PUBLICATIONS DU DÉPARTEMENT DE MATHÉMATIQUES DE LYON

ALAIN BOUVIER

Survey on Locally Factorial Krull Domains

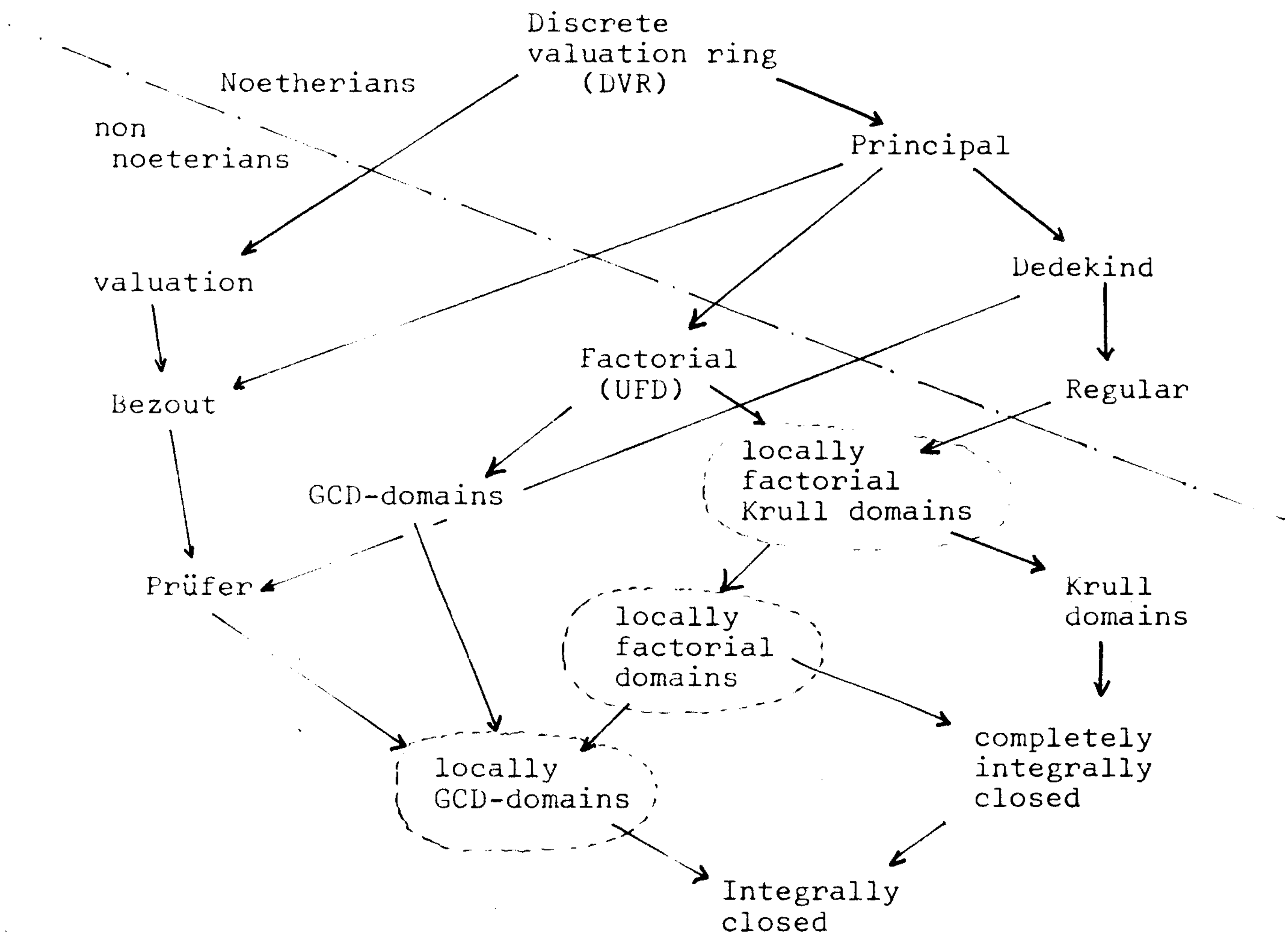
Publications du Département de Mathématiques de Lyon, 1980, tome 17, fascicule 1
, p. 1-33
<http://www.numdam.org/item?id=PDML_1980__17_1_1_0>

Publications du
Département de
Mathématiques
Lyon 1980 t.17-1

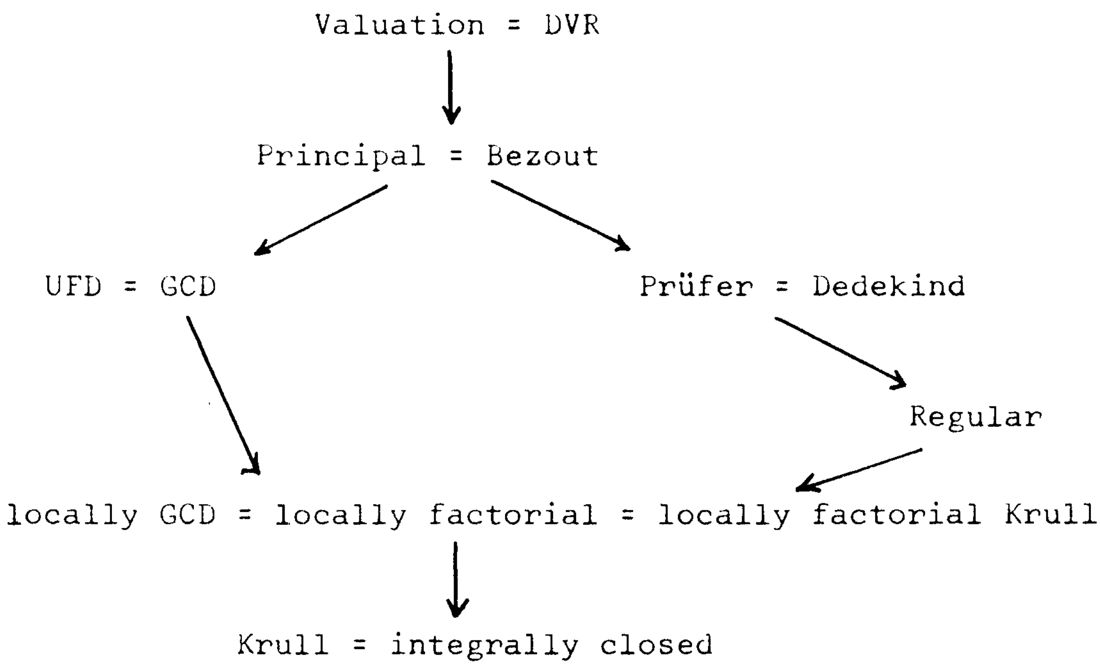
SURVEY ON LOCALLY FACTORIAL KRULL DOMAINS

Alain Bouvier

In the following diagram, we consider some classes of rings which have been introduced to study integrally closed domains:



Assuming furthermore that the domains are noetherian the above diagram collapses and becomes the diagram below.



In this note we shall prove some interesting properties of locally factorial Krull domains; namely:

1) They are the Krull domains such that every divisorial ideal is invertible; in comparison, Dedekind domain have the property that non-zero ideals are invertible and UFD's that every divisorial is principal.

2) For a locally factorial Krull domain A , the Picard group $\mathrm{Pic}(A)$ is equal to the class group $c\ell(A)$; thus the quotient group $c\ell(A)_{|\mathrm{Pic}(A)}$ indicates how far the Krull domain A is from being a locally factorial domain.

3) In order to prove a ring A is factorial it is convenient to prove that A is locally factorial and satisfies few more conditions (for instance A is noetherian on Pic(A) = 0). Several of these results, like in [GR], are proved for noetherian domains; what is yet true without this hypothesis?

Before the study of locally factorial Krull domains (§3) we indicate a few properties of domains satisfying local conditions: locally GCD-domains and locally UFD. We end this lecture by posing some questions related to this topic.

I want to thank D.D. Anderson for the stimulating letters he sent to me, P. Ribenboim and A. Geramita for their help while I was preparing this lecture.

§1. Preliminaries

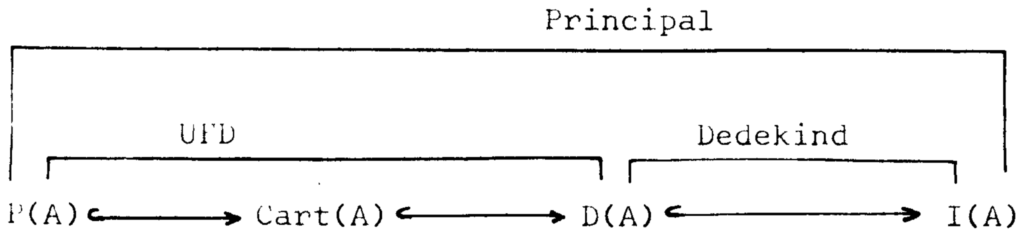
To make this paper easy to read, we summarize in this first paragraph the terminology and notations used in the following ones. More details can be found in [B], [BG], or [F].

(1-1) Let A be a domain. We write $K = \mathrm{Frac}(A)$ its field of quotients, $\mathrm{Spec}(A)$ its prime spectrum, $\mathrm{Max}(A)$ its maximal spectrum and $X^{(1)}(A)$ the set of height one prime ideals of A .

A non null (fractional) ideal is called <u>divisorial</u> or a <u>v-ideal</u> if it is the intersection of any family of principal ideals. Let $D(A)$ be the set of such ideals and $I(A)$ the monoid of non null ideals in A . We write $\mathrm{Div}(A)$ for the set of <u>divisors</u> of A ; that is, the quotient of $I(A)$ by the <u>Artin congruence</u> defined by $I \equiv J$ if and only if $A: I = A:J$. Let $\mathrm{div}: I(A) \to \mathrm{Div}(A)$ be the canonical surjection. We write $I^{-1} = A:I$ and since $I_v = (I^{-1})^{-1}$ is a v-ideal, we call it the <u>v-ideal associated</u> to I . If P is a prime, we sometimes write $P^{(n)}$ instead of $(P^n)_v$. We say I is a <u>v-ideal of finite type</u> if $I = (\sum_1^n a_i A)_v$ for some $a_i \in K$. Let $D_t(A)$ be the set of v-ideals of finite type, $P(A)$ the set of principal ideals and $\mathrm{Cart}(A)$ the <u>Cartier group</u> of invertible ideals of A; one has the following inclusion:

$$P(A) \hookrightarrow Cart(A) \hookrightarrow D_t(A) \hookrightarrow D(A) \hookrightarrow I(A)$$

A domain A is a Dedekind domain if and only if $Cart(A) = I(A)$ or if and only if $D(A) = I(A)$.



We will complete this diagram later (§3).

We say a ring A [resp. an ideal I of A] satisfies <u>locally</u> a property P if each A_P [resp. each IA_P] satisfies the property P for every $P \in Spec(A)$. For instance, a ring A is <u>locally a UFD</u> if every A_P for $P \in Spec(A)$ is an UFD. An ideal I is invertible if and only if I is finitely generated and locally principal.

(1-2) Let $(A_i)_{i \in I}$ a family of subrings of a field K . We say this <u>family satisfies (FC) ("finiteness condition")</u> if every non real element $x \in K$ is a unit in every A_i but finitely many of them.

A domain A is a <u>Krull domain</u> if there exists a family $(V_i)_{i \in I}$ of discrete valuation rings in $Frac(A)$, satisfying the finiteness condition and such that $A = \bigcap_i V_i$.

Recall that if A is a Krull domain, then $A = \bigcap_{P \varepsilon X^{(1)}(A)} A_P$, the A_P are discret valuation rings, the family $(A_P)_{P \varepsilon X^{(1)}(A)}$ satisfies (FC) and if $(V_i)_{i \varepsilon I}$ is another family of discrete valuation rings satisfying the same conditions, then for every $P \varepsilon X^{(1)}(A)$, their exists $i \varepsilon I$ such that $A_P = V_i$. The A_P are called the <u>essential valuation rings of A</u> .

If A is a Krull domain, then Div A is a free abelian group with $\{\text{div } P\}_{P \varepsilon X^{(1)}(A)}$ as a basis; the subgroup of <u>principal divisors</u> is denoted by Prin(A) , the quotient group $\text{Div}(A)|_{\text{Prin}(A)} = c\ell(A)$ is called the <u>class group of A</u> and the canonical image of Cart(A) in $c\ell(A)$ the <u>Picard group of A</u> denoted by Pic(A) . In a Dedekind domain, $c\ell(A) = \text{Pic}(A)$.

Let Y be a subset of $X^{(1)}(A)$; the ring $A_Y = \bigcap_{P \varepsilon Y} A_P$ is a Krull ring called <u>a subintersection of A</u> ; for instance one can prove every ring of quotients is a subintersection.

(1-3) Let A be a Krull ring, A_Y a subintersection, $\bar{Y}$ the set complement of Y in $X^{(1)}(A)$ and $G_p(\bar{Y})$ the group generated by the canonical image of $\bar{Y}$ in $c\ell(A)$, then there exists a canonical map $c\ell(A) \to c\ell(A_Y)$ and CLABORN [C] proved that the following sequence of groups is exact:

[CLAB] $0 \to G_p(\bar{Y}) \to c\ell(A) \to c\ell(A_Y) \to 0$

In particular, if $S^{-1}A$ is a ring of quotients of A_1 then $c\ell(A) \to c\ell(S^{-1}A)$ is surjective and its kernel is generated by the images of the height one primes which meet S .

(1-4) Special notations and terminology

An overring of a domain A is a domain B such that $A \subset B \subset \mathrm{Frac}(A)$. Following Gilmer, we write $A(x)$ the quotient ring $S^{-1}A[x]$ where S is the set of polynomials whose coefficients generate the unit ideal. If S is a multiplicatively closed set of ideals of A , then A_S will denote the S-transform of the generalized quotient ring of A with respect to S (see [ARB]).

A domain A is coherent if the intersection of two ideals finitely generated is an ideal finitely generated.

All the rings are domains; for instance the valuation rings or the Krull rings are actually domains.

§2. Local Properties

(2-1) GCD-domains

A GCD-domain is a domain A satisfying the following equivalent conditions:

i) $aA \cap bA$ is principal for any $a,b \in \mathrm{Frac}(A)$;

ii) $A: (A: aA + bA)$ is principal for any $a,b \in \mathrm{Frac}(A)$;

iii) Every finitely generated v-ideal is principal (i.e. $D_t(A) = P(A)$).

For instance, UFD's, Bezout rings and so valuation rings are GCD-domains. If A is a noetherian or a Krull domain one checks that A is a GCD-domain if and only if A is a UFD.

A GCD-domain A is integrally closed and $\mathrm{Pic}(A) = 0$. But an integrally closed domain A with $\mathrm{Pic}(A) = 0$ is not necessarily a GCD-domain. Let $f \in A[x]$; the v-ideal of A generated by the coefficients of f is a principal ideal, denoted $c(f)_v$ and called the v-content of f. If $c(f)_v = aA$, then $f = af^{\#}$ with $c(f^{\#})_v = A$. A polynomial $f \in A[x]$ such that $c(g)_v = A$ is called a v-primitive polynomial. Any $P \in \mathrm{Spec}(A[x])$ such that $P \cap A = (0)$ contains a v-primitive polynomial.

Properties of v-contents can be found in [Mc] and [T]. For instance, if A is a GCD-domain and $f \in K[x]$, then there exists $a \in c(f)^{-1}$ such that

$$fK[x] \cap A[x] = afA[x] \quad .$$

If A is a GCD-domain, the rings $A[x]$, $A(x)$ and any localizations of A are also GCD-domain.

(2-2) Locally GCD-domains

(2-2-1) Lemma. For a domain A, the following assertions are equivalent

i) A_M is a GCD-domain for any $M \in Max(A)$;

ii) A_P is a GCD-domain for any $P \in Spec(A)$;

iii) $aA \cap bA$ is locally principal for any $a,b \in A$.

Proof. i)$\Rightarrow$ii). It is enough to notice that if P is a prime and M a maximal ideal containing P then $aA_P \cap bA_P = A(A_M)_{PA_M} \cap b(A_M)_{PA_M}$ is principal. The remainder parts are clear. $\square$

Definition. Any ring A which satisfies any one of the equivalent conditions of the above lemma is called a locally GCD-domain.

For instance, GCD-domains, regular rings, and Prüfer rings are locally GCD-domains. The following result is obvious:

(2-2-2) Proposition. Let A be a locally GCD-domain; then A is integrally closed and $S^{-1}A$, $A[x]$ and $A(x)$ are locally GCD-domains.

The Picard group of a GCD-domain is null; but any commutative group G can be the Picard group of a locally GCD-domain: consider a Dedekind domain A such that Pic(A) = $c\ell(A)$ = G , [C]. Furthermore:

(2-2-3) Proposition. A locally GCD-domain is a GCD-domain if and only if

a) Pic(A) = 0 ;

b) $aA \cap bA$ is finitely generated for any $a,b \in A$.

Proof. If $aA \cap bA$ is finitely generated, then, since it is locally principal, it is invertible and so principal (Pic(A) = 0). □

(2-2-4) Corollary (Zaffrullah). Let A be a locally GCD-domain such that

a) Pic(A) = 0 ;

b) $(A_M)_{M \in MaxA}$ satisfies the (FC) .

Then, A is a GCD-domain.

Proof. By (2.2.3), it is enough to prove $aA \cap bA$ is finitely generated for any $a,b \in A$, which is nothing more than [GI; 37-3]. □

Of course, (b) is not a necessary condition for a ring to be a GCD-domain: consider for instance, k[X,Y] where k is a field.

Let A be a semi-local domain; then the following assertions are equivalent:

i) A is locally a GCD-domain;

ii) A is a GCD-domain.

(2-2-5) Remarks. Let A be a locally GCD-domain. Then:

a) A is a Krull ring if and only if

* $(A_P)_{P \in X^{(1)}(A)}$ satisfies the (FC);

* A_M is a UFD for any $M \in \mathrm{Max}(A)$.

b) If A is noetherian, then necessarily A is a Krull ring and furthermore A is a UFD if and only if $\mathrm{Pic}(A) = 0$.

A ring A is called **reflexive** [MT_1] if every submodule N , of a finitely generated torsion-free A-module M is a reflexive A-module (i.e. $N^{**} \simeq N$) .

(2-2-6) Lemma (Matlis). Let (A,M) be a local ring and $x \in \mathrm{Frac}(A)/A$ a non element; then $\mathrm{Ann}_A(x)$ is an irreducible ideal.

Proof. see [MA_1] .

(2-2-7) Corollary. Let A be a locally reflexive and locally GCD-domain; then A is a Prüfer domain.

Proof. We have to prove A_M is a valuation domain for any $M \in Max(A)$; so we can suppose A is a local, reflexive, GCD-domain. Let $a,b \in A$ and let $c \in A$ be an element such that $aA \cap bA = cA$. If $x = \frac{1}{c} + A \in Frac(A)/A$, then $Ann_A(x) = cA$; by (2-2-6), cA is irreducible and so $cA = aA$ or $cA = bA$, i.e. $aA \subset bA$ or $bA \subset aA$. □

The next proposition generalizes [GS, 11-5].

(2-2-8) Proposition. Let A be a coherent ring; the following conditions are equivalent

i) A is a locally a GCD-domain;

ii) the intersection of two non zero principal ideals of A is an invertible ideal.

Proof. (i) => (ii). Let a,b be two non zero elements of A ; then $aA \cap bA$ is finitely generated and locally principal;

(ii) => (i) $aA_M \cap bA_M = (aA \cap bA)A_M$ is principal because $aA \cap bA$ is invertible. □

(2-2-9) Notice. In [AA] D.D. Anderson and D.F. Anderson study a class of rings, called generalized GCD-domains or G-GCD-domains. Instead of $aA \cap bA$ principal, they only want to have $aA \cap bA$ invertible. So a GCD-domain is a G-GCD-domain and a G-GCD-domain is locally a GCD-domain. Connections between G-GCD-domains and other kinds of rings and properties of G-GCD-domains can be found in [AA].

(2-2-10) Problem 1. Is a locally GCD-domain of dimension 1 a Prüfer ring?

2-3 Locally factorial domains

A domain A is locally factorial (or locally UFD) if it satisfies the following equivalent conditions:

i) A_M is factorial for any $M \in Max(A)$;

ii) A_P is factorial for any $P \in Spec(A)$.

Factorial domains and regular domains are locally UFD. A locally factorial domain is not necessarily noetherian nor is it necessarily a Krull domain; a Krull domain, like $k[x,y,u,v]/(xy-uv)$, is not a locally UFD (see [BO_2]). If A is locally UFD then it is a locally GCD-domain. For any abelian group G there exists a locally factorial domain A such that $Pic(A) = c\ell(A) = G$.

(2-3-1) Proposition. Let A be a locally factorial domain, then

a) A is an intersection of discrete valuation rings;

b) A is completely integrally closed;

c) $S^{-1}A$, $A[x]$ and $A(x)$ are locally factorial domains;

d) Any non zero prime contains a height one prime;

e) If I is a finitely generated ideal which is locally divisorial; then I is invertible.

Proof. a) Because $A = \bigcap_{M \in Max(A)} A_M = \bigcap_{P \in X^{(1)}(A)} A_P$.

b) and c) are clear.

d) If $Q \in Spec(A)$, then QA_Q contains a height one prime PA_Q and so Q contains the height one prime P .

e) Let $M \in Max(A)$; if $I \not\subset M$ then $IA_M = A_M$; if $I \subset M$, then IA_M , which is divisorial in the UFD A_M , is principal. So I is locally principal. □

(2-3-2) Proposition. Let A be a locally factorial domain, then

a) A is a Krull ring if and only if every principal non zero ideal has finitely many minimal primes.

b) If A is noetherian or if $(A_M)_{M \in Max(A)}$ satisfies the (FC), then A is a Krull ring.

Proof. a) Follows (2-3-1a) and (b) is a consequence of (a). □

Krull rings which are locally UFD will be studies in §3.

(2-3-3) Proposition (Matlis). Let A be a locally factorial domain; then A is a Dedekind domain if and only if A is reflexive.

Proof. * If A is a Dedkind domain, then A is reflexive [MT_2, Th. 40].

* Suppose A is a reflexive domain; so for any $M \in Max(A)$, the ring A_M is reflexive [MT_2, Th. 31] and so, by (2-2-2), A_M is a valuation ring which proves A is a Prüfer domain. We have only to prove that A is a noetherian ring.

Let $M \in Max(A)$; the valuation ring A_M is a UFD and so a discrete valuation ring. Thus $\dim A = 1$. Furthermore, A_M is reflexive and integrally closed; by [MA_1, 3-5], M is finitely generated. By Cohen's theorem, A is a noetherian ring. □

(2-3-4) Proposition. Let A be locally factorial and suppose $Pic(A) = 0$. The following assertions are equivalent.

i) A is a UFD ;

ii) A is a Krull ring and $aA \cap bA$ is finitely generated for any $a,b \in A$.

iii) The height one primes of A are finitely generated.

Proof. (ii) => (iii). In a Krull ring, any height one prime P is divisorial, and so is the intersection of two principal ideals and hence finitely generated.

(iii) => (i). Let $Q \in Spec(A)$; then Q contains a height one prime P by (2-3-1-d). Such a P is finitely generated and locally divisorial, so it is invertible (2-3-1-c) and then principal because $Pic(A) = 0$. Any domain A in which every prime contains a principal prime in a UFD (see [K]). □

(2-3-4) Corollary. Let A be locally factorial with $Pic(A) = 0$. If one of the following properties is satisfied then A is a UFD :

a) A is noetherian;

b) A is a coherent Krull domain;

c) (A_M) satisfies the (FC) and $aA \cap bA$ is finitely generated for any $a,b \in A$;

d) $(A_P)_{P \in X^{(1)}(A)}$ satisfies the (FC) and $aA \cap bA$ is finitely generated for any $a,b \in A$.

Of course we could add several other sufficent conditions. Let us notice that the last one (d) is also necessary.

(2-3-5) Notes. ✱ Let A be a regular domain, A is a UFD if and only if $Pic(A) = 0$

✱ Let $A = \bigoplus_{n \geq 0} A_n$ be a noetherian graded Krull ring where A_0 is a field. Then A is factorial if and only if A is locally factorial, since if $M = \bigoplus_{n \geq 1} A_n$, then [BO2] $c\ell(A_M) = c\ell(A)$.

✱ In view of some of the results in this section it would be interesting to know the answer to the following questions asked by I. Papick: What are the coherent Krull domains?

* We don't know examples of a locally factorial domain A with Pic(A) = 0 and which is not a UFD.

§3. Locally Factorial Krull Domains

Like Dedekind domains or Prüfer domains, the locally factorial Krull domains can be characterized by many different conditions. Almost all of them can be found in [A_2], [AAJ], [AM], [GI] and [L] where the locally Krull domains are called <u>π-domains</u>.

(3-1) Examples of locally factorial Krull domains.

Any factorial domain and any regular domain is a locally factorial Krull domain.

Let A be a locally factorial Krull domain; then $A[x_i]_{i\in I}$ is a locally factorial Krull domain.

For any commutative group G and any $n \geq 1$ there exists a locally factorial Krull domain A such that $\dim A = n$ and $c\ell(A) = G$. Indeed: if $n = 1$, this is nothing more than Claborn's theorem [C]; if $n \geq 1$ we have only to consider $A = R[x_1, x_2, \ldots, x_n]$ (on $R[x_1, x_2, \ldots]$ if n is infinite) where R is a Dedekind domain with $c\ell(R) = G$.

(3-2) <u>Proposition</u>. Let A be domain; the following assertions are equivalent:

i) A is a locally factorial Krull domain;

ii) Every divisorial ideal of A is uniquely a product of height one prime ideals of A ;

iii) Every non zero prime ideal of A contains an invertible prime ideal;

iv) A is a locally factorial ring in which the height one primes are finitely generated;

v) A is a Krull ring and the height one primes are invertible;

vi) A is a Krull ring and $\mathrm{Pic}(A) = c\ell(A)$;

vii) A is a locally GCD-Krull ring;

viii) A is a Krull ring and the product of two divisorial ideals is divisorial;

ix) $A(x)$ is factorial.

<u>Proof</u>. (i) => (ii) Let I be a divisorial ideal of A and let us write $I = A:(A:\Pi P_i^{e_i})$ in the Krull ring A. Let M be a maximal ideal; then $IA_M = A_M:(A_M:\Pi P_i^{e_i} A_M)$. Because A_M is factorial $IA_M = (\Pi P_i^{e_i})A_M$ and then $I = \Pi P_i^{e_i}$. Of course, such a product is unique.

(ii) => (iii) is immediate.

(iii) => (iv) Note that every height one prime is invertible and so finitely generated. Now, let $M \in \mathrm{Max}[A]$ and $PA_M \in \mathrm{Spec}(A_M)$; then PA_M contains an invertible prime QA_M which is principal. So, every prime in A_M contains a principal prime, thus A_M is factorial.

(iv) => (v). By (2-3-1-b) $A = \bigcap_{P \in X^{(1)}(A)} A_P$ and (2-3-1-e) the height one prime are invertible. So we only have to prove that the family $(A_P)_{P \in X^{(1)}(A)}$ satisfies (FC). Let $B = A_S$ where S is the ideal system generated by the height one primes and QB a prime

B . By (2-3-1-d), Q contains a height one prime P ; but P is invertible so $PB = B$ and a fortiori $QB = B$. That means $B = K$. Now let x be a non zero element in A ; there exists height one primes $P_1, \ldots, P_n$ and integres e_i such that $P_1^{e_1} \ldots P_n^{e_n} \subset xA$. So the set of height one primes containing x is finite.

(v) => (vi) Let I be a divisorial ideal, there are height one primes $P_1, \ldots, P_n$ and integers $e_1, \ldots, e_n$ such that $A: I = A: \prod_i P_i^{e_i}$. But $\prod_i P_i^{e_i}$ which is invertible is a divisorial ideal and so $I = \prod_i P_i^{e_i}$ is also an invertible ideal.

(vi) => (vii) Let M be a maximal ideal and $a, b \in A$; the ideal $aA \cap bA$ is divisorial and so is invertible; thus $aA_M \cap bA_M$ is principal.

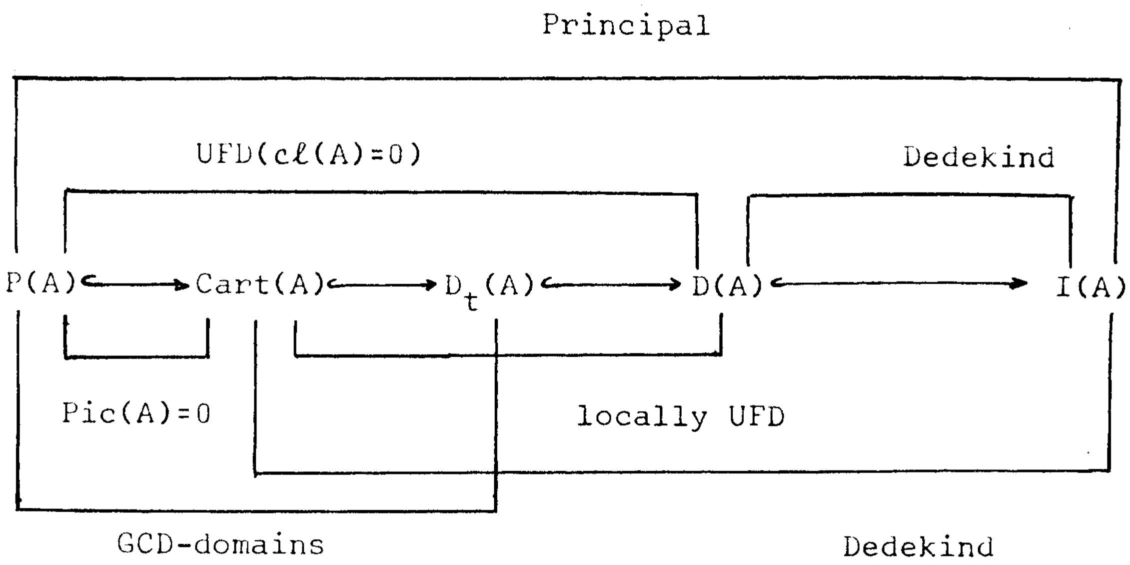
(vii) => (i) Let M be a maximal ideal; the ring A_M is a Krull ring and so it is a UFD.

(vi) => (viii) Because the set of divisorial ideals is a group.

(viii) => (ix) The ring $B = A(x)$ is a locally factorial Krull ring. Furthermore, in this ring, the invertible ideals are principal (see [A_1]). So $c\ell(B) = \mathrm{Pic}(B) = 0$.

(ix) => (v) $A \to A(x)$ is faithfully flat; so [F; 6-10] A is a Krull domain. Let P a height one prime in A ; the ideal $PA(x)$ is principal and so [F; 6-10] P is invertible. □

We can now complete the diagram given in §1.



Let A be a locally factorial Krull domain; then A is factorial [resp. Dedekind] if and only if Pic(A) = 0 [resp. dim A = 1].

(3-3) Proposition. Let A be a locally factorial Krull ring.

a) If A is not factorial then A has infinitely many height one primes which are not principal.

b) For any height one P one has $P^{(n)} = P^n$ for any n .

c) Any divisorial ideal is pure of height one and a product of height one prime ideals.

d) Any primary divisorial ideal is a power of a height one prime.

Proof. a) See [BO_1] .

b) is clear.

c) Any divisorial ideal is a product of height one primes.

d) If I is a primary divisorial ideal and if $I = \prod_1^s P_i^{e_i}$ where the P_i are height one prime, then $\sqrt{I} = P_i = P_1 \dots P_s$. If I is primary, then $s = 1$. □

(3-5) Proposition (D.D. Anderson). Let A be a locally factorial Krull domain and B an overring of A . The following assertions are equivalent:

i) B is A-flat;

ii) B is a generalized quotient ring of A ;

iii) B is a subintersection;

If B satisfies any of the above conditions, then B is a locally factorial Krull domain.

Proof. (i) => (ii) => (iii) see [ARB, Th 1.3 and 2.2].

(iii) => (i). Let $B = \bigcap_{P \in Y} A_P$ be a subintersection and S the multiplicatively closed set of ideals of A generated by the height one primes which do not belong to Y . Then $B = A_S$.

In A , any height one prime P is invertible; so PB is invertible and thus divisorial. Let $I \in S$; if $IB \neq B$ then $\operatorname{div} IB = \operatorname{div} \prod P_i^{e_i} B$ with $P_i \in Y$ and

$I \subset IB \cap A \subset P_i B \cap A = P_i$ which is impossible because $I \in S$.

Let us prove now that $B_Q = A_{Q \cap A}$ for any $Q \in \mathrm{Spec}(B)$ (see [RI]). Let $x = \frac{b}{s} \in B_Q$ and $I_1, I_2 \in S$ such that $sxI_1 \subset A$ so $sI_2 \subset A$. Then $sxI_1I_2 \subset A$. Let $P = A \cap Q$. We have $I_1I_2 \not\subset P$ (if not $B = I_1I_2B \subset PB \subset Q$) . Let b be an element in $I_1I_2 - P$. Then $sxb \in A$.

But $s \notin Q$ and $b \notin Q$ so $sb \notin Q$; since $sI_2 \subset A$ and $b \in I_1I_2$ we have $sb \in A$, so $sb \in A - P$ and $x \in A_P$. □

(3-6) For a Krull ring A, Claborn [C] has shown: every subintersection is a ring of quotients if and only if $c\ell(A)$ is a torsion group. So, if A is a Dedekind domain with a class group which is not a torsion group, then there exists a subintersection B which is not a ring of quotients ; notice that A and B are locally factorial Krull domains. Thus the condition "B is a quotient ring of A" is not equivalent to the conditions in (3-5). In [BO_2, p. 47] we asked for an explicit example. Using a construction of Eakin and Heinzer [EH] we now give such an example

(3-6-1) <u>Lemma</u> (Eakin and Heinzer). Let $n \geq 1$ be an integer. There is a Dedekind domain A_n such that $\mathbb{Z}[x] \subset A_n \subset \mathbb{Q}[x]$ and $c\ell(A_n) = \mathbb{Z}^n$.

Proof. The main idea of the proof is to choose a suitable simple transcendental extension of $\mathbb{Q}$. Let $p > 0$ be a prime integer, v the p-adic valuation with $\mathbb{Z}_{(p)}$ as the valuation ring and $\mathbb{Z}_p$ the ring of p-adic integers. Let $t \in \mathbb{Z}_p$ be a transcendental element with p-adic value one. Then, $V = \mathbb{Z}_p \cap \mathbb{Q}[t]$ is a discrete rank one and an extension of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}[t]$ with $\mathbf{F}_p$ as the residue field. Now, for any integer $k \geq 1$, the $\mathbb{Q}$-automorphisms σ_k of $\mathbb{Q}(t)$ defined by $t \mapsto p^k t$ give distinct extensions V_k of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}[t]$. Now, let $V_1,\dots,V_{n+1}$ be $n+1$ distinct rank one discrete valuation rings of $\mathbb{Q}(t)$ prolonging the p-adic valuation v and let $v_1,\dots,v_{n+1}$ be their valuations. The ring

$$A_n = V_1 \cap \dots \cap V_{n+1} \cap \mathbb{Q}[t]$$

is a Krull ring and $A_n[\frac{1}{p}] = \mathbb{Q}[t]$.

Let $(W_i)_{i \in I}$ be the family of essential valuation rings of $\mathbb{Q}[t]$; their residue fields are algebraic extensions of $\mathbb{Q}$ [B, p. 106], so none of them are equivalent to any of the V_j and thus the essential reduction rings of A_n are the V_j's and the W_i's.

Because the V_i are the essential reductions ring of A_n containing p, in order to see that A_n is a Dedekind domain (i.e., $\dim A_n = 1$), we only have to prove that the $Q_i = M_i \cap A_n$ are maximal where M_i is the maximal ideal of V_i . It follows from $\mathbb{F}_p \to A_n|Q_i \to V_i|M_i = \mathbb{F}_p$ that $A_n|Q_i$ is a field. Because $c\ell(A_n[\frac{1}{p}]) = 0$, by [CLAB], the class group $c\ell(A)$ is generated by the $c\ell(Q_i)$ for $i=1,\ldots,n+1$. If $G = \bigoplus_{i=1}^{n+1} \mathbb{Z} \operatorname{div} Q_i$, then

$$c\ell(A) = G_{|G \cap \operatorname{Prin}(A)}$$

Now, let $d = \sum_{i=1}^{n+1} \operatorname{div} Q_i$. Because $pA_n = Q_1^{d_1} \ldots Q_{n+1}^{d_{n+1}}$, we have $d_i = v_i(p) = 1$, so $pA_n = Q_1 \ldots Q_{n+1}$ and $d = \operatorname{div}(Q_1 \ldots Q_{n+1}) = \operatorname{div}(p) \in \operatorname{Prin}(A)$. That proves $d\mathbb{Z} \subset \operatorname{Prin}(A_n) \cap G$. Suppose now that $\delta = \sum a_i \operatorname{div} Q_i = \operatorname{div}(f)$ with $f \in \mathbb{Q}[t]$, be an element in $\operatorname{Prin}(A_n) \cap G$; from $fA_n = Q_1^{a_1} \ldots Q_{n+1}^{a_{n+1}}$ we get f is a unit in $\mathbb{Q}[t]$, so $f \in \mathbb{Q}^*$ and $a_i = v_i(f) = v(f)$. Now, $\delta = v(f) \sum \operatorname{div} Q_i = v(f)d \in d\mathbb{Z}$. Then

$$c\ell(A_{n+1}) = (\bigoplus_{i=1}^{n+1} \mathbb{Z} \operatorname{div} Q_i)/d\mathbb{Z} \simeq \mathbb{Z}^n \quad . \qquad \square$$

Note. By a slight modification of the proof, Eakin and Heinzer show that for any commutative finitely generated group G , one can find a Dedeking domain A such that $\mathbb{Z}[x] \subset A \subset \mathbb{Q}[x]$ and $c\ell(A) = G$.

(3-6-2) Example. With the previous notation, take

$$A = A_1 = V_1 \cap V_2 \cap \mathbb{Q}[t] \quad \text{and} \quad B = V_1 \cap \mathbb{Q}[t] \ .$$

The ring A is a Dedeking domain with class group $\mathbb{Z}$ and B is a subintersection of A. (Notice B is A-flat). If there exists a multiplicatively closed subset S of A such that $B = S^{-1}A$, then for every $s \in S$ the prime ideal Q_2 will be the only one height one prime in A containing s. So, we get $sA = Q_2^{\alpha}$ with $\alpha \geq 1$ and $\alpha c\ell(Q_2) = 0$ which is impossible in $\mathbb{Z}$. This gives the explicit example previously asked for. □

(3-7) The following result generalizes [T, 2.2].

Proposition. A locally factorial Krull ring is factorial if and only if every irreducible element generates a primary ideal.

Proof. Let A be a locally factorial Krull ring in which even irreducible elements generates a primary ideal. Let $P \in X^{(1)}(A)$; because A_P is a DVR, there exists $a \in A$ such that $PA_P = aA_P$; in the Krull ring A we can write $a = b_1 \dots b_n$ where the b_i are irreducible in A. But a is irreducible in A_P ; so $aA_P = b_1 \dots b_n A_P$ implies all the b_i but one are units in A_P . So one can suppose $aA_P = b_1A_P$ with b_1 irreducible in A ; so aA is a primary ideal and

$$P = PA_P \cap A = aA_P \cap A = aA \qquad \square$$

(3-8) Let A be a Krull domain and $G(A)$ be the group defined by

$$G(A) = c\ell(A)_{|\mathrm{Pic}(A)} \cdot$$

From (3-2), one has: <u>A is locally factorial if and only if $G(A) = 0$</u> . For instance, let

$$A = \mathbb{R}[x,y]/(x^2+y^2-1) \quad ; \quad \text{then [F, p. 113]}$$

$$c\ell(A) = \mathrm{Pic}(A) = \mathbb{Z}_{|2\mathbb{Z}}$$

and so, $G(A) = 0$.

Since the size of $G(A)$ indicates how a Krull rings is from a locally factorial ring, the study of the group $G(A)$ is natural. For instance, in [BEO] (Prop. 1-9) an example is given of a noetherian Krull domain A such that

- $G(A)$ is a torsion group;
- $c\ell(A)$ is not a torsion group;
- $\mathrm{Pic}(A) \neq c\ell(A)$.

As was noted by D.D. Anderson [A_2], for a Krull domain A, the following assertions are equivalent:

i) G(A) is a torsion group;

ii) For each rank one prime P of A, $P^{(n)}$ is invertible for some $n > 0$;

iii) If I and J are invertible ideals, then there exists an integer $n > 0$ such that $I^n \cap J^n$ is invertible.

Questions. 1) Is every abelian group G equal to G(A) for some Krull domain A ?

2) Let $A \to B$ be a morphism of Krull rings. What are the relations between G(A) and G(B) where $B = S^{-1}A$ or $A[x]$ or $A \to B$ is a flat or faithfully flat morphism?

Few properties of the group G(A) can be found in [F] and the noetherian Krull domains for which G(A) is a torsion group are studies in [BEO] and [M]. This study could be done for Krull rings.

(3-9) We end this paper with three questions from [AAJ]:

1) Let A be a Krull domain; are the following assertions equivalent:

i) A is locally factorial;

ii) $P \cap Q = PQ$ for any two distinct height one primes;

iii) The product of two height one primes is a divisorial ideal?

2) Let A be a Krull ring such that

i) Pic(A) = 0 ;

ii) $P \cap Q = PQ$ for any two distinct height one primes. Is A a UFD?

3) Let A be a locally factorial Krull ring. Are the following assertions equivalent:

i) dim A $\leq$ 2 ;

ii) $P \cap Q = PQ$ for any incomparable primes;

iii) Every ideal is a product of primary ideals?

When A is a noetherian domain, the answers to these three questions are positive (see [AAJ]).

REFERENCES

[ARB] J.T. ARNOLD and J.W.BREWER, On flat overrings, ideal transform and generalized transforms on a commutative ring, J. of Algebra 18 (1971), pp. 254-263.

[A_1] D.D. ANDERSON, Some remarks on the ring R(X) comment. Math. Univ. St. Pauli XXVI 2(1977), pp. 137-140.

[A_2] D.D. ANDERSON, π-domains, overrings and divisorial ideals. Glasgow Math. J. 19 (1978), pp. 199-203.

[AA] D.D. ANDERSON and D.F. ANDERSON, Generalized GCD-domains, to appear.

[AAJ] D.D. ANDERSON, D.F. ANDERSON and E.W. JOHNSON, Some ideal-theoretic conditions on a neotherian ring, to appear.

[AM] D.D. ANDERSON and J. MATIJEVIC, Graded π-rings, Can. J. Math. XXXI, No. 3 (1979), pp. 449-457.

[BEO] M.C. BELTRAMETTI and F.L.ODETTI, On the projectively almost-factorial varieties, Ann. Math. Pura e Applicadia 113 (1977), pp. 255-263.

[B] N. BOURBAKI, Algebre commutative, Chap. 1 à 7 Hermann.

[BO_1] A. BOUVIER, Sur les ideaux premiers de hauteur 1 d'un anneau de Krull, C.R. Acad. Sc. Paris 284 A (1977), pp. 727-729.

[BO_2] A. BOUVIER, Quatre exposés sur les anneaux de Krull, Instituto de Matematica, Roma (1977).

[BG] A. BOUVIER et G. GERMAIN, Anneaux de Krull
1. Notions de base (1977)
2. Normalité (1979)
UER de Math, Université Lyon 1, Villeurbanne, France.

[C] L. CLABORN, Every abelian group is a class group, Pac. J. Math. 18 (1966), pp. 219-222.

[EH] P. EAKIN and W. HEINZER, More non euclidean PID's and Dedekind domains with prescribed class group, Proc. Amer. Math. Soc. 40-1 (1973) pp. 66-68.

[F] R. FOSSUM, The divisor class group of a Krull domain, Springer Verlag Band 74 (1973).

[GI] R. GILMER, Multiplicative ideal theory, Marcel Dekker (1972).

[GR] S. GRECO, Singli ideale frazionari invertibili, Rend Sem. Math. Univ. Padova 36 (1966), pp. 315-333.

[GS] S. GRECO and P. SALMON, Topics in m-adic topologies, Springer Verlag (1971).

[K] I. KAPLANSKY, Commutative Rings, The University of Chicago Press, 2nd edition (1976).

[L] K. LEVITZ, A characterization of general ZPI-rings, Proc. Amer. Math. Soc 32 (1972), pp. 376-380 II Pacific J. Math. 42 (1972), pp. 147-151.

[M] P. MAROSCIA, Anelli Semifattoriali e varietá intersezioni complete.

[MA] J. MATIJEVIC, Three local conditions on a graded ring. Trans. Amer. Math. Soc. 205 (1975), pp. 275-284.

[MT_1] E. MATLIS, Reflexive domains, J. of Algebra 8 (1968), pp. 1-33.

[MT_2] E. MATLIS, Torsion-free modules, Chicago Lectures in Math., The University of Chicago Press (1972).

[Mc] S. McADAM and D.E. RUSH, Schreeier Rings, Bull. London Math. Soc. 10 (1978), pp. 77-80.

[RI] F. RICHMAN, Generalized quotient rings, Proc. AMS 16 (1965), pp. 796-799.

[RU] D.E. RUSH, Pic(R) and the R-flatness of R[x]/I, J. of Pure and Applied Alg. 14 (1979), pp. 307-310.

[T] G. TAMONE, Su una condizione di fattorialita debole e l'annullamento del gruppo di Picard, Ann. Math. pura ed App. CXII (1977), pp. 285-304.

[TA] H.T. TANG, Gauss lemma, Proc. Amer. Math. Soc. 35 (1972), pp. 372-376.

[Z] M. ZAFRULLAH, On a result of Gilmer, J. London Math. Soc. (i) 16 (1977), pp. 19-20.

Département de Mathematiques
Université Claude Bernard (Lyon 1)
43, Bd. du 11 Nov. 1918
69 100 Villeurbanne, France

ADDENDA.

D.D. ANDERSON and M. ZAFRULLAH who have seen a preliminary version of this paper have sent to me some answers and remarks.

* Answer to problem 1: A locally GCD-domain A of dimension 1 is a Prüfer domain.

Indeed, if M is a maximal ideal of A , then A_M is a one dimensional GCD-domain. So, by a result from SHELDON, P. in "Prime ideals in GCD-domains" Canad. J. Math 26 (1974), 98-107, A_M is a Bezout domain, and then, a valuation domain.

Furthermore, ZAFRULLAH noticed: "A locally GCD domain A is a Prüfer domain if and only if Spec(A) is a tree".

* Examples of not factorial locally factorial domains A with Pic(A) = 0 .

1) Let A be an almost Dedekind domain which is not a Dedekind domain (such domains can be found in [GI]). Then, A is a Prüfer domain of dimension 1, non-noetherian and A_M is a discrete valuation domain for every $M \in Max(A)$. So, A is locally factorial and by (2-3-1), A(x) is also locally factorial. But D.D. ANDERSON has proved, in "Some remarks on the ring R(x)" in Comment. Math. Univ. St. Pauli XXVI 2 (1977) 137-140 that Pic(A(x)) = 0 . Because A is not a Krull domain, by (3-2), A(x) is not factorial and so it provides the example.

2) In "On Prüfer v-multiplication domains" (to appear), MOTT and ZAFRULLAH give an example of a non-GCD-domain A which is locally factorial and which is a Schreier domain. But in "On pre-Schreier domains" (to appear), ZAFRULLAH has proved that the Picard group of a Schreier domain is null. So A provides another family of examples.

* (2-3-5) can be generalized in the following way:

Let $A = \bigoplus_{n \geq 0} A_n$ be a graded Krull domain; then, A is factorial if and only if A is locally factorial and $Pic(A_o) = 0$.

Indeed, a Krull domain is semi-normal and by a result from D.F. ANDERSON in "Semi-normal graded rings" (to appear), one has $Pic(A) = Pic(A_o)$ so the result is a consequence of (3-2).

* The study of properties of the group $G(A) = C\ell(A) | Pic(A)$ will appear in a forthcoming paper: "The G-group of a Krull domain" by A. BOUVIER.

A. BOUVIER
DEPARTEMENT DE MATHEMATIQUES
UNIVERSITE CLAUDE BERNARD
43, bd du 11 novembre 1918
69622 VILLEURBANNE CEDEX