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GALVIN TREE-GAMES

by

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Let $\mathcal{T}$ denote the class of all rooted trees having no infinite path. For T_1, T_2 in $\mathcal{T}$, the Galvin-tree game $[T_1:T_2]$ between two players called WHITE and BLACK is played on a 'board' consisting of disjoint copies of T_1 and T_2 with a white pawn at the root of T_1 and a black pawn at the root of T_2. As in chess, the players move alternately and WHITE moves first. On his turn a player may push either one of the two pawns from its current position to an adjacent node one step up the corresponding tree (away from the root). The game ends when one of the pawns reaches a terminal position or 'queens', and the winner is the player whose name is the same as the colour of the 'queening' pawn. The game ends after a finite number of moves and there are no draws (if both trees have a single vertex we declare this a win for WHITE), and so, by a standard elementary argument, one of the two players must have a winning strategy. Since the game is essentially a race to the top, it seems intuitively obvious (??) that the first move is a considerable advantage for the game $[T:T]$, and that WHITE should win. GALVIN proved that this is indeed the case, but his proof was not quite so obvious and so he circulated it as a problem for other mathematicians.

We will describe Galvin's proof, but first we consider the special case of finite trees. The following proof for this case was known to Galvin and was independently given by EHRENFEUCHT.

For a finite tree T consider an auxiliary game, QS[T], between two players QUICK and SLOW. The 'board' now consists of a single copy of T with a pawn at the root. QUICK moves first, and on his move he MUST push the pawn up the tree and pay his opponent $1. The moves alternate, but, on his turn, SLOW may EITHER push the pawn and pay QUICK $1, OR he may simply 'pass' and give the move back to QUICK with no financial penalty. As before, the game ends when the pawn reaches a terminal node. QUICK's objective, not surprisingly, is to lose as little as possible, while SLOW tries to gain as much as possible. (The game SQ[T] is exactly the same except that SLOW has the first move.)

It is clear that SLOW cannot win more than $N in this game, where N is the length of the longest path from the root to a terminal node, and hence there is a largest integer v (<= N) and a strategy g_s for SLOW which ensures that he wins at least $v, no matter how QUICK plays. The integer v is the *value* of the game QS[T], and QUICK has a strategy, g_Q, which ensures that he does not lose more than $v.

We claim that WHITE has a winning strategy for the game

[T:T] by using g_B to move the black pawn and g_Q to move his own pawn. More precisely, on his first turn or after any push of the white pawn by BLACK, WHITE moves his pawn as if he were QUICK playing the strategy g_Q in the game QS[T]; after a BLACK push of the black pawn, WHITE first checks to see if g_B calls for a move or not by SLOW in the game QS[T] against the black pawn; if it does, then this is the move WHITE should make in the actual game [T,T]; but, if g_B tells SLOW to 'pass', then WHITE moves his own pawn (using g_Q) as if SLOW had just passed in the game QS[T] against the white pawn.

To prove the claim, let us suppose, for contradiction, that the black pawn 'queens' first. Let W_w and W_b (resp. B_w and B_b) denote the number of times that WHITE (resp. BLACK) moves the white and black pawns during the play. Since WHITE is using g_Q for his own pawn, and since the white pawn has not yet 'queened', we have $W_w - B_w < v$. Also, since he is using g_B against the black pawn, $B_b - W_b >= v$. This implies that $W_w + W_b < B_w + B_b$, and this is impossible since WHITE moves first!

The above strategy does not actually tell WHITE *how* to play the game [T:T], but it is not difficult to translate this into an effective version. First define two functions E, F on the nodes of T by the following rules: (i)*If x is a terminal node of T, then put* $E(x) = F(x) = 0$. (ii)*If x is not terminal and if E and F have been defined at every successor node x' of x (we write x'Sx), then put*

$$E(x) = \min\{F(x')+1 : x'Sx\},$$
$$F(x) = \max\{E(x), \max\{E(x')-1 : x'Sx\}\}.$$

This defines E, F on T; in fact it is easily seen that E(x) and F(x) are just the values for the games $QS[T_x]$ and $SQ[T_x]$, where T_x denotes the subtree above the root x. We claim that WHITE can play the game [T:T] in such a way as to ensure that, at any stage of the game with the white pawn at node x and the black pawn at node y:

(1) *before any move of BLACK,* $F(x) <= E(y) - 1$;
(2) *before any move of WHITE, EITHER (a)* $E(x) <= E(y)$, *OR (b) BLACK just moved the black pawn and* $F(x) <= F(y)$.

This follows easily from the above definitions of the functions E and F. And, if WHITE plays in this manner,it is not possible to reach a position (x,y) with y terminal and x non-terminal (since E(x), F(x) are non-negative and are zero only at a terminal node). It will be noticed also that, if he plays this way, then WHITE need only push the black pawn immediately after BLACK has done so.

Incidentally, the above strategy for WHITE can be applied equally well in the game $[T_1,T_2]$ when the trees are different, provided only that $E(r_1) <= E(r_2)$, where r_i is the root of T_i. But the argument only works in the case of

FINITE trees. Of course, it is possible to define the functions E, F even for infinite trees - simply replace the 'max' by 'sup' in the definition of F. To see that the above strategy does not work for infinite trees consider the trees illustrated in diagram 1:

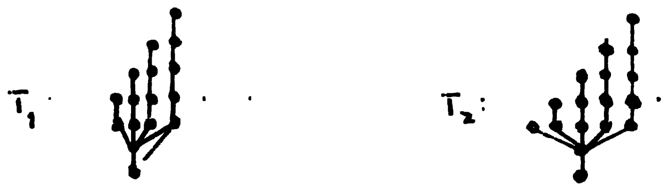

diagram 1.

In this case, we do have $E(T_1) = E(T_2)$ (= ω+1), but BLACK wins the game $[T_1, T_2]$)!

Here is Galvin's proof for arbitrary trees in **T**.

THEOREM: *WHITE can win [T:T].*

PROOF. As we have already remarked, one of the players does have a winning strategy. We shall assume that BLACK has one, say *s*, and derive a contradiction.

Let T_n ($n < \omega$) be a copy of T, with a pawn p_n at the root. Let P_n be an attendant sitting in front of T_n with an alarm clock A_n by his side. P_0 is something of a dummy - every time his alarm, A_0, rings he pushes the pawn p_0 one step up the tree T_0 and this causes the alarm A_1 to sound. For $n > 0$, the player P_n is given a copy of the strategy *s* which he uses as if he were playing BLACK in the game $[T_{n-1}, T_n]$. Whenever his alarm, A_n, rings, P_n uses *s* to push either p_{n-1} or p_n; when he pushes p_{n-1} this causes the alarm A_{n-1} to ring, and when he pushes his own pawn p_n this causes the alarm A_{n+1} to ring. To start the action, we ring the alarm A_0. What happens? Note that, when A_n rings and $n > 0$, then either P_{n-1} has pushed p_{n-1}, or P_{n+1} has pushed p_n; in either case, it appears to P_n that WHITE has just made a move in $[T_{n-1}, T_n]$ to which he can respond using *s*. The action continues until one of the pawns reaches a terminal node. It is possible, of course, that there should be an infinite wave of ringing bells. But this can happen only a finite number of times, and we reset the alarm A_0 to ring after each such infinite wave. Eventually, one of the pawns, say p_n, 'queens'. But this is impossible, since P_{n+1} was using the winning strategy *s* to ensure that p_{n+1}queened' before p_n.

The above proof of GALVIN is short and elegant, but it does not tell WHITE *how* to play the game [T:T]. GRANTHAM [1] subsequently gave a proof of Galvin's theorem which describes an actual winning strategy for WHITE. This is somewhat analogous to the proof given above for finite trees, but considerably more complex. In the case of infinite T, the functions E, F described earlier have to be replaced by sequences (of $|T|^{+}$) functions.

Let us denote by $[T_1,T_2]^*$, the restricted game $[T_1,T_2]$ in which WHITE is only allowed to push the black pawn immediately after BLACK has pushed it (so that, in particular, he must push his own pawn on the first move). It is easily verified that, in the above proof of Galvin's theorem, it appears to P_{n+1} that his immaginary opponent for the game $[T_n,T_{n+1}]$ is actually playing the restricted version of the game (after P_{n+1} pushes p_n he is next called upon to move only after a further push of this pawn by P_n), and so WHITE actually has a winning strategy for this restricted game. Define binary relations $\leqslant$, $\equiv$ and $<$ on $\boldsymbol{T}$ by

$$T_1 \leqslant T_2 \iff \text{WHITE wins } [T_1,T_2]^*.$$

$$T_1 \equiv T_2 \iff T_1 \leqslant T_2 \leqslant T_1; \quad T_1 < T_2 \iff T_1 \leqslant T_2 \not\leqslant T_1$$

The argument above shows that $\leqslant$ is reflexive; in fact, the proof of Galvin's theorem really shows that the relation $\leqslant$ is well founded (i.e. there is no infinite sequence T_n such that BLACK wins $[T_n,T_{n+1}]$), since the trees T_i in the proof do not have to be distinct. GRANTHAM [1] has shown that $\leqslant$ is also transitive so that $\equiv$ is an equivalence on $\boldsymbol{T}$; in fact, it is a congruence modulo $\leqslant$, so that $\leqslant$ is a well ordering of $T/\equiv$. Thus to each tree of $\boldsymbol{T}$ we can associate a definite ordinal and a natural question is to ask for the order type of special subsets of $\boldsymbol{T}$; for example, GRANTHAM has calculated the ordinal number which corresonds to the set $\boldsymbol{T}_\kappa$ consisting of all those members of $\boldsymbol{T}$ of cardinality κ, where κ is infinite - it is the ordinal exponentiation of κ^+ raised to the power κ^+ iterated κ^+ times!

VARIATIONS

LAVER suggested a variation of GALVIN's game, denoted by $[T:(T)^\kappa]$, where κ is an infinite cardinal. This is similar to the game $[T:T]$, except that now there is just one tree with a white pawn at the root but κ trees each having a black pawn at the root. The rules are as before, but BLACK wins if any one of the black pawns 'queens' first. The game is no longer finite (but it ends in fewer than κ^+ moves), and to help BLACK further he is given the move at every limit stage of the play. Laver asked if WHITE also wins this game. Galvin's proof can be adapted to show that BLACK does not have a winning strategy, but, since the game is not finite, this no longer implies that WHITE can win! However, MARTIN (see [1]) has given a proof that WHITE can force a win in Laver's game.

Another variation is to allow multiple moves. Let $[T:T]_{n,m}$ denote the game which is played just as before, except that now, on WHITE's turn to move, he must push either of the pawns a total of n steps up the trees, and BLACK must push a total of m steps. Mixed moves are allowed. There is one exception to this rule, and that is that if a player can 'queen' his pawn, he may do so without having to use his full allocation of moves. Surprisingly, the first move may not be an advantage! For example, If T is the tree shown in diagram 2, then BLACK wins the game $[T:T]_{4,3}$! However, in the examples illustrating this phenomenum, n is less than 3m/2, and GRANTHAM [1] conjectures that this is the critical ratio. However, as far as I know, nobody has yet answered Grantham's question whether WHITE always wins $[T:T]_{100,2}$?

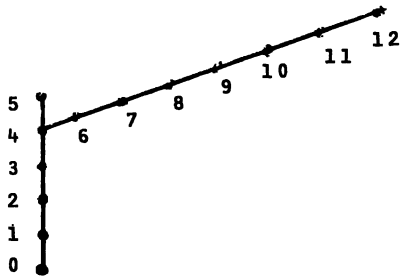

diagram 2.

Finally, we mention that GRANTHAM [1] also considers Galvin-type games on the enlarged class of trees in which there are paths of lengths ω and $\omega+1$.

REFERENCES

[1] Steve Grantham, GALVIN'S "PAWNS UP THE TREES" GAME, University of Calgary, Department of Mathematics, Research Paper No. 478, March 1981. (A fuller version will appear in the A.M.S. Memoirs, 1985.)

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