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NOTE ON CATEGORIES OF INDECOMPOSABLE MODULES

by Manabu HARADA

Let  $R$  be a ring with identity and  $M$  a unitary right  $R$ -module which is a directsum of indecomposable, injective modules. E. Matlis [13] posed the following question : for any direct summand  $L$  of  $M$ , is  $L$  also a directsum of indecomposable injective modules ? Recently, U.S. Kahlon [9] and K. Yamagata [16] studied this problem under an assumption that the singular submodule of  $L$  is equal to zero.

In this short note, we shall show that if the singular submodule of  $L$  is equal to zero, then the affirmative answer of Matlis' problem is an immediate consequence from [6] and [10]. Especially, in the section 4, we shall give simpler proofs of generalized Kahlon' results [9]. In sections 2 and 3, we shall give some supplementary results of [7] and [8] .

I. DEFINITIONS

Let  $R$  be a ring with identity. We assume that all modules in this note are unitary right  $R$ -modules. Let  $M$  be an  $R$ -module. If  $\text{End}_R(M) = S_M$  is a local ring (the Jacobson radical is a unique maximal ideal among left and right ideals),  $M$  is called *completely indecomposable*.

Let  $\mathcal{A}$  be the induced full sub-category from all completely indecomposable modules  $M_\alpha$  in the category of right  $R$ -modules  $\mathcal{M}_R$ , namely every object in  $\mathcal{A}$  is a direct sum of some family of  $\{M_\alpha\}$  (see [6], § 3). Let  $M^1, M^2$  be objects in  $\mathcal{A}$  and  $M = \sum_{\mathbb{I}\alpha} \oplus M_\alpha^i ; M_\alpha^i \in \{M_\alpha\}$  . We put  $[M^1, M^2] \cap \mathcal{A}' = \{f \in \text{Hom}_R(M^1, M^2), p_\beta f i_\beta : M_\beta^1 \rightarrow M_\beta^2$

is non-isomorphic for all  $\beta \in I^1, \beta' \in I^2$ , where  $i_{\beta'}^1 : M_{\beta'}^1 \rightarrow M^1$  is the injection and  $p_{\beta}^2 : M^2 \rightarrow M_{\beta}^2$  is the projection}. Then  $\mathcal{J}'$  is an ideal in  $\mathcal{A}$  and  $\mathcal{A}/\mathcal{J}'$  is a completely reducible  $C_3$ -abelian category by [6], Theorem 7. If  $M^1 = M^2$ , we denote  $[M^1, M^1] \cap \mathcal{J}'$  by  $\mathcal{J}'$ .

Let  $A$  and  $f$  be an object and a morphism in  $\mathcal{A}$ , respectively. By  $\bar{A}$  and  $\bar{f}$  we denote the residue classes of  $A$  and  $f$  in  $\mathcal{A}/\mathcal{J}'$ . Let  $A \supset B$  be in  $\mathcal{A}$  and  $i$  the inclusion of  $B$  to  $A$ . If  $\bar{i}$  is isomorphic in  $\mathcal{A}/\mathcal{J}'$ , we say  $B$  is a *dense submodule* of  $A$  (see [7], p. 310-311). We assume  $A = C \oplus D$  as  $R$ -modules and let  $e$  the projection of  $A$  onto  $C$ . By  $\bar{C}$  we denote  $\text{Im } \bar{e}$  in  $\mathcal{A}/\mathcal{J}'$ , even though  $C$  is not in  $\mathcal{A}$ . Next, we assume that  $A \supset B$  are in  $\mathcal{A}_R$  and  $B = \sum_K \oplus T_{\alpha}$  as  $R$ -modules. If  $\sum_{K'} \oplus T_{\alpha}$  is a direct summand of  $A$  for any finite subset  $K'$  of  $K$ , then we say that  $B$  is a *finitely direct summand* of  $A$  (with respect to the decomposition  $\sum_K \oplus T_{\alpha}$ ). It is clear that every directsum of injective modules is a finitely direct summand of its extension module.

We summarize here definitions of the exchange property given in [6], [7], [8] and [10].

Let  $\{M_{\alpha}\}_I$  and  $\{N_{\beta}\}_J$  be sets of completely indecomposable modules. We put  $M = \sum_I \oplus M_{\alpha}$  we recall Condition II given in [6], §3.

II (Take out). For any subset  $I'$  of  $I$  and any other decomposition  $M = \sum_J \oplus N_{\beta}$ , there exists a subset  $\{N_{\phi(\gamma)}\}_{\gamma \in I'}$ , of  $\{N_{\beta}\}_J$  such that  $M_{\gamma} \approx N_{\phi(\gamma)}$  for all  $\gamma \in I'$  and  $M = \sum_{I'} \oplus N_{\phi(\gamma)} \oplus \sum_{\alpha \in I-I'} \oplus M_{\alpha}$ .

II' (Put in). For the same assumption as above, there exists a subset  $\{N_{\psi(\gamma)}\}_{\gamma \in I'}$ , such that  $M_{\gamma} \approx N_{\psi(\gamma)}$  for all  $\gamma \in I'$  and  $M = \sum_{\alpha \in I'} \oplus M_{\alpha} \oplus \sum_{\beta \in J-\psi(I')} \oplus N_{\beta}$  where  $\phi$  and  $\psi$  are one-to-one mappings of  $I'$  into  $J$ .

If we replace the subset  $I'$  by  $I-I'$ , then II and II' are equivalent by Azumaya's theorem [1]. Furthermore, Azumaya [1] showed that II and II' are satisfied for any finite subset  $I'$  of  $I$ .

We remark that if a given decomposition  $M = \sum_I \oplus M_\alpha$  satisfies II or II', then any decomposition  $M = \sum_J \oplus N_\beta$  does the same property. Because, let  $M = \sum_K \oplus T_\delta$  be another decomposition with  $T_\delta$  indecomposable. Then there exists an automorphism  $\sigma$  of  $M$  such that  $\sigma(N_\beta) = M_{\pi(\beta)}$  by Azumaya's theorem, where  $\pi$  is a one-to-one mapping of  $J$  to  $I$ . We apply II or II' for the decompositions  $M = \sum \oplus M_{\pi(\beta)} = \sum \oplus \sigma(T_\delta)$ . Then we have  $M = \sum_{\gamma \in J'} \oplus \sigma(T_{\phi(\gamma)}) \oplus \sum_{\beta \in J-J'} \oplus M_{\pi(\beta)}$  or  $M = \sum_{\beta \in J'} \oplus M_{\pi(\beta)} \oplus \sum_{\delta \in K-\psi(J')} \oplus \sigma(T_\delta)$ . Hence,  $M = \sigma^{-1}(M) = \sum_{J'} \oplus T_{\phi(\gamma)} \oplus \sum_{\beta \in J-J'} \oplus N'_\beta$  or  $M = \sum_{J'} \oplus N'_\beta \oplus \sum_{\delta \in K-\psi(J')} \oplus T_\delta$ .

We note II and II' are independent for fixed two decompositions

$M = \sum_I \oplus M_\alpha = \sum_J \oplus N_\beta$  and a given subset  $I'$  of  $I$ . For example, we assume there exist non-isomorphic monomorphisms  $f_i$  of  $M_i$  to  $M_{i+1}$  for all  $i \in K \subseteq I$ . We put  $M'_i = \{m_i + f_i(m_i) \mid m_i \in M_i\} \in M_I \oplus M_{i+1}$ ,  $m_i \in M_i$ . Then  $M = M'_1 \oplus M_2 \oplus M'_3 \oplus \dots \oplus M_0 = M_1 \oplus M'_2 \oplus M_3 \oplus M'_4 \oplus \dots \oplus M_0$ , where  $M_0 = \sum_{\alpha \in I-K} \oplus M_\alpha$ . It is clear that  $N' = M'_1 \oplus M'_3 \oplus \dots \oplus M_0$  has the property II for the second decomposition in the above. However, by the proof of [6], Lemma 9 we know that if  $N$  had the property II', then  $\{f_i\}$  would be a locally semi-T-nilpotent system (see (see [7], § 1 for the definition). Similarly,  $N = M_1 \oplus M_3 \oplus \dots \oplus M_0$  has the property II' for the first decomposition, however if  $N$  had the property II, then  $\{f_i\}$  would be a locally semi-T-nilpotent system.

We say that a direct summand  $T$  of  $M$  has the *exchange property in  $M$*  if for any decomposition  $M = \sum_J \oplus U_\delta$  ( $U_\delta$  are not necessarily indecomposable),

$M = T \oplus \Sigma \oplus U'_\delta$  and  $U_\delta \cong U'_\delta$  for all  $\delta \in J$ . Especially, if  $T$  has the above property, whenever all  $U_\delta$  are indecomposable, we say  $T$  has the *exchange property in  $M$  for indecomposable modules*. We refer the reader for terminologies to [6] and [7].

## 2. DIRECT SUMMANDS

First we recall some of main theorems in [7] and [10].

**THEOREM 1** ([7],[10]). - *Let  $M$  be a direct sum of a family of completely indecomposable modules  $\{M_\alpha\}_I$ . Then the following statements are equivalent.*

- 1)  $M$  satisfies the property of "take out"
- 2) Every direct summand of  $M$  has the exchange property in  $M$
- 3) Every direct summand of  $M$  has the exchange property in  $M$  for indecomposable modules
- 4)  $\{M_\alpha\}_I$  is a locally semi- $T$ -nilpotent system
- 5)  $\mathcal{J}$  is the Jacobson radical  $\mathcal{J}$  of  $S = \text{End}_R(M)$
- 6) Every finitely direct summand  $M'$  of  $M$  such that  $M' = \sum_K T_\alpha$  is a direct summand of  $M$  for any  $[K]$  and any family  $\{T_\alpha\}$
- 6') 6) is valid for any  $K$  with  $K = \aleph_0$ , and
- 7) 6') is valid whenever all  $T_\alpha$  are completely indecomposable,
- 8)  $S/\mathcal{J}$  is a regular ring (and self injective as a one sided module) and every idempotents in  $S/\mathcal{J}$  are lifted to  $S$ , where  $|K|$  is the cardinal number of  $K$ .

Proof. 2)  $\longleftrightarrow$  4) is proved by [8], Corollary to Proposition 1. We note 1)  $\rightarrow$  4)

in § 1. 2)  $\rightarrow$  3)  $\rightarrow$  1) is clear. 4)  $\longleftrightarrow$  5) is proved by [10], Theorem.

6)  $\rightarrow$  6')  $\rightarrow$  7) is trivial. 7)  $\rightarrow$  4). Let  $\{M_{\alpha i}\}_1^{\infty}$  be any countable sub-family of  $\{M_{\alpha}\}_I$  and let  $\{f_i\}_1^{\infty}$  be a family of non-isomorphisms  $f_i : M_i = M_{\alpha i} \rightarrow M_{i+1} = M_{\alpha i+1}$ . Put  $M' = \sum_1^{\infty} \oplus M_i'$ , where  $M_i' = \{m_i + f_i(m_i) \mid m_i \in M_i\}$ . Since

$$\sum_1^n \oplus M_i' \oplus M_{n+1} = \sum_1^{n+1} \oplus M_i, \quad M' \text{ is a finitely direct summand of } M. \text{ Hence, } M'$$

is direct summand of  $\sum_1^{\infty} \oplus M_i (=M_0) \subseteq M$  by 7). We know from [8], Theorem 2

that  $M'$  is a dense submodule of  $M_0$ . Therefore,  $M' = M_0$  by [6], lemma 7, which

means that  $\{M_i\}_1^{\infty}$  is a locally semi-T-nilpotent (cf. [6], the proof of Lemma 9).

4)  $\rightarrow$  6). Let  $M' = \sum_K \oplus T_{\alpha}$  be a finitely direct summand of  $M$ . We may assume from 2) that all  $T_{\alpha}$  are indecomposable. Now, we consider the above modules in

$\mathcal{A}/\mathcal{J}'$ . Let  $i$  be the inclusion of  $M'$  to  $M$ . Since  $M'$  is a finitely direct summand of  $M$  and  $\mathcal{A}/\mathcal{J}'$  is a  $C_3$ -abelian,  $\bar{i}$  is the inclusion of  $\bar{M}'$  into  $\bar{M}$  and  $\bar{M}' = \sum_K \oplus \bar{T}_{\alpha}$ . Then  $\bar{M}'$  is a coretract of  $\bar{M}$  by [6], Theorem 7.  $\{T_{\alpha}\}_K$  is a

locally semi-T-nilpotent system. Therefore,  $i$  is a coretract of  $M$  by 5)

(cf. [7], the proof of Proposition 2). 5)  $\longleftrightarrow$  8) It is clear from [6], Lemma 7

and 13 Corollary to Lemma 6.

Remark 1. In the above proof of 4)  $\rightarrow$  6) we only make use of a fact that

$\{T_{\alpha}\}_K$  is a locally semi-T-nilpotent system.

Next, we study a general type of Matlis' problem. The following theorem combines [10] and [14].

THEOREM 2. - Let  $M$  be a directsum of completely indecomposable modules  $M_{\alpha}$  ;

$M_{\alpha} = \sum_I \oplus M_{\alpha}$  and  $\{M_{\beta}\}_J$  the sub-family of countably generated  $R$ -modules  $M_{\beta}$

of  $\{M_{\alpha}\}_I$ . We assume  $\{M_{\gamma}\}_{I-J}$  is a locally semi-T-nilpotent system. Then

every direct summand of  $M$  is in  $\mathcal{A}$ .

Proof. Let  $M = N_1 \oplus N_2$  and  $K = I - J$ . Each  $N_i$  contains a dense submodule  $T_i = \sum_{L_i} \oplus M'_{\gamma_i}$  such that  $M \approx T_1 \oplus T_2$  by [6], Theorem 1 for  $i = 1, 2$ , where  $M'_{\gamma_i}$  is isomorphic to some  $M_\alpha$ ;  $\alpha \in I$ . We divide  $L_i$  into two partitions  $L_i = J_i \cup K_i$  such that for  $\gamma_i \in J_i$  (resp.  $K_i$ )  $M'_{\gamma_i}$  is isomorphic to some  $M_\alpha$ ;  $\alpha \in J$  (resp.  $K$ ). Since  $\{M'_{\gamma_i}\}_{K_i}$  is locally semi-T-nilpotent,  $T'_i = \sum_{K_i} \oplus M'_{\gamma_i}$  is a direct summand of  $N_i$  by Remark 1 and [7], proposition 2, say  $N_i = T'_i \oplus N'_i$ . Furthermore,  $T'_1 \oplus T'_2$  has the exchange property in  $M$  by [7], Theorem 2. Hence,  $N'_1 \oplus N'_2 \approx \sum_{J'} \oplus M'_\beta \oplus \sum_{K'} \oplus M'_\gamma$ . We consider those modules in  $\mathcal{A}/\mathcal{J}'$ . Then  $\bar{M} = \bar{T}'_1 \oplus \bar{T}'_2 \oplus \sum \oplus \bar{M}'_{\gamma_i} \oplus \sum \oplus \bar{M}'_{\gamma_i}$ . On the other hand,  $\bar{M} = \bar{N}_1 \oplus \bar{N}_2 = \bar{T}'_1 \oplus \bar{N}'_1 \oplus \bar{T}'_2 \oplus \bar{N}'_2 = \bar{T}'_1 \oplus \bar{T}'_2 \oplus \sum_{J'} \oplus \bar{M}'_\beta \oplus \sum_{K'} \oplus \bar{M}'_\gamma$ , where  $M'_\delta \approx M_\delta$ . Hence  $\sum_{J_1} \oplus \bar{M}'_{\gamma_i} \oplus \sum_{J_2} \oplus \bar{M}'_{\gamma_i} \approx \sum_{J'} \oplus \bar{M}'_\beta \oplus \sum_{K'} \oplus \bar{M}'_\gamma$ . Since all  $M'_{\gamma_i}$  in the left side are countably generated,  $K' = \emptyset$  by [6], Theorem 7. Therefore,  $N'_1$  is in  $\mathcal{A}$  by [14] or [7], Proposition 3. We have completed the proof.

In Theorem 2 if  $N_1$  is injective,  $NN_1$  is in  $\mathcal{A}$  by [4], [9] or [15] without any assumption. Similarly

PROPOSITION 1. - Let  $M$  be in  $\mathcal{A}$  and  $N$  a direct summand of  $M$ . If  $N$  is projective,  $N$  is in  $\mathcal{A}$ .

Proof. By [11], Theorem 1,  $N$  is a directsum of countably generated  $R$ -submodules  $P_\alpha$ . Furthermore,  $P_\alpha$  is in  $\mathcal{A}$  by [7], Proposition 3. Hence,  $N$  is in  $\mathcal{A}$ .

The following corollary was given with an assumption that  $J(P)$  is small in  $P$  by [12], Theorem 5.5 and [7], Proposition 5.

COROLLARY. - Let  $M$  be in  $\mathcal{A}$  and  $R$ -projective. Then every direct summand  $P$  of  $M$  is in  $\mathcal{A}$ .

It is also clear.

We give a property of dense submodules.

PROPOSITION 2. - Let  $M$  be in  $\mathcal{A}$  and  $N$  a direct summand of  $M$ . Then there exists a submodule  $N'$  of satisfying the following properties :

1)  $N'$  is in  $\mathcal{A}$ , 2)  $N'$  is a finitely direct summand of  $N$  and 3) If  $T = \sum_J \oplus T_\alpha$  is finitely direct summand of  $N$ ,  $T$  is isomorphic to a direct summand of  $N'$ , where  $T_\alpha$ 's are indecomposable.

Especially, every countable generated  $R$ -submodule of  $N$  is isomorphic to a submodule of  $N'$ . Every submodule  $N''$  of  $N$  satisfying 1) and 2) is isomorphic to a direct summand of any dense submodule of  $N$ .

Proof. Let  $N'$  be a dense submodule of  $N$ . Then  $N'$  satisfies 1) and 2) by [7], Proposition 2. Let  $e$  and  $e_\alpha$  be projections of  $M$  to  $N$  and  $T_\alpha$  with respect to given decompositions. Then  $ee_\alpha = e_\alpha$  for all  $\alpha \in J$ . We consider those modules in  $\mathcal{A}/\mathcal{I}$ . Since  $T$  is a finitely direct summand of  $N(\subseteq M)$ ,  $\bar{T} = \sum_J \oplus \bar{T}_\alpha \subseteq \text{Im } \bar{e} = \bar{N}'$ . Hence,  $T$  is isomorphic to a direct summand of  $N'$  by [6], Theorem 7. We easily see that for two finitely generated submodules  $T_1 \supseteq T_2$  in  $N$ , we can find a direct summand  $T'_i$  of  $N$  such that  $T'_i \supseteq T_i$ ,  $T'_1 \supseteq T'_2$  and  $T'_i$  is a finite directsum of indecomposable modules for  $i = 1, 2$  (cf. [7], the proof of Proposition 3). Since  $T'_1 \supseteq T'_2$ , we can find a monomorphism of  $T_1$  to  $N'$  which is an extension of a given monomorphism of  $T_2$  to  $N'$  by [6], Theorem 7.



Hence, every countably generated  $R$ -submodule of  $N$  is isomorphic to a submodule of  $N'$  by the standard argument. Let  $N''$  be a submodule satisfying 1) and 2). Then  $\bar{N}' \subseteq \bar{N}''$ . Hence, the last statement is clear from [6], theorem 7.

### 3. EXCHANGE PROPERTY

It seems to the author that the difficulty of the exchange property in  $M$  comes from the following facts. Let  $M$  be in  $\mathcal{A}$  and  $M = N_1 \oplus N_2 \oplus N_3$  as  $R$ -modules. It is well known from [2] that if  $N_1$  and  $N_2$  have the exchange property in  $M$ , then so is  $N_1 \oplus N_2$ , however the converse is not true. Furthermore, even if neither  $N_1$  nor  $N_2$  has the exchange property in  $M$ , it is possible that  $N_1 \oplus N_2$  does.

We note that if a direct summand  $L$  of  $M$  has the exchange property, then  $L$  is in  $\mathcal{A}$ . The following theorem is a slight generalization of some parts in Theorem 1.

**THEOREM 3.** - *Let  $M$  be in  $\mathcal{A}$  and  $M = N_1 \oplus N_2$ . Let  $f$  be the projection of  $M$  onto  $N_1$ . Then  $fJ'f = fJf$  if and only if every direct summand of  $N_1$  has the exchange property in  $M$ . In the case  $N_2$  also has the exchange property in  $M$ , where  $J'$  is the ideal defined in §1 and  $J$  is the Jacobson radical of  $S = \text{End}_R(M)$ .*

*Proof.* "Only if" . Let  $M = \sum_I \oplus M_\alpha$  and  $M'_\alpha$ 's are completely indecomposable. We can find a subset  $J$  of  $I$  such that  $\bar{M}_J = \sum_J \oplus \bar{M}_\alpha \approx \text{Im } \bar{f}$  in  $\mathcal{A}/J'$  by [6], Theorem 7. Let  $e$  be the projection of  $M$  to  $M_J$ . Then  $fS/fJ' \approx eS/eJ'$ .

Hence, there exist  $a \in eSf$ ,  $b \in fSe$  such that  $ba \equiv f \pmod{\mathcal{J}'}$ . Put  $f-ba = n \in \mathcal{J}'$ , then  $n \in f\mathcal{J}'f = f\mathcal{J}f$ , which is the radical of  $S_{N_1} = \text{End}_R(N_1)$ . Hence,  $ba$  is an automorphism of  $fS$  as an  $S$ -module. Therefore,  $eS = f_1S \oplus f_2S$  and  $f_1S \stackrel{a}{\cong} fS$ ,  $f_2S = \text{Ker } b$  and  $f_i^2 = f_i$ . Since  $b$  induces  $eS/e\mathcal{J}' \cong fS/f\mathcal{J}'$ ,  $f_2S = f_2\mathcal{J}' \subseteq \mathcal{J}'$ . Hence,  $f_2 = 0$  by [1], Theorem 1 or [6], Lemma 7 and  $eS \cong fS$ , which implies  $N_1 \cong M_J$ . Therefore,  $\{M_\alpha\}_J$  is a locally semi-T-nilpotent system by Theorem 1. Thus, we have proved "only if" from [8], Corollary to Theorem 2. "if".  $N_1 = \sum_K \oplus M'_Y$  and  $\{M'_Y\}$  is a locally semi-T-nilpotent system by [8], Corollary to Theorem 2. Hence,  $f\mathcal{J}'f = f\mathcal{J}f$  by Theorem 1 and [6], Lemma 5. The remaining part is clear from [8], Theorem 2.

**COROLLARY .** - Let  $M$  and  $N_1$  be as above. If for every monomorphism  $g$  in  $S_{N_1}$ .  $\text{Im } g$  is a direct summand of  $N$  (i.e.  $gS_{N_1} = eS_{N_1}$ ,  $e^2 = e$ ), then  $N_2$  and every direct summand of  $N_1$  have the exchange property in  $M$ . Especially, if  $N_1$  is quasi-injective,  $N_i$  has the exchange property in  $M$  for  $i = 1, 2$ , (cf. [4]).

**Proof.** Let  $M = N_1 \oplus N_2$  and  $f$  be the projection of  $M$  to  $N_1$ . We take any element  $a$  in  $f\mathcal{J}'f$ . Then  $\text{Ker } (1-a) = 0$  by [1], Theorem 2 and  $\text{Im } (1-a) = \text{Im}((1-a)|_{N_1}) \oplus N_2$ . Since  $\text{Im}((1-a)|_{N_1})$  is a direct summand of  $N_1$  by the assumption,  $\text{Im}(1-a)$  is a direct summand of  $M$ . On the other hand,  $\text{Im } (1-a)$  is a dense submodule of  $M$  by [7], Theorem 2 and hence,  $M = \text{Im } (1-a)$ . Therefore,  $f-a$  is an automorphism of  $N_1$ , which implies  $f\mathcal{J}'f = f\mathcal{J}f$ . Hence,  $N_i$  has the exchange property by the theorem. The remaining part of the corollary is immediate from the above.

In Theorem 4 below, we shall show the converse of Corollary in a special case.

Remark 2. [6], Proposition 10 and [9], Theorem I are special cases of Corollary I.

It is shown in [8], Remark in p. 52 that the exchange property does not imply the locally semi-T-nilpotency. In a special case we have

PROPOSITION 3. Let  $\{M_i\}^\infty$ , be a set of completely indecomposable modules such that  $M_i$  is monomorphic, but not isomorphic to  $M_{i+1}$  (cf. [6], p. 340 and [8], Corollary 3). 1) Let  $M = \sum_1^\infty M_i = N_1 \oplus N_2$ . Then  $N_1$  has the exchange property in  $M$  if and only if either  $N_1$  or  $N_2$  is a directsum of indecomposable modules  $\{M'_i\}$  which is a semi-T-nilpotent system, (in this case, a finite directsum of  $M'_i$ ). 2) We further assume that each  $M_i$  itself is a locally T-nilpotent system and  $M = \sum M'_\alpha$ ;  $M'_\alpha \approx M_i$  for some  $i$  and  $M = N_1 \oplus N_2$ . Then we have the same statement in 1).

Proof. 1) "If part" is clear from [8], Theorem 2. We assume that  $N_1$  has the exchange property. Then  $N_i$  is in  $\mathcal{A}$ : say  $N_i = \sum_{K \in J^i} T_k^i$ , where  $T_k^i \approx M_m$  for some  $m$ . if  $J^i$  were infinite for  $i = 1, 2$ , we would have a contradiction from the assumption and [8], Lemma 2. 2) We can prove it similarly to 1).

PROPOSITION 4. - Let  $M = \sum_I M_\alpha$  and  $M_\alpha$  be isomorphic to a completely indecomposable module  $M_1$  for all  $\alpha \in I$ . Let  $M = N_1 \oplus N_2$ . Then  $N_1$  has the exchange property in  $M$  if and only if  $M$  itself is a locally T-nilpotent system or either  $N_1$  or  $N_2$  is isomorphic to a finite directsum of  $M_1$ .

Proof. It is clear from [8], Lemma 2.

COROLLARY. - Let  $P$  be a completely indecomposable and projective module and

$M = \sum_I P_\alpha$  ;  $P_\alpha \approx P$ . Let  $M = N_1 \oplus N_2$ . Then  $N_1$  has the exchange property in  $M$  if and only if either  $N_1$  or  $N_2$  is semi-perfect or equivalently  $J(N_i)$  is small in  $N_I$  for  $i = 1$  or  $2$ .

Proof. It is clear from Proposition 4 and [7], Theorem 7.

#### 4. MODULES WITH ZERO SINGULAR SUBMODULES.

In this section, we study Matlis' problem and give simpler proofs of slightly generalized results of [9], Theorems 2 and 3.

Let  $N$  be an  $R$ -module. We denote the singular submodule of  $N$  by  $Z(N)$ , namely  $Z(N) = \{n \in N, (o:n) \text{ is large in } R\}$ . The following lemma is well known and essential in this section.

LEMMA. - Let  $\{N_\alpha\}_I$  be a set of indecomposable injective modules. If  $Z(N_\alpha) = 0$  any homomorphism of  $N_\alpha$  to  $N_\alpha$ , is either isomorphic or zero. Especially, if  $Z(N_\alpha) = 0$  for all  $\alpha \in I$ ,  $\{N_\alpha\}$  is a locally  $T$ -nilpotent system.

Let  $M$  be in  $\mathcal{A}$  and  $M = N_1 \oplus N_2$ . Then we note that  $N_1$  always contains direct summands which are finite directsums of indecomposable modules (cf. [6], Corollary 1 in p. 334).

PROPOSITION 5. - Let  $M$  be in  $\mathcal{A}$  and  $M = N_1 \oplus N_2$ . We assume that for any decomposable direct summand  $T_1, T_2$  of  $N_1$  non-zero elements in  $\text{Hom}_R(T_1, T_2)$  are always isomorphic. Then  $N_i$  is in  $\mathcal{A}$  for  $i = 1, 2$ .

Proof. Let  $\sum_J T_\beta$  be a dense submodule of  $N_1$ , where  $T_\beta$ 's are indecomposable. Every  $T_\beta$  is a direct summand of  $N_1$  by [6], Proposition 2. Hence  $\{T_\beta\}_J$  is a locally T-nilpotent system by the assumption. Therefore,  $N_1 = \sum_J T_\beta$  and  $N_2$  is in  $\mathcal{A}$  by Remark 1, [7], Proposition 2 and [8], Theorem 2.

COROLLARY. - ([16], Theorem 4). Let  $\{M_\alpha\}_I$  be a family of indecomposable injective modules and  $M = \sum_I M_\alpha$ . If  $M = N_1 \oplus N_2$  and  $Z(N_1) = 0$ , then  $N_i$  is in  $\mathcal{A}$  for  $i = 1, 2$  (cf. [9], Theorem 3).

Proof. It is clear from Lemma and Proposition 5.

THEOREM 4. - Let  $\{E_\alpha\}_I$  be a family of injective and indecomposable modules and  $E = \sum_I E_\alpha$ . Then the following statements are equivalent.

- 1)  $\{E_\alpha\}_I$  is a locally semi-T-nilpotent system
- 2) Any extension module  $M$ , in  $\mathcal{A}$ , of  $E$  contains  $E$  as a direct summand
- 3) There are no proper and essential extensions, in  $\mathcal{A}$ , of  $E$  and
- 4)  $\text{Im } g$  is a direct summand of  $E$  for any monomorphism  $g$  in  $S_E$ .

Proof. 4)  $\longrightarrow$  1) is proved by Theorem 1 and Corollary 1 to Theorem 3. 1)  $\longrightarrow$  4) by the assumption and Theorem 1  $\text{Im } g = \sum_K E'_\beta$ ;  $E'_\beta \approx E_\alpha$  for some  $\alpha$ . Since  $E'_\beta$  is injective,  $\text{Im } g$  is a finitely direct summand of  $M$ . On the other hand,  $\{E'_\beta\}_K$  is a locally semi-T-nilpotent system by Theorem 1. Hence,  $\text{Im } g$  is a direct summand of  $M$  from Remark 1. 1)  $\longrightarrow$  2) is clear from Theorem 1. 2)  $\longrightarrow$  3) is trivial. 3)  $\longrightarrow$  1) We assume that  $\{E_\alpha\}_I$  is not locally semi-T-nilpotent.

Then we have a sub-family  $\{E_i\}_1^\infty$  of  $\{E_\alpha\}_I$  such that there exist non-isomorphisms  $f_i: E_i \rightarrow E_{i+1}$  and an element  $x$  in  $E$  with a property :  $f_n f_{n-1} \dots f_1(x) \neq 0$  for all  $n$ . We note  $\text{Ker } f_i \neq 0$  for all  $i$ , since  $E_i$  is injective and indecomposable. Put  $E'_i = \{x_i + f_i(x_i) \mid x_i \in E_i \oplus E_{i+1}\} \subseteq \sum_1^\infty E_j$  and  $E = \sum_1^\infty E_j \oplus E_0$ . Then  $E_i \cap (\sum_j E'_j) \supseteq \text{Ker } f_i \neq 0$ . Hence,  $\sum_j E'_j \oplus E_0$  is essential in  $E$ . It is clear  $x \in E - (\sum_j E'_j \oplus E_0)$ . Let  $E^*$  be an injective hull of  $E$ . Then we can extend an isomorphism  $\phi$  of  $\sum_j E'_j \oplus E_0$  onto  $E$  to a monomorphism of  $E^*$ . Hence  $\phi(\sum_j E'_j \oplus E_0) = E \subseteq \phi(E) = \sum_I \phi(E_\alpha) \in \mathcal{A}$ .

COROLLARY 1. - We assume further in Theorem 4 that all  $E_\alpha$  are noetherian.

Then we obtain all of 1)  $\sim$  4) in Theorem 4.

Proof. Let  $E_1, E_2$  and  $E_3$  be injective, indecomposable and noetherian modules, and  $f_i: E_i \rightarrow E_{i+1}$  non-isomorphisms. Then  $\text{Ker } f_i \neq 0$  and  $\text{Im } f_1 \cap \text{Ker } f_2 \neq 0$  if  $f_1 \neq 0$ , since  $E_2$  is uniform. Hence,  $\text{Ker } f_1 \subseteq \text{Ker } f_2 f_1$  if  $f_1 \neq 0$ . Therefore,  $\{E_\alpha\}_I$  is a T-nilpotent system.

COROLLARY 2. Let  $M$  be in  $\mathcal{A}$  and  $L$  a submodule of  $M$ . We assume that  $Z(L) = 0$  and  $L$  is a directsum of injective modules. Then  $L$  is a direct summand of  $M$  (cf. [9], Theorem 2).

Proof. Since every injective submodule of  $M$  is in  $\mathcal{A}$  by Corollary to Theorem 3.  $L$  is a direct summand of  $M$  by Lemma and Theorem 4.

Remark 3. Let  $\{E_\alpha\}_I$  be a family of indecomposable, injective modules. In

general,  $\{E_\alpha\}_I$  is not locally semi-T-nilpotent and hence,  $\Sigma \oplus_I E_\alpha$  is not quasi-injective. Furthermore, even if all  $E_\alpha$  is not injective. If either  $E = \Sigma \oplus_I E_\alpha$  is (quasi-)injective or  $Z(E) = 0$ ,  $\{E_\alpha\}_I$  is a locally semi-T-nilpotent system. However, the converse is not true as follows. Let  $K$  be a commutative, local Frobenius ring with  $Z(K) \neq 0$ ,  $M_{21} = \sum_1^\infty \oplus K_i$ ;  $K_i \approx K$ ,  $M_{32} = \text{Hom}_K(M_{21}, K) = \prod K_i$  and  $M_{31} = K$ . Then

$$R = \begin{pmatrix} K & & 0 \\ M_{21} & K & \\ M_{31} & M_{32} & K \end{pmatrix} = T(K, K, K; M_{ij})$$

is a ring (cf. [5], p. 23). Put  $e_1 = T(1, 0, 0; 0)$  and  $e_3 = T(0, 0, 1; 0)$ , then  $\text{Hom}_K(Re_1, K) \approx e_3R$  is  $R$ -injective. Since  $e_3Re_3 = K$  is local,  $e_3R$  is indecomposable and  $e_3R$  itself is a T-nilpotent system. It is clear that  $Z(e_3R) \supseteq (Z(K), 0, 0) \neq 0$ . Put  $S_i = (M_{31}, \prod_{j=i}^\infty K_j, K) \subseteq e_3R$ . Then  $(0: S_1)_R \subsetneq (0: S_j)_R$  if  $i < j$ . Hence,  $\sum_1^\infty \oplus e_3R$  is not injective by [3]. However, I do not know whether  $\sum_1^\infty \oplus e_3R$  is quasi-injective or not. If we can construct a self-injective and perfect, but not  $\Sigma$ -injective ring  $S$  (or a self-injective and local, but not  $\Sigma$ -injective ring with  $Z(S) = 0$ ), then  $\Sigma \oplus_I S$  is not quasi-injective but  $\{S\}_I$  is a T-nilpotent system.

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