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MULTIPLIERS FOR THE HECKE ALGEBRA OF A REAL SEMI SIMPLE LIE GROUP

by Patrick DELORME

INTRODUCTION.

Let G be a real semi-simple Lie group, connected, with finite center and K a maximal compact subgroup of G. In this paper, we study multipliers of the Hecke algebra $\mathcal{D}(G)_{(K)}$ of smooth, compactly supported functions on G, which ar left and right K-finite. By a multiplier we mean a linear endomorphism commuting with the left and right actions of the algebra. Essentially we construct a sub-algebra of the algebra of multipliers of $\mathcal{D}(G)_{(K)}$ (th. 3). This result has been originally proved by Arthur (cf. [1], th. III, 4.2), but his proof rests on a Paley-Wiener theorem for real semi-simple Lie groups, whose proof is very hard (cf. [1], th. III.4.1). Our construction of multipliers for $\mathcal{D}(G)_{(K)}$ is simple and elementary. Let us explain it in more details.

Let $\underline{g}$ be the Lie algebra of G, $\underline{g} = \underline{k} \oplus \underline{p}$ a Cartan decomposition of $\underline{g}$ with Cartan involution θ, $\underline{g}_{\mathbb{C}}$ the complexified Lie algebra of $\underline{g}$. We set $\underline{u} = \underline{k} \oplus i\,\underline{p}$, $\underline{q} = i\,\underline{u}$. Then $\underline{g}_{\mathbb{C}} = \underline{u} \oplus \underline{q}$ is a Cartan decomposition of $\underline{g}_{\mathbb{C}}$ (viewed as a real Lie algebra).

Let $\underline{h}_{\emptyset}$ be the Lie algebra of a maximally split θ-stable Cartan subgroup of G. Then $\underline{h}_{\emptyset} = \underline{t}_{\emptyset} \oplus \underline{a}_{\emptyset}$, where $\underline{t}_{\emptyset} = \underline{h}_{\emptyset} \cap \underline{k}$, $\underline{a}_{\emptyset} = \underline{h}_{\emptyset} \cap \underline{p}$. Moreover $\underline{a} = i\,\underline{t}_{\emptyset} \oplus \underline{a}_{\emptyset}$ is a Cartan subspace of $\underline{q}$, and $(\underline{h}_{\emptyset})_{\mathbb{C}}$ is a Cartan subalgebra of $\underline{g}_{\mathbb{C}}$. We denote by $W_{\mathbb{C}}$ the Wheyl group of the pair $(\underline{g}_{\mathbb{C}}, (\underline{h}_{\emptyset})_{\mathbb{C}})$ which acts on $\underline{a}$.

Now we denote by $G_{\mathbb{C}}$ the connected, simply connectec Lie group with Lie algebra $\underline{g}_{\mathbb{C}}$, by U the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra.

Let $\mathscr{E}(G_{\mathbb{C}}/U)$ (resp. $\mathscr{E}'(U \setminus G_{\mathbb{C}}/U)$) be the space of smooth functions on $G_{\mathbb{C}}/U$ (resp. the space of compactly supported distributions on G, biinvariant under U).

From the spherical Paley-Wiener (cf. [4]), for each τ in $\mathscr{E}'(\underline{a})^{W_{\mathbb{C}}}$ (compactly supported, W_C-invariant distributions on $\underline{a}$) there exists a unique $\tilde{\tau}$ in $\mathscr{E}'(U \setminus G_{\mathbb{C}}/U)$ whose spherical Fourier transform in equal to the usual Fourier transform of τ , $\hat{\tau}$. The right convolution by $\tilde{\tau}$ determines a continuous endomorphism T_{τ} of $\mathscr{E}(G_{\mathbb{C}}/U)$ which commutes with the left translations by elements of $G_{\mathbb{C}}$. We show in theorem 1 that every such map is a right convolution by an element of $\mathscr{E}'(U \setminus G_{\mathbb{C}}/U)$ i.e. is one of the T_{τ} . Now, from the Flensted-Jensen's correspondance between certain functions on dual symmetric spaces (cf. [2]), there is an inejction η of $\mathscr{D}(G)_{(K)}$ in $\mathscr{E}(G_{\mathbb{C}}/U)$., with nice properties. It is easy to show that each T_{τ} leaves stable the image of η , hence $T^{\eta}_{\tau} = \eta^{-1} \circ T_{\eta} \circ \eta$ is a well defined endomorphism of $\mathscr{D}(G)_{(K)}$. From the properties of η , it is easy to see that T^{η}_{τ} commutes with the left and right actions of the enveloppind algebra $U(\underline{g})$ of $\underline{g}$.

We show in theorem 2 that it is enough to ensure that T^{η}_{τ} is a multiplier for $\mathscr{D}(G)_{(K)}$. Finally, we have defined a map $(\tau \to T^{\eta}_{\tau})$ from $\mathscr{E}'(\underline{a})^{W_C}$ into the algebra of multipliers of the Hecke algebra $\mathscr{D}(G)_{(K)}$.

Now we identify $Z(\underline{g})$, the center of $U(\underline{g})$, with $S(\underline{a})^{W_{\mathbb{C}}}$. Then we show in theorem 3 that, for any element ϕ of $\mathscr{D}(G)_{(K)}$ and any principal series representation (π, H_{π}) of G with infinitesimal character χ_{ν} $(\nu \in \underline{a}^*_{\mathbb{C}})$, $\pi(T^{\eta}_{\tau}\,\phi) = \hat{\tau}(\nu)\pi(\phi)$.

This achieves the comparison with the multipliers constructed by Arthur in [1], th. III.4.2.

In paragraph 1, we introduce the general conventions.

In paragraph 2, we introduce the Flensten-Jensens correspondance and etablish some of its properties needed in the sequel.

In paragraph 3, we study the $G_{\mathbb{C}}$-endomorphisms of $\mathscr{E}(G_{\mathbb{C}}/U)$ (th. 1).

In paragraph 4, we construct certain multipliers of the Hecke algebra $\mathscr{D}(G)_{(K)}$ and etablish some of their properties.

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