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SOME PROPERTIES OF THE BORNOLOGICAL SPACES

Manuel Valdivia

N. Bourbaki, [1, p.35] notices that it is not known if every bornological barrelled space is ultrabornological. In [2] we proved that if E is the topological product of an infinite family of bornological barrelled spaces, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces, which are not ultrabornological. We also gave some examples of barrelled normable non-ultrabornological spaces. In [3] we gave an example of a bornological barrelled space E, such that E is not inductive limit of Baire espaces. We prove in this article that the example given in [3] is not inductive limit of barrelled normed spaces. Other result given here is the following: If E and F are two infinite dimensional Banach spaces, such that the conjugate of F is separable, there exists a family in E of precompact absolutely convex sets $\{B_s : s\in S\}$ such that, for every $s\in S$, $E_{B_s}=F$, $E_{\overline{B}_s}$ is the second conjugate of F, being $\overline{B}_s$ the closure of B_s in E, and E is the inductive limit of the family $\{E_{B_s} : s\in S\}$.

The vector spaces we use here are defined over the field K of the real or complex numbers. We mean under "space" a separated locally convex space. If T is the topology of a space E we shall write $E[T]$ sometimes instead of E. If A is a bounded absolutely convex set of E, then E_A denotes the normed space over the linear hull of A, with the norm associated to A. We say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy (convergent) sequence for the Mackey convergence in E if there is a bounded closed absolutely convex set B in E such that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy (convergent) sequence in E_B. We say that E is locally complete if every Cauchy sequence for the Mackey convergence in E is convergent in E. We represent by $\hat{E}$ the completion of E. If $\mathcal{F}$ is the family of all locally complete subspaces of $\hat{E}$, which contain E, its intersection is a locally complete space $\tilde{E}$ and we call it the locally completion of E. We say that a subspace E of F is locally dense if, for every $x\in F$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E, which converges to x in the Mackey sense. We say that a space E is a Mackey space if it is provided with the Mackey topology.

We shall need the following result ,[2]: a) <u>Let E be a locally dense subspace of a space F. If E is bornological, then F is bornological</u>.

<u>THEOREM 1</u>. <u>If E is a bornological space, then $\tilde{E}$ is bornological</u>.

<u>Proof</u>: Let $\{E_i : i\in I\}$ be the family of all bornological subspaces of $\tilde{E}$, containing E. We show now that $\{E_i : i\in I\}$ with the inclusion relation is an inductive ordered set. Indeed, let $\{E_j : j\in J\}$ be a totally ordered subfamily of $\{E_i : i\in I\}$ and we set $F=\cup\{E_j : j\in J\}$. Since F is a Mackey space and E_j is dense in F, for every $j\in J$, we have that F is the inductive limit of the family $\{E_j : j\in J\}$ and, therefore, F is bornological. By Zorn's lemma, there exists a bornological subspace G of $\tilde{E}$

containing E, which is maximal, refered to the family $\{E_i : i\in I\}$. We shall see now that G coincides with $\tilde{E}$. Indeed, if $G\neq\tilde{E}$, G is not locally complete and, therefore, there exists a vector x in $\tilde{E}$, $x\notin G$, such that if M is the linear hull of $G\cup\{x\}$, then G is locally dense in M and, by result a), we have that M is bornological, which contradicts the maximality of G in $\{E_i : i\in I\}$. q.e.d.

For the proof of Theorem 2 we shall need the following results: b) Let E be a barrelled space. If F is a subspace of E, of finite codimension, then F is barrelled, [4]. c) If E is a metrizable barrelled space, then it is not the union of an increasing sequence of closed, nowhere dense and absolutely convex sets, [5]. d) Let E and F be two spaces so that F is a Pták space. If u is an almost continuous linear mapping from E into F and the graph of u is closed, then u is continuous, [6, p.302].

THEOREM 2. Let E be a non-complete (LB)-space. Let x_o be a point of $\hat{E}$ which is not in E. If G is the linear hull of $E\cup\{x_o\}$ with the by $\hat{E}$ induced topology, then G is not the locally convex hull of barrelled normed spaces.

Proof: Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of subspaces of E such that $\cup\{E_n : n=1,2,..\}=E$. Let T_n be a topology on E_n finer than the topology of E_n, such that $E_n[T_n]$ is a Banach space and E is the inductive limit of $\{E_n[T_n]\}_{n=1}^{\infty}$.* If B_n is the unit ball in E_n let $\{\lambda_n\}_{n=1}^{\infty}$ be a strict increasing sequence of positive numbers such that $\lambda_n B_n \subset \lambda_{n+1}B_{n+1}$ and $\cup\{\lambda_n B_n : n=1,2,..\}=E$. We suppose that G is the locally convex hull of the family $\{G_i : i\in I\}$ of normed barrelled spaces. Since E is dense in G there exists an element i_o in I such that $G_{i_o}\cap E$ is dense in G_{i_o}. Let j be the injective mapping of $G_{i_o}\cap E$, with the by G_{i_o} induced topology in E. If A_n is the closure of $j^{-1}(\lambda_n B_n)$ in $G_{i_o}\cap E$, then $\cup\{A_n : n=1,2,..\}=G_{i_o}\cap E$. According to result b), we have that $G_{i_o}\cap E$ is barrelled and, by result c), there exists a positive integer n_o such that A_{n_o} is a neighbourhood of the origin in $G_{i_o}\cap E$. Let L be the linear hull of $j^{-1}(\lambda_{n_o}B_{n_o})$ with the by G_{i_o} induced topology. Let k be the canonical mapping of L into E_{n_o}. Since k, considered from L into E is the restriction of j to L we have that k is continuous from L into E and, therefore, the graph of k is closed in $L\times E_{n_o}$. Obviously, k is almost continuous and, according to result d), u is continuous since E_{n_o} is a Banach space. Since L is dense in $G_{i_o}\cap E$ we have that L is dense in G_{i_o} and, therefore, we can take a point $y_o\in G_{i_o}$, $y_o\notin G_{i_o}\cap E$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in L converging to y_o in G_{i_o}. If u is the canonical mapping of G_{i_o} in G, we have that

$$\lim_{n\to\infty} u(y_n) = \lim_{n\to\infty} k(y_n) = y_o\notin E$$

and since E_{n_o} is a Banach space it results that

$$\lim_{n\to\infty} u(y_n) = \lim_{n\to\infty} k(y_n) = y_o\in E_{n_o}\subset E$$

which is a contradiction. q.e.d.

* We put now E_n instead $E_n[T_n]$.

In [7, p.434] G. Kothe gives an example of a non-complete (LB)-space which is defined by a sequence of Banach spaces such that there exists a bounded set A in E which is not a subset of E_n, n=1,2,.. If B is the closed absolutely convex hull of A, then E_B is not a Banach space, (see [3], proof of Theorem 2), and, therefore, E is not locally complete. We take $x\notin E$, $x\in\hat{E}$. If G is the linear hull of $E \cup \{x_o\}$, with the by $\hat{E}$ induced topology, then G is barrelled and, by result a), G is bornological and, according to Theorem 2, G is not inductive limit of normed barrelled spaces.

In [8] Markushevich proves the existence of a generalized basis for every Banach separable space of infinite dimension, (see also [9] p. 116). In the following Lemma we give a proof of the existence of basis of Markushevich, which is valid for Fréchet spaces, and we shall need it after. Given a space E we represent by E' the topological dual of E and by $\sigma(E',E)$ and $\beta(E',E)$ the weak and strong tooologies on E', respectively.

<u>LEMMA</u>. <u>Let E be a separable Fréchet space of infinite dimension. If G is a total subspace of E' $[\sigma(E',E)]$ of countable dimension, there exists a biorthogonal system $\{x_n,u_n\}_{n=1}^{\infty}$ for E, such that $\{x_n\}$ is total in E and the linear hull of $\{u_n\}_{n=1}^{\infty}$ coincides with G.</u>

<u>Proof</u>: In E let $\{y_n\}_{n=1}^{\infty}$ be a convergent to the origin total sequence and let B be the closed absolutely convex hull of this sequence. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of non-zero elements of K, such that $z_n=\lambda_n y_n$, $||z_n||\leqslant 1/n$, being $||.||$ the norm in the Banach space E_B. Let f be the mapping of ℓ^2 into E_B such that if $\{a_n\}_{n=1}^{\infty}\in\ell^2$, then

$$f(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n z_n.$$

Since

$$||f(\{a_n\}_{n=1}^{\infty})|| \leqslant \sum_{n=1}^{\infty} |a_n|.||z_n|| \leqslant (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}(\sum_{n=1}^{\infty}||z_n||^2)^{1/2}$$

$$\leqslant (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}(\sum_{n=1}^{\infty}1/n^2)^{1/2}$$

we have that f is well defined and it is continuous. Let U be the closed unit ball in ℓ^2 and we set f(U)=A. Then the Hilbert space $\ell^2/f^{-1}(0)$ can be identified with E_A and, therefore, E_A is a Hilbert space. Obviously, E_A is total in E and thus E' is weakly dense in $(E_A)'$. Let $\{v_n\}_{n=1}^{\infty}$ be a Hamel basis in G. In $(E_A)'$ $[\beta((E_A)',E_A)]$ we apply the orthonormalization method od Gram-Schmidt and we obtain an orthonormal sequence $\{u_n\}_{n=1}^{\infty}$ from $\{v_n\}_{n=1}^{\infty}$. If $\{x_n\}_{n=1}^{\infty}$ is the sequence in E_A such that $\langle u_n,x_n\rangle=1$, $\langle u_n,x_m\rangle=0$, $n\neq m$, n, m=1,2,.., then the biorthogonal system $\{x_n,u_n\}_{n=1}^{\infty}$ verifies that $\{x_n\}_{n=1}^{\infty}$ is total is E and $\{u_n\}_{n=1}$ has G as linear hull.

q.e.d.

<u>THEOREM 3</u>. <u>Let E and F be Banach spaces of infinite dimension. If $F'[\beta(F',F)]$ is</u>

separable, there exists in E a family $\{B_s : s \in S\}$ of precompact absolutely convex sets such that $\bigcup\{B_s : s \in S\} = E$, $E_{B_s} = F$ and $E_{\overline{B}_s}$ is the second conjugate of F for every $s \in S$, and E is the locally convex hull of the family $\{E_{B_s} : s \in S\}$.

Proof: We shall use the symbol $||\cdot||$ for the norm of every normed space. By the **Lemma** we can choose a Markushevich basis $\{x_n, u_n\}_{n=1}^{\infty}$ for F such that $\{u_n\}_{n=1}^{\infty}$ is total in $F'[\beta(F',F)]$ and $||x_n|| = 1$, $n=1,2,\ldots$. Let $\{\lambda_n\}_{n=1}$ be a strictly increasing sequence of positive integers such that $||u_n|| \leqslant \lambda_n$. Let S be the family of all the sequences in E such that if $s = \{y_n\}_{n=1}^{\infty} \in S$ then $\{y_n\}_{n=1}^{\infty}$ is topologically free and $||y_n|| \leqslant 2^{-n}\lambda_n^{-2}$. Let f_s be the mapping of F into E such that if $x \in F$ then

$$f_s(x) = \sum_{n=1}^{\infty} \langle u_n, x\rangle y_n.$$

Since

$$||f_s(x)|| \leqslant \sum_{n=1}^{\infty} ||u_n|| \cdot ||x|| \cdot ||y_n|| \leqslant ||x|| \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n}$$

we get that f_s is well defined and it is continuous. If $x \neq 0$ there exists a positive integer n_o such that $\langle u_{n_o}, x\rangle \neq 0$ and there exists a $w \in E'$ such that $\langle w, y_{n_o}\rangle = 1$, $\langle w, y_n\rangle = 0$, $n \neq n_o$, since $\{y_n\}_{n=1}^{\infty}$ is topologically free. Then

$$\langle f_s(x), w\rangle = \sum_{n=1}^{\infty} \langle u_n, x\rangle . \langle w, y_n\rangle = \langle u_{n_o}, x\rangle . \langle w, y_{n_o}\rangle = \langle u_{n_o}, x\rangle \neq 0$$

and, thus, f_s is injective. If U is the closed unit ball in F let $f_s(U) = B_s$. We shall see that B_s is precompact. Indeed, given $x \in U$ it results that

$$f_s(x) = \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} \langle u_n, x\rangle \lambda_n 2^n y_n = \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} \langle u_n, x\rangle z_n,$$

$$||z_n|| = \lambda_n 2^n ||y_n|| \leqslant \lambda_n 2^n . 2^{-n} \lambda_n^{-2} = \lambda_n^{-1}$$

$$\sum_{n=1}^{\infty} |\lambda_n^{-1} 2^{-n} \langle u_n, x\rangle| \leqslant \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} ||u_n|| \cdot ||x|| \leqslant \sum_{n=1}^{\infty} 2^{-n} = 1$$

and, therefore, $f_s(x)$ is in the closed absolutely convex hull of the sequence $\{z_n\}_{n=1}^{\infty}$. Since $||z_n|| = \lambda_n^{-1}$, $\{z_n\}_{n=1}^{\infty}$ converges to the origin in E and, thus, B_s is precompact. Obviously F coincides with E_{B_s}. We shall show that E is the locally convex hull of the family $\{E_{B_s} : s \in S\}$. In E let V be an absorbent absolutely convex set such that $V \cap E_{B_s}$ is a neighbourhood of the origin in E_{B_s} for every $s \in S$. We suppose that V is not a neighbourhood of the origin in E. We take $t_1 \in E$, $w_1 \in E'$ such that $||t_1|| \leqslant 2^{-1}\lambda_1^{-3}$, $\langle w_1, t_1\rangle = 1$. Since $V \cap w_1^{-1}(0)$ is not a neighbourhood of the origin in $w^{-1}(0)$, considered as subspace of E, we take $t_2 \in w_1^{-1}(0)$ and $w_2 \in E'$ such that $t_2 \notin V$, $||t_2|| \leqslant 2^{-2}\lambda_2^{-3}$, $\langle w_2, t_1\rangle = 0$, $\langle w_2, t_2\rangle = 1$. We suppose we have constructed $\{t_p, w_p\}_{p=1}^{n}$ and, according to the fact that $V \cap w_1^{-1}(0) \cap w_2^{-1}(0) \cap \ldots \cap w_n^{-1}(0)$ is not a neighbourhood of the origin in $w_1^{-1}(0) \cap w_2^{-1}(0) \cap \ldots \cap w_n^{-1}(0)$, we take t_{n+1} belonging to this last subspace such that $t_{n+1} \notin V$, $||t_{n+1}|| \leqslant 2^{-(n+1)}\lambda_{n+1}^{-3}$, $w_{n+1} \in E'$ such that $\langle w_{n+1}, t_p\rangle = 0$, $p=1,2,\ldots,n$, $\langle w_{n+1}, t_{n+1}\rangle = 1$. If $r_n = \lambda_n t_n$, then $||r_n|| \leqslant 2^{-n}\lambda_n^{-2}$ and $\{r_n\}_{n=1}^{\infty}$ is topologically free since $\langle w_n, r_n\rangle = \lambda_n$, $\langle w_m, r_n\rangle = 0$,

$m \neq n$, m, n=1,2,.., and, thus, $r=\{r_n\}_{n=1}^{\infty} \varepsilon\, S$ $V \cap E_{B_r}$ is a neighbourhood of the origin in E_{B_r} and $r_n=f_r(x_n)\varepsilon B_r$ and, therefore, $\{t_n\}=\lambda_n^{-1} r_n$ converges to the origin in E_{B_r}, which is contradiction with $t_n \notin V$, $n \geqslant 2$. From the way we chose t_1 it results that $\bigcup\{B_s; s\varepsilon S\}=E$. We shall show now that $E_{\overline{B}_s}$ is the second conjugate of F. If F" is the bidual of F and $s=\{y_n\}_{n=1}^{\infty}\varepsilon S$, let g_s be the mapping from F" into E such that if $x \varepsilon F''$

$$g_s(x)=\sum_{n=1}^{\infty}\langle u_n,x\rangle y_n .$$

Since $\{y_n\}_{n=1}^{\infty}$ is topologically free and $\{u_n\}_{n=1}^{\infty}$ is total in $F'[\beta(F',F)]$ and also in $F'[\sigma(F',F'')]$, following the same patterns as we did for f_s we prove that g_s is injective. Obviously, f_s is the restriction of g_s to F. If U^* is the closure of U in $F''[\sigma(F'',F')]$ and we prove that $g_s(U^*)=\overline{B}_s$ then $E_{\overline{B}_s}$ is the second conjugate of F. Let z be a point of U^*. In $F''[\sigma(F'',F')]$, U^* is metrizable and U is dense in U^* and, therefore, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in U $\sigma(F'',F)$-converging to z. Given an arbitrary $\epsilon>0$, we can find two positive integers n_o and p_o such that

$$\sum_{n=n_o+1}^{\infty} 2^{-n+1}\lambda_n^{-1} < \epsilon/2$$

$$|\langle u_n, z-z_p\rangle| < \epsilon \Big/ \sum_{n=1}^{n_o} 2^{-n+1}\lambda_n^{-2}, \quad n=1,2,\ldots,n_o, \quad p \geqslant p_o .$$

Then, it follows for $p \geqslant p_o$

$$|g_s(z)-g_s(z_p)| = \Big|\sum_{n=1}^{\infty}\langle u_n, z-z_p\rangle y_n\Big|$$

$$\leqslant \sum_{n=1}^{n_o}|\langle u_n,z-z_p\rangle|.||y_n||+\sum_{n=n_o+1}^{\infty}|\langle u_n,z-z_p\rangle|.||y_n||$$

$$\leqslant \sum_{n=1}^{n_o}\Big(\epsilon\,||y_n|| \Big/ \sum_{n=1}^{n_o}2^{-n+1}\lambda_n^{-2}\Big)+\sum_{n=n_o+1}^{\infty}||u_n||.||z-z_p||.||y_n||$$

$$\leqslant \epsilon/2+\sum_{n=n_o+1}^{\infty}\lambda_n .2.2^{-n}\lambda_n^{-2} < \epsilon ,$$

and since $g_s(z_p)=f_s(z_p)\varepsilon B_s$ we have that $g_s(z)\varepsilon\overline{B}_s$. On the other hand, if x' is a point of $\overline{B}_s$ we can find a sequence $\{x'_n\}_{n=1}^{\infty}$ in B_s converging to x'. If y'_n is the point of U such that $f_s(y'_n)=x'_n$, then, since U^* is $\sigma(F'',F)')$-compact we can choose a subsequence $\{y'_{n_p}\}_{p=1}^{\infty}$ of $\{y'_n\}_{n=1}^{\infty}$ $\sigma(F'',F')$-converging to a point y' of U^*. Then $g_s(y')=x'$ and, therefore $g_s(U^*)=\overline{B}_s$. q.e.d.

In [10] we prove the following result : e) _Let_ F _be a sequentially complete infinite-dimensional space with the following properties_: 1) _There is in_ F _a bounded countable total set_ . 2) _There is in_ $F'[\sigma(F',F)]$ _a countable total set which is equicontinuous in_ F. 3) _If_ u _is an injective linear mapping from_ F _into_ F, _with closed graph, then_ u _is continuous_. _Then if_ E _is an infinite-dimensional Banach space it results that_ E _is the inductive limit of a family of spaces equal to_ F_b _spanning_ E.

According to the Lemma and using the same kind of proof as we did for result e) it is possible to prove Theorem 4.

THEOREM 4. Let E and F be two infinite-dimensional Banach spaces. If $F'[\beta(F',F)]$ is separable, there exists in E a saturated family $\{B_s : s\epsilon S\}$, directed by inclusion, of precompact absolutely convex sets such that $\bigcup\{B_s : s\epsilon S\}=E$, E is the locally convex hull of the family $\{E_{B_s} : s\epsilon S\}$, $E_{B_s}=F$ and $E_{\bar{B}_s}$ is the second conjugate of F for every $s\epsilon S$.

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