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RAMSEY THEOREMS AND THE PROPERTY OF BAIRE

by Bernd VOIGT

1. Introduction.

My talk is concerned with so called infinite partition theorems, like, typically, $\omega \rightarrow (\omega)^{\omega}_{2}$ which are trivially false in ZFC , but which are consistent with ZF, at least, if large cardinals exist.

I will concentrate on a few, in my opinion typical, examples.

2. A partition theorem for 2^{ω}.

Consider 2^{ω} , the powerset of ω, endowed with the Tychonoff product topology. So this is the Cantor space.

THEOREM 1 [PV xx].

For every partition $2^{\omega} = \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{r-1}$ *into finitely many sets having the property of Baire there exist infinitely many mutually disjoint and nonempty subsets* $A_k \in 2^{\omega}$, $k < \omega$, *and there exists* $i < r$ *such that* $\bigcup_{k \in K} A_k \in \mathcal{B}_i$ *for all nonempty* $K \subseteq \omega$.

In some sense, this partially answers of Erdös [Er 75,] who asked whether there exists a cardinal κ such that for every partition $2^{\kappa} = \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{r-1}$ there exist infinitely many mutually disjoint and nonempty sets $A_k \in 2^{\kappa}$, $k < \omega$, and there exixts $i < r$ such that $\bigcup_{k \in K} A_k \in \mathcal{B}_i$ for all nonempty $k \subseteq \omega$.

Erdös conjectured the answer to be negative. The previous theorem indicates that the answer depends on set theoretic axioms :

COROLLARY. Con(ZF) *implies* Con(ZF+DC+already $\kappa = \omega$ *has his property*). □

This is due to a result of Shelah and Stern who showed that Con(ZF) implies Con(ZF+DC+every set $\mathcal{B} \subseteq 2^{\omega}$ has the property of Baire (cf. [Rai 84]).

I should mention that theorem 1 is related to the so called "dual Ramsey theorem" of Carlson and Simpson [CS 84] . What Carlson and Simpson prove is an infinite (topological) generalization of the Graham-Rothschild partition theorem for n-parameter sets. It can be formulated by saying that *Borel sets of* k-*parameter words are Ramsey.*

In [PV xx] Prömel and I extended this by showing that *Baire sets of* k-*parameter words are Ramsey.* Theorem 1 is a particular instance of this result.

3. An ordering theorem for 2^{ω}

A relation $R \subseteq 2^{\omega} \times 2^{\omega}$ is a *Baire relation* iff R has the property of Baire with respect to product topology on $2^{\omega} \times 2^{\omega}$.

THEOREM 2 [PSV 84]. *For every Baire order* $\leq$ *on* 2^{ω} *there exists a* 2^{ω}*-sublattice* $\mathcal{L} \subseteq 2^{\omega}$ *(i.e., join- and meet-preserving) such that* $\leqslant \upharpoonright \mathcal{L}$ *is a lexicographic ordering (either, with* $0 < 1$ *or with* $1 < 0$).

In particular, this result shows in a nice way that well orderings of 2^{ω} are not Baire.

4. A canonizing partition theorem for 2^{ω}

Recall that a mapping $\Delta : \mathcal{X} \to \mathcal{Y}$ between topological spaces is a *Baire mapping* provided that $\Delta^{-1}(\mathcal{O})$ has the property of Baire for all open subsets $\mathcal{O} \subseteq \mathcal{Y}$ (preimages of open sets are Baire).

In contrast to Ramsey type theorems, which seek for monochromatic substructures, so called *canonizing partition theorems* try to find nice patterns. The following is a typical example :

THEOREM 3 [PSV 84] .

For every Baire mapping $\Delta : 2^\omega \to \mathscr{X}$, *where* $\mathscr{X}$ *is a metric space, there exists sets* $A \subseteq B \subseteq \omega$ *with* $B \setminus A$ *being infinite such that the restriction* $\Delta \upharpoonright \{C \in 2^\omega | A \subseteq C \subseteq B\}$ *either is a constant or a one-to-one mapping.*

I should remark that in general this result is not valid with respect to topologically nice equivalence relations $R \subseteq 2^\omega \times 2^\omega$, e.g., the Borel-equivalence relation $A \approx B$ iff $(A \setminus B) \cup (B \setminus A)$ is finite is a good example. However, still nice canonizing patterns can be obtained, cf. [PSV 84] .

5. A canonizing theorem for $[\omega]^\omega$

With respect to partitions of infinite dimensional objects (typically, $[\omega]^\omega$ the infinite subsets of ω) the situation is not yet that clearly understood (cf. section six). The reason is that there exist meager sets which are not Ramsey neglegible (Ramsey null) and thus, assuming the axiom of choice, there exist Baire sets which are not Ramsey (cf. [GP 73]).

So I will restrict my attention to Borel measurable mappings, although the results remain valid for a somewhat larger class of mappings.

THEOREM 4 [PV yy] .

Let $\Delta : [\omega]^\omega \to \mathscr{X}$ *be Borel measurable, where* $\mathscr{X}$ *is a metric space. Then there exists* $A \in [\omega]^\omega$ *and there exists* $\mathscr{I} \subseteq [A]^\omega$ *such that for all* $X, Y \in [A]^\omega$ *it follows that*

$$\Delta(X) = \Delta(Y) \quad \text{iff} \quad \{k \in X | X \cap k \in \mathscr{I}\} = \{k \in Y | Y \cap k \in \mathscr{I}\}.$$

Loosely speaking, the theorem asserts that for every $X \in [A]^\omega$ the image $\Delta(X)$ only depends on a certain subset of X, moreover, there exists a very specific continuous procedure for determining this subset.

Theorem 4 generalizes the Erdös-Rado canonization theorem (dealing with mappings $\Delta : [\omega]^k \to \omega$) [ER 50] , as well as a result of Pudlak and Rödl [PR 82] concerning continuous mappings $\Delta : [\omega]^\omega \to \omega$

6. Ellentuck type theorems.

In this, final, section I would like to make some remarks about partitions of infinite dimensional objects.

These remarks apply to

- $[\omega]^\omega$ (Ellentuck [Ell 74])
- ω-parameter words (Carlson-Simpson [CS 84])
- ascending ω-parameter words (Milliken [Mil 75], Carlson [Car xx])
- closed infinite dimensional vector spaces over finite fields (Carlson [Car yy]).
- Dowling lattices ([Voi 84]).

Typically, one may think about $[\omega]^\omega$.

Generally speaking, the situation is the following :

For each of the structures mentioned above there exists a set X (infinite) such that infinite dimensional objects can be represented by, certain, ω-sequences X^ω .

Let us denote by X^ω_ω the set of those sequences which represent an infinite dimensional object.

It turns out that $X^\omega_\omega \subseteq X^\omega$ is a G_δ-subset (w.r.t. Tychonoff product topology). So it is a nice polish space.

The *Tychonoff-cones*

$$\mathcal{T}(F,u) = \{G \in X^\omega_\omega | F(i) = G(i) \quad \text{for all} \quad i < n \}$$

form a basis for this topology.

Also, there exists a continuous composition $X^{\omega}_{\omega} \times X^{\omega}_{\omega} \to X^{\omega}_{\omega}$ with the interpretation that F.G represents the G-subobject of F. So, $F.X^{\omega}_{\omega} = \{F.G \mid G \in X^{\omega}_{\omega}\}$ is the set of all infinite dimensional subobjects of F.

Unfortunately, the subspace embeddings $\hat{F} : X^{\omega}_{\omega} \to X^{\omega}_{\omega}$ with $\hat{F}(G) = F.G$ are, in general, not open.

In particular, there exist (assuming AC) nowhere dense sets $N \subseteq X^{\omega}_{\omega}$ which are not Ramsey, i.e., $N \cap F.X^{\omega}_{\omega} \neq \emptyset$ and $N \cap (X^{\omega}_{\omega} \setminus F.X^{\omega}_{\omega}) \neq \emptyset$ for all $F \in X^{\omega}_{\omega}$.
This leads to consider the *Ellentuck topology* on X^{ω}_{ω}, which is, by definition, the coarsest refinement of the Tychonoff topology such that all subspace embeddings $\hat{F} : X^{\omega}_{\omega} \to X^{\omega}_{\omega}$ become open.

A *basis* is given by the Ellentuck neighbourhoods $F.\mathcal{C}(G,n)$, $F,G \in X^{\omega}_{\omega}$, $n \in \omega$.
An *Ellentuck type theorem* for X^{ω}_{ω} asserts :

"Baire sets w.r.t. Ellentuck topology are Ramsey".

This implies results for Tychonoff topology, namely,

"Analytic sets (in particular : Borel sets) w.r.t. Tychonoff topology are Ramsey".

And, in some sense, this is best possible, as, using AC there exists $\Sigma^1_2 \cap \Pi^1_2$ (PCA $\cap$ CPCA-) sets which are not Ramsey.

Let me mention that all these Ellentuck type theorems can be proved using just one pattern, based on the so called Carlson Simpson theorem. For details I refer [Voi 84].

Still, the situation is somewhat unsatisfactory. It seems that the Ellentuck topology is a tool for proving theorems and not the ultimate goal.

I would suggest to consider the following question, which deals with the Tychonoff product topology (and this is certainly the appropriate topology). Recall that for topological spaces $\mathcal{X}$ a set $B \subseteq \mathcal{X}$ has the *Baire property in the restricted sense* iff for all $A \subseteq \mathcal{X}$ the intersection $A \cap B$ has the property of Baire with respect to A.

So, in contrast to the Baire property (in the original sense) the Baire property in the restricted sense is kind of hereditary.

Question.

Is it true that every set $B \subseteq [\omega]^{\omega}$ which has the property of Baire in the restricted sense already is Ramsey ? Of course, the same question can be asked about the other structures that I mentioned above.

I would expect the answer to be affirmative.

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