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ON THE COMPLETE INTEGRAL CLOSURE OF A MORI DOMAIN

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It is well known that the complete integral closure A^* of a domain A need not be completely integrally closed (R. Gilmer and W. Heinzer). It turns out that even if A is Mori, then A^* is not necessarily completely integrally closed or Mori, thus answering a question of Professor Valentina Barucci (Università di Roma "La Sapienza"). On the positive side, for any Mori domain A , the domain A^{**} is completely integrally closed (in short c.i.c.). If A is Mori and root-closed, then A^* is c.i.c. We recall that by a result of V. Barucci, if A is Mori and $(A:A^*) \neq 0$, then A^* is Krull.

In order to construct a Mori domain A such that $A^* \neq A^{**}$, we remark that for any domain A , we have: $A^* = \bigcup_{x,y} (A_0[x,y]_y \cap A)^*$, where A_0 is the prime ring contained in A .

This remark leads us to consider certain subrings of $D[X,Y]_y$ for a domain D and to define power functions. Furthermore, as being complete integral closed and the Mori property are multiplicative properties of a domain, we deal here not just with ring power functions but also with semigroup power functions.

Let x, y be elements of a cancellative semigroup S with unit. We define the function $\Psi_{S;x,y} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ as follows: $\Psi_{S;x,y}(m) = \sup \left\{ n \in \mathbb{N} : (x^m/y^n) \in S \right\}$ (Here x^m/y^n belongs to the localization S_y of S). A function $\Phi : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ will be called a semigroup power function if $\Phi = \Psi_{S;x,y}$ for some

nonzero elements x, y in a cancellative semigroup S . A ring power function is a function of the form $\Psi_{S;x,y}$, where $S = A \setminus \{0\}$ for some domain A .

We denote by \mathbb{M} the semigroup $\{X^m Y^n : m \in \mathbb{N}, n \in \mathbb{Z}\}$, where X, Y are indeterminates. Let M be a subsemigroup of \mathbb{M} containing X, Y . We denote by Ψ_M the function $\Psi_{M;X,Y}$. On the other hand, for any function $\Phi: \mathbb{N} \rightarrow \mathbb{R} \cup \{\infty\}$, we denote by Λ^Φ the set of all elements X^m / Y^n in \mathbb{M} such that $n \leq \Phi(m)$; for any domain D , we denote by D^Φ the domain $D[\Lambda^\Phi]$.

It is easy to obtain the following characterization of semigroup power functions:

THEOREM 1 Let $\Phi: \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ be a function. The following conditions are equivalent:

- 1) Φ is a ring power function.
- 2) Φ is a semigroup power function.
- 3) For all m, n in \mathbb{N} it holds: $\Phi(m+n) \geq \Phi(m) + \Phi(n)$.
- 4) For any domain D , it holds: $\Phi = \Psi_{D^\Phi; X, Y}$.
- 5) For any domain D , it holds: $\Lambda^\Phi = M \cap D^\Phi$.

EXAMPLES of power functions: Let $c \geq 0$ in \mathbb{R} . We denote by τ_c the function $[cn]$. For $c > 0$, we define $\sigma_c(n)$ as the greatest integer which is $< cn$ for $n > 0$ and set $\sigma_c(0) = 0$. We also define $\sigma_0(n) = 0$ for all $n \geq 0$, thus $\sigma_0 = \tau_0$. We denote by ℓ_c the function $\ell_c(n) = [c(n - \log(n+1))]$ ($n \in \mathbb{N}$). By Theorem 1, σ_c , τ_c and ℓ_c are power functions.

Given a class \mathcal{E} of cancellative semigroups with unit, a \mathcal{E} -power function is a power function of the form $\Psi_{S;x,y}$, where

S is a semigroup in \mathcal{E} . For example, we will deal here with Mori semigroup power functions, etc. (The Mori property for cancellative semigroups is defined similarly to the Mori ring property). We use a similar terminology for ring power functions. Any Mori semigroup power function which is not identically ∞ is necessarily finite.

For any function $\Phi: \mathbb{N} \rightarrow \mathbb{R}$, we denote by $\Delta\Phi: \mathbb{N} \rightarrow \mathbb{R}$ the function $\Delta\Phi(n) = \Phi(n+1) - \Phi(n)$ and by $\delta\Phi: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ the function $\delta\Phi(m, n) = \Phi(m+n) - \Phi(m) - \Phi(n)$. For example, condition 3) of the Theorem 1 for the case that Φ is finite, can be stated in the form: $\delta(\Phi) \geq 0$.

We now characterize More semigroup power functions:

THEOREM 2 Let Φ be a finite power function. The following conditions are equivalent:

- (i) Φ is semigroup Mori.
- (ii) The function Φ has the following two properties:
 - (1) $\Delta\Phi$ is bounded.
 - (2) Any infinite set $I \subseteq \mathbb{N}$ has a finite subset F such that for all $m \in \mathbb{N}$ it holds:

$$\min \left\{ \delta\Phi(m, k) : k \in I \right\} = \min \left\{ \delta\Phi(m, k) : k \in F \right\} .$$
- (iii) The semigroup Λ^Φ is Mori.

Condition (ii) (2) of the preceding theorem is equivalent to the CC^\perp in the semigroup M/MY^r for every r .

For any finite power function Φ , we denote $\sup_n \Phi(n)/n$ by c_Φ . Clearly for $\Phi = \iota_c, \sigma_c$ or τ_c we have: $c = c_\Phi$. It is easy to show that $\lim_{n \rightarrow \infty} \Phi(n)/n = c_\Phi$ and $c_\Phi \leq \sup \Delta\Phi$ for any

power function Φ . Thus, if $\Delta\Phi$ is finite, in particular if Φ is a finite semigroup Mori power function, then c_Φ is finite.

For any finite power function Φ , we define the following two functions from \mathbb{N} to $\mathbb{N} \cup \{\infty\}$:

$$\Phi^*(m) := \sup_{r \in \mathbb{N}} \inf_{k \in \mathbb{N}} [\Phi(km+r)/k], \quad \hat{\Phi}(m) := \sup_{k \in \mathbb{N}} [\Phi(km)/k].$$

Both $\hat{\Phi}$ and Φ^* are power functions. Moreover, Φ is root-closed (as a semigroup or as a ring power function) if and only if $\Phi = \hat{\Phi}$. Similarly, Φ is c.i.c. if and only if $\Phi = \Phi^*$ (again this holds in both senses: as a semigroup or as a ring power function).

We see that we can translate semigroup or ring properties into properties of power functions and conversely. For example, we can characterize Mori root-closed or c.i.c. power functions. Indeed, let $\mathcal{S} := \{\sigma_c : c \geq 0\}$ and $\mathcal{T} := \{\tau_c : c \geq 0\}$. It can be shown that $\mathcal{S} \cup \mathcal{T}$ is the set of all root-closed power functions with c_Φ finite and \mathcal{T} is the set of all c.i.c. power functions with c_Φ finite. Also, $\mathcal{S}_{\mathbb{Q}} \cup \mathcal{T}_{\mathbb{Q}}$ is the set of finite root-closed Mori power functions and $\mathcal{T}_{\mathbb{Q}}$ is the set of all finite c.i.c. Mori power functions (here $\mathcal{S}_{\mathbb{Q}}$ is the set of all σ_c with c rational and similarly for $\mathcal{T}_{\mathbb{Q}}$). As before everything here is in both senses).

The set of all factorial finite (semigroup or ring) power functions equals $\mathcal{T}_{\mathbb{Q}}$.

For the Mori property we have a two-sided translation just for semigroup power functions by Theorem 2 above and we conject that a semigroup Mori power function is also ring Mori (the converse is clear).

We have the following

THEOREM 3 Let $\Phi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function with the following properties:

- (1) For all m, n in \mathbb{N} it holds: $\Phi(m+n) \geq \Phi(m) + \Phi(n)$.
- (2) The sequence $\Delta\Phi(n)$ converges.
- (3) $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \delta\Phi(m, n) = \omega$.

Then K^Φ is Mori for any field K . In particular, $[\Phi]$ is a Mori power function.

The conditions of the last theorem are fulfilled by the function $\Phi(n) := c(n - \log(n+1))$ for any $c > 0$, so all the functions ι_c for $c > 0$ are ring Mori.

Taking into account Theorem 3 and further properties as e.g. $\hat{\iota}_c = \iota_c^* = \sigma_c$, we can obtain our counterexamples:

Let K be a field. Let $c > 0$ and let $A := K^{\iota_c}$, thus A is Mori. We have:

- (1) For c rational, $A^* = \bar{A}$ is Mori, but is not c.i.c.
- (2) For c irrational, $A^* = \bar{A}$ is not Mori, but is c.i.c.
- (3) For positive constants a and b , where a is rational and b is irrational, the domain $B := \left[K^{\iota_a} \right]^{\iota_b}$ is Mori, but B^* is neither Mori, nor c.i.c. Notice that $B \cong K^{\iota_a} \otimes_K K^{\iota_b}$.

In particular, we see that the integral closure of a Mori domain is not necessarily Mori, thus answering a question of Professor Evan G. Houston (University of North Carolina at Charlotte). We recall that by a result of V. Barucci, the integral closure of a Mori domain is not necessarily c.i.c.