:1 = [3, 7, 15, 1, 292, 1, 1, 1,...]$; this explains why $[3, 7] = 22:7$ and $[3, 7, 15, 1] = 355:113$ are particularly good approximations to π, both overestimates. But there are few general results about any other incommensurable ratios other than quadratic surds, though sometimes merely looking at an expansion suggests some observation. For example $(\sqrt{2}+\sqrt{3}):1$ (this is *not* a quadratic surd) has the expansion $[3, 6, 1, 5, 7, ...]$ so $[3, 6, 1]$ wil be a good approximation; but this is the same as $[3, 7]$, the standard approximation to π. Hence we should expect $\sqrt{2}+\sqrt{3}$ to be close to and bigger than $\pi = 3{\cdot}14159...$, and it is: $\sqrt{2}+\sqrt{3} = 3{\cdot}14626...$. Once again, an explanation cannot but help us understand more about this coincidence: here, an octagon inscribed in the circle with unit radius has area $2\sqrt{2}$; a hexagon circumscribed around has area $2\sqrt{3}$; and their average will be close to the area of the circle, which is π.

7. DECIMAL FRACTIONS

> "... proportional relationships must be expressed as decimal fractions initially in any investigation of their significance ... however no ancient architect could have designed in decimal fractions. In so far as he conceived his design in terms of arithmetic proportions, he must have used expressions equivalent to vulgar fractions" (pp.74-5).

Decimal fractions were not introduced in the West until the late sixteenth century, when they were popularised by Viete, Stevin, and others. Greek astronomers did use a Babylonian sexagesimal system (i.e., to base 60, just as, under the same Babylonian influence, we still measure angles and time), but not before the time of Hipparchus in the second century B.C., and then only for scientific, in particular astronomical, calculations.

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[12] A detailed history of the mathematics involved, from antiquity up to the 18th century, can be found in Herz-Fischler, *A Mathematical History of Division in Extreme and Mean Ratio.* A mathematical explanation of some of the evidence for the golden section that some people induitably find is given in Herz-Fischler, 'How to find the golden number without really trying', and I have written a less technical description of this in another article of the same name in a journal, *The Mathematical Review,* intended for schools. In a nutshell, the explanation is as follows: Suppose we are presented with data in the form of pairs of numbers whose ratio is random. Write a typical pair as m & M, where m is the smaller and, more precisely, suppose that the numbers m/M (i.e. smaller to larger) are uniformly distributed between 0 & 1. But if we now analyse $M/(m+M)$ (i.e. larger to whole), instead of m/M, we will find that the data lies within the interval from $\frac{1}{2}$ to 1, and are, in fact, compressed around $\cdot 6180... = \frac{1}{2}(\sqrt{5} - 1)$, the reciprocal of the golden section! (Note that $m/M = M/(m+M)$ if and only if $m/M = \cdot 6180...$; the map defined by $f(x) = (1 + x)^{-1}$ which sends m/M to $M/(m+M)$ is a contraction of $[0,1]$ with fixed point $\frac{1}{2}(\sqrt{5} - 1)$.) And if the numbers m/M in fact lie in some smaller interval (for example, if m and M are different but not too different, so m/M keeps away from 0 & 1), then this compression effect in $M/(m+M)$ will be even more exagerated.

It is worth describing a subtraction algorithm for generating decimal ratios, similar to the anthyphairetic procedure of Section 1. We start with a choice of basic length b, e.g. a metre, and a base, e.g. 10. Then, to measure a given line a_0, we perform the following subtractions:

$$
\begin{array}{llll}
a_0 - m_0 \times b & = a_1 & \text{with} & 0 \le a_1 \le b \ , \\
a_1 - m_1 \times b/_{10} & = a_2 & \text{with} & 0 \le a_2 \le b/_{10} \ , \\
a_2 - m_2 \times b/_{10^2} & = a_3 & \text{with} & 0 \le a_3 \le b/_{10^2} , \\
\text{etc.,} & & &
\end{array}
$$

and the 'length' of a_0 corresponds to the ratio $a_0{:}b$, now characterised by the sequence of numbers $m_0, m_1, m_2, m_3, \ldots$ which are conventionally written $m_0{\cdot}m_1m_2m_3\ldots$.

These numbers $m_0, m_1, m_2, \ldots$ have certain properties and can be manipulated in certain ways; we learn about these slowly, and sometimes painfully, and then this knowledge forms the basis of our understanding of numerical phenomena. Similarly the numbers $[n_0, n_1, n_2, \ldots]$ associated with the continued fraction expansion have certain, completely different, properties and they also can be manipulated in other, completely different ways. It is beyond doubt that some features of ratios are only clearly revealed in the continued fraction description of ratios, and it is those features that I am describing and exploiting here.

7. FURTHER HISTORICAL REMARKS

"Plato expressed the ratio[13] 256:243 without reducing it to a fraction." (p.75).

The technique described here is presented with the intention of helping those, today, who need to analyse data, to facilitate their search for different kinds of rules of proportion that may be contained therein, and it stands or falls on this criterion. However it may be of interest to conclude with some incidental remarks on the role of these kinds of expansions in a speculative reconstruction of pre-Euclidean Greek mathematics.

It is a curious and remarkable fact that although ratios are frequently referred to by Plato, Aristotle, and even Euclid, the idea of ratio, as opposed to proportion (i.e. a meaning for '*a*:*b*', rather than '*a*:*b*::*c*:*d*') is not clearly defined, either in Euclid's *Elements* or anywhere else in the surviving corpus of Greek mathematics, and the techniques that we use to handle ratios — fractions, and decimal or sexagesimal numbers — may not have been available to them at that time. All that we have is a vague description of the word at *Elements* V, Def. 3: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind", and a further hint of what an early definition might have been in Aristotle, *Topics* 158b29ff: "For the areas and the bases have the same *antanairesis*; such is the definition of the same ratio [or proportion?]". Instead, most of our surviving evidence refers to the celebrated later Eudoxan definition of proportion, in *Elements* V, Def. 5, which is believed to date from c.350 B.C. Now this word *antanairesis* used by Aristotle is closely related to, and may even be synonymous with, the word *anthyphairesis* found in Euclid's *Elements* to describe the so-called Euclidean algorithm[14]. In my book, *The Mathematics of Plato's Academy: A New Reconstruction*, I develop in detail the proposal that there were several fundamentally different though equivalent

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13 Coulton uses the word 'proportion' here but, in strict mathematical usage, 'ratio' is perhaps more correct (see Euclid *Elements* V, Defs. 3 and 6), though everyday usage is far from consistent; we ask 'what proportion of A is B?', and the usual Latin translation of *logos* is *proportio*. An analogy may illustrate the difference between the words: to define a ratio is like defining a human being; a proportion is like defining twins, where we compare appropriate characteristics to verfy they are identical. Such an analogy is of course, itself, a proportion (*analogon*): ratio:proportion::human being:twin.

14 The Greek verb *anthuphairein* occurs in Euclids *Elements* VII 1 (twice) and 2, and X 2 (twice) and 3. The process it describes can be rendered into English as 'reciprocal subtraction'.

ways of defining ratio before the development of the Eudoxan proportion theory, of which the most highly developed was the 'anthyphairetic' or continued fraction expansion I have been discussing here, and I show that a substantial amount of the *Elements* and other fourth century mathematical references can be interpreted in this light; for example, I suggest that Plato's mathematical *logistikê* may have been the study of these different kinds of ratio.

However, even if my new interpretation were to be accepted, I do not think that it would have any significant bearing on practical questions of the design and construction of Greek buildings. As Plato observes, at *Philebus* 56D-57A:

> SOCRATES: Are there not two kinds of arithmetic (*arithmêikê*) that of the people and that of the philosophers?... And how about the arts of reckoning (*logistikê*) and measuring as they are used in building and trade when comparing with philosophical geometry and elaborate computations (*logismos katameletomenos*) — shall we speak of each as one or two?
> PROTARCHUS: I should say that each of them was two.

BIBLIOGRAPHY

COULTON J.J., "Towards Understanding Greek Temple Design: General Considerations", *Annual of the British School of Athens,*70 (1975), 59-99; Corrigenda, *ibid.,* 51 (1976), 149-50.

FLETCHER T., "Approximating by vectors", *Mathematics Teaching* 63 (1973), 4-9 & 64 (1973), 42-44.

FOWLER D.H., *The Mathematics of Plato's Academy: A New Reconstruction,* Oxford, Clarendon Press, 1987; revised paperback reprint, 1991.

FOWLER D.H., "Logistic and fractions", pp133-147 in Benoit, P., Chemla K., & Ritter J., edd., *Histoire de Fractions, Fractions d'Histoire,* Basel etc., Birkhäuser, 1992.

FOWLER D.H., "How to find the golden number without really trying", *Mathematics Review,* 3_4 (April 1993), 2-7.

FREYL L., "Médiétés et approximations chez Vitruve", *Revue Archéologique,* 1990, 285-330.

FREYL L., "Pour un modèle du chapiteau ionique vitruvien", *Revue Archéologique,* 1992, 37-63.

FREYL L., "La transmission d'un canon: les temples ioniques", to appear in *Le Projet de Vitruve: Object, Destination et Reception du De Architectura,* Rome, Ecole Française de Rome.

HEATH T.L., *A History of Greek Mathematics,* 2 vols., Oxford, Clarendon Press, 1921, reprinted New York, Dover 1981.

HERZ-FISCHLER R., *A Mathematical History of Division in Extreme and Mean Ratio,* Waterloo (Ontario), Wilfred Laurier University Press, 1987.

HERZ-FISCHLER R,, "How to find the golden number without really trying", *Fibonacci Quarterly,* 19 (1981), 406-10.