# Upsetting the Foundations for Mathematics

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**Résumé :** Commençant par une revue sommaire des types de questions qu'une fondation des mathématiques devrait poser, cet article présente premièrement une critique des fondements basés sur la théorie des ensembles, puis propose l'idée que plusieurs fondements catégoriques, reliés les uns aux autres, seraient plus avantageux, et finalement indique une méthode pour retrouver la théorie des ensembles à travers une approche catégorique.

**Abstract:** Starting with a review of the kinds of questions a foundation for mathematics should address, this paper provides a critique of set theoretical foundations, a proposal that multiple interconnected categorical foundations would be an improvement, and a way of recovering set theory within a categorical approach.

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# 1 Introduction

Expressing all of mathematics in set theoretical terms is like writing love poetry in assembly language. A lot of the nuance is lost: even if it can be done it is not clear we should want to.

Mathematical objects like numbers, patterns, symmetries, and collections arise as abstractions from human interaction with physical objects and from human actions like drawing and construction. They are communicated culturally so they become social constructs and not just in the mind of the mathematician. Operations on natural numbers arise from understanding of the potential consequences of physical actions on a counting board, not from consideration of what the stones were made of. The rules for reasoning about mathematical objects come from the contexts in which we use them and the purposes we propose for the actions we model with our operations. The most problematic aspect of set theory as a foundation from this point of view is its insistence on extensionality.

Categorical approaches to foundations can reflect the importance of context. By considering mathematical structures in several different categories we can focus on different aspects. Relating different foundational frames of reference helps us understand how mathematical innovation comes about. Internal logical constructs in categories can provide the logical foundation justifying our reasoning. Knowledge of how completely the logical constructs are preserved by change of base functors will help us understand the level of necessity of our mathematical statements.

# 2 What are foundations for?

In order to investigate the efficacy of differing approaches to foundations of mathematics we need first to understand what we want foundations for and what we might mean by foundations. As has been pointed out by Marquis [Marquis 1995] category theory and set theory address different foundational issues.

Foundations can deal with several philosophical questions:

1. Ontological questions:
   (a) To what extent are mathematical objects real?
   (b) Do we discover or invent new structures?

   (c) Are mathematical objects immutable?

2. Epistemological questions:
   (a) How do we come to know about mathematical truths?
   (b) How do we account for mathematical intuition?
   (c) How do we accept something as part of mathematics?
   (d) How do we distinguish mathematics from other disciplines?

3. Logical questions:
   (a) What constitutes valid reasoning about mathematical objects?
   (b) What connectives, what quantifiers, and what forms of induction are allowed and what are their properties?
   (c) What are the limitations on mathematical reasoning?
   (d) What are the unintended consequences of our forms of reasoning?
   (e) What modalities apply? How necessary are mathematical truths?

4. Cognitive and methodological questions
   (a) What are the fundamental mathematical objects of study?
   (b) What are derived concepts?
   (c) What are the fundamental patterns of thought in mathematics? Lakov and Núñez [Lakov and Núñez 2000]call these basic metaphors.
   (d) What mathematical concepts are self-evident?

We want to understand foundational questions so that we know how firm the ground under our mathematical reasoning is. We also want to know so that we can teach fundamental notions early and so we can avoid confusion in our students. A form of Ockham's razor is suggested by Mayberry in [Mayberry 1994, 18]

> Above all, in the foundations of mathematics we must shun sophistication and strive for simplicity. . . . Highly developed and refined mathematical concepts are therefore out of place in foundations, and philosophical 'depth' is to be avoided there in the interests of perspicuity and logical simplicity.

This puts a constraint on a foundational approach which may rule out categorical foundations like the ones proposed by Makkai [Makkai 1998]because the development of higher dimensional analogs of categories requires so much machinery. It also puts into question the usefulness of many set theoretic foundations. For most of mathematics what we use is a utilitarian naïve set theory which does not use full comprehension, regularity, or any large cardinal axioms and which does not insist that all entities we discuss be sets. Perhaps a categorical foundation would also have a similar utilitarian naïve form.

Those favoring set theoretic foundations (like [Simpson 1996]and [Friedman 2002])prefer a hierarchy of concepts reflected in the bottom up approach of modern set theory: all concepts in mathematics are to be built from the simplest available material, often the empty set. This puts the ontological commitments at a minimum, but it makes connections between the mathematics so founded and the real world more mysterious. Categorical foundations tend to have a more gestalt character: by looking at a category of all groups, for instance, we are assuming that we can organize part of our understanding of an external world by focusing on one part of it (groups and group homomorphisms). If that makes an ontological commitment it is to something much larger than just the empty set and the sets which we can construct from it.

Set theoretic foundations are also claimed by their proponents to be coherent and extremely natural. It is precisely the complete naturality of founding mathematics in set theory that this paper questions.

Categorical foundations is a subject with about 40 years of development. It has not yet reached a fully mature form, but has a substantial start. Categories provide a rich and varied logical foundation for mathematics which includes the set theoretic semantics of first order logic as one of its examples but also addresses semantics of non-classical logics (intuitionistic, fuzzy, linear, computational, modal). Connections between proof theory and higher order categorical logic have been known since early work of Lambek.

For many parts of mathematics, but not all, category theoretic approaches have provided a methodological foundation and a guide to practice. Fundamental categorical constructions (product, sum, function space formation, pull back, quotient, adjoint functors, algebras and coalgebras) find utility in most parts of mathematics. A focus on the question of what the correct maps are between structures is also a categorical influence. The ubiquity of adjoint functors may well provide an underlying cognitive metaphor.

Providing a categorical foundation is like painting a picture of a reality: it provides a viewpoint with particular emphasis from the artist, specifying some aspects of the reality, but not attempting to be exhaustive. What we expect to find is a glimpse of the truth, not a definitive answer. We invent new ways to look at and organize the mathematical landscape; this leads us to discover new aspects of mathematical truth.

# 3 Why elements don't suffice

For many reasons taking elementhood as the one basic relationship and making all mathematical objects of the same kind misses the essence of the enterprise.

**Context matters.** Mathematical innovation often comes from looking at familiar mathematical objects in novel ways giving rise to new questions to ask. The questions may have different answers depending on the context in which they are asked. What is true about the number 2, for instance, depends on which number system you are considering 2 to be in and what aspect of "twoness" you wish to concentrate on.

Similarly, considering $\mathbb{R}$ as a field is different from considering it as a vector space, group, topological space, or as a geometric continuum. Choosing what other mathematical items to relate a structure to changes the focus in our gaze on the structure.

Intentionality (how an object relates to others) is at least as important as extentionality (what is the object made up of) for determining what mathematical objects are. Focus is a pragmatic consideration not captured in set theoretic foundations. Progress in mathematics often comes from seeing old friends in new contexts, so the ability to change focus in our foundations is important, particularly if we want to account for innovation as well as for truth.

**Mathematical objects have types.** Mathematical statements have a context and a universe of discourse; statements in mathematics are about something, not just free floating. The choice of context and universe of discourse often has the effect of imposing a type theory. Mathematical objects and statements about them tend to have types associated with them.

Sometimes the statements depend only on how the objects are related to each other, not on the specific types of the objects. Thus these types may be polymorphic: the membership relation, for instance, has a type $- \times \mathcal{P}(-)$ where $-$ can be any type you wish and $\mathcal{P}(-)$ is the powerobject of that type: form matters more than content here, but for something which doesn't match the form the question of membership becomes essentially meaningless.

Still, even with polymorphic types, predicates are about objects of specified types. Similarly functions must have specified domains and codomains. This is a change from how functions are usually specified in axiomatic set theory, where the focus is on the assignment of a unique image for each element in the domain, often not specifying the codomain. A function can be defined by with an untyped variable $y$ and an untyped predicate $\Phi$ satisfying

$$\forall_{x \in D} \exists !_y \Phi(x, y)$$

Insisting on specified types and codomains has the consequence that the usual axiom of Replacement in axiomatic set theory ends up having the same effect as the axiom of Separation.

As Cohen points out in [Cohen 1966, 53-55], both of these axioms in Zermalo-Frankel Set theory allow the predicates involved to vary over all possible sets, not just those which have been seen before, making them a powerful means of constructing new sets, but making them very nonconstructive. A similar nonconstructivity occurs in categorical logic since we do not constrain how the predicates of a given type are derived, only that they must have the given type.

**Distinct things exist in the world not distinguished by their elements.** People exist as individuals, not as collections of molecules. Biological species are determined by how their representatives interact with those of other species rather than by the listing of their members. Kinship structures in anthropology describe which relationships are considered significant enough to name, not sets of ordered pairs of individuals with those relationships. To the extent that mathematics describes things in the real world it needs to make the same kind of distinctions that the real world does.

Even within mathematics there are many constructs not determined by their elements: Collections of random variables are known through their joint distributions, not through their exact description as sets of measurable functions on a sample space. Vector bundles are described through both elements and how they are bound together. Locales give a pointless approach to topology. Classifications of problems in technique of integration and differential equations use pattern recognition based on how different tools relate to them, not on lists of examples. Constructions in Euclidean geometry specify actions relating points, lines, and planes rather than specifying sets determined by their elements.

Even for a mathematical object as basic as the system of rational numbers we gain most of our understanding not through knowing what the members are exactly, but rather with knowing some ways to represent them symbolically (say as fractions, repeating decimals, terminating continued fractions, etc.), ways to tell if two different representations are of the same number, ways to relate the sizes of rationals, and ways to add, subtract, multiply and divide them. If we insisted on knowing what the elements of $\mathbb{Q}$ were we would have to think of large equivalence classes rather than simple, but not unique, symbolic representations. The situation with Cauchy $\mathbb{R}$ is even worse: knowing any one real number $r$ would require knowing all of the uncountable number of distinct Cauchy sequences of rationals converging to $r$. The Dedekind reals are just as unfathomable.

**Abstraction can be based on interaction with the world.** Foundations must allow for abstraction in different ways: Natural Numbers arise as an abstraction from finite sets. A natural number is what remains when you decide that all that matters about a finite set is the number of elements it has, not what those elements are. At the start we only consider rather small finite sets, ones we can relate to as small collections of physical objects we can move around, copy, and arrange into suggestive shapes. Once we understand such small collections we abstract to what the similar properties ought to be for finite collections too large to physically handle. We develop the system of natural numbers when we start thinking about how natural numbers relate to each other and how we can prove things about natural numbers and natural number operations. Induction gets recognized as the key rather late in the game. Using finite Von Neuman ordinals to represent natural numbers is primarily useful as a way to make a set theoretic exemplar concrete.

Our concept of real number is an abstraction from measurement and approximation: actions involving comparing and relating different real numbers. When I teach calculus I tell my students that what we actually know about a real number is how to approximate it arbitrarily well. If I want them to understand why calculus works so well I do not start by specifying a set theoretic construction of the real numbers. If I want to know how much of the calculus I teach might carry over to real numbers objects in another topos I may care a lot more about the specific constructions used (particularly since the Cauchy and Dedekind reals are distinct in many contexts).

Other mathematical objects have been defined by form or function: triangles, random variables, infinitesimals (from those in Leibniz to those in synthetic differential geometry [Kock 1981] and nonstandard analysis [Robinson 1996]),complex numbers, ultrapowers. Mathematical notions often start out life as convenient fictions and then become objects of study in themselves. Mac Lane develops some of these ideas in [Mac Lane 1990].

## 4 How a categorical foundation might help

When we decide to consider a mathematical object as being an object in a category we are defining a context and a point of view. It is the nature of category theory to determine what is known about a structure in a category by considering how it relates to other structures in that same category. Further information is obtained by giving functors connecting that category to others. The success of algebraic topology starts with its connection of topological spaces with groups in various ways through functors which accurately reflect some of the truths about both the spaces and the groups involved. Galois theory shows how a pair of adjoint functors between categories derived from the category of fields and the category of groups can provide insight into long standing problems and not just structural questions. This is evidence of the utility of categorical viewpoints in mathematics, but not necessarily of utility as a foundation for mathematics.

In part to provide such a foundation, since the early 1970's a variety of categorical structures have been studied which allow for mathematics to be done internally: elementary topoi (see [Kock and Wraith 1972, Lawvere 1972, Freyd 1972, Johnstone 1977, Goldblatt 1979, Bell 1988]). quasitopoi ([Penon 1977, Wyler 1991] ), various approaches using monoidal structures ( [Stout 1992, Höhle 1991, Monro 1986] ) and fibered categories [Jacobs 1999].

Much of this work focuses on how to do higher order categorical logic. It uses internal representation of various kinds of subobjects and various logical connectives arising from the structures in the category. All examples I can think of use objects in a category as types and have all statements typed according to which objects are under consideration.

The logic arises from the structure of the category and the kind of subobjects taken for predicates. The internal predicate logic may be intuitionistic, may be non-commutative, may have higher order constructs or not. It is not imposed by the mathematician based on traditional philosophical positions, though the particular examples may be chosen to illustrate how certain non-classical logics can arise. The intuitionism in topos based mathematics does not come from a constructivist position about what mathematical objects are, what truth means, and how we know about mathematics, but rather from the behavior of the truth functional connectives in typical examples.

Viewing logic from within a category gives clear mechanisms for recognizing inference rules from adjointness situations. Those rules of inference give rise to a notion of proof which can be systematically analyzed. The types in the theory arise from the objects in the category and interpretation of expressions in the typed language of the category as specific subobjects gives a notion of validity. Model theory can be done internally without reference to set based interpretations and structures. Categorical logic does not depend on set theory for either its syntax or its semantics.

One problem with categorical constructions is that they typically only specify objects up to isomorphism. We do not talk about *the* natural numbers, for instance, but rather about *a* natural numbers object. Thus categorical foundations can be seen as having a descriptive function rather than an ontological one. We are not building *the natural numbers* but rather describing how to recognize objects which behave like the natural numbers in many different settings. This is not building a foundational object on which we can base our mathematics, but rather providing a prototype of the situation in which induction can be done successfully. If we want the kind of categoricity which specifies unique objects we will need additional structure beyond that given by categories.

## 5 Why no single category can suffice

If, as I am, we are looking at foundations for mathematics as giving viewpoints in which we can see aspects of mathematical reality, we should not be looking for one foundation for mathematics but rather looking for what mathematics can be in a variety of related foundational contexts. Since we expect that there is one mathematical reality, we should also be looking for how those different foundations relate to each other.

**Why any single foundation is insufficient** Any one abstraction from reality limits our vision. In some ways this is good because it provides focus. Focus leads to a certain simplicity in our work in mathematics and allows for a depth of understanding, but any focus requires blocking out some aspects of the situation. In other ways it limits our creativity.

At any one point in history we cannot predict all of the contexts in which a mathematical object can be studied: we need to allow for progress through novel settings. Golden ages in the history of mathematics in the past have arisen from the use of novel viewpoints: discovery of the axiomatic method in ancient Greece, the arithmetization of analysis, founding of non-Euclidean geometry, axiomatization of set theory, foundation of Non-standard analysis, use of modular forms in number theory. If we come to the conclusion that we have found the final answer on foundations of mathematics, the subject will progress ignoring that answer and prove us wrong. (Kant's failure to recognize the possibility of a non-Euclidean geometry provides one of history's prominent examples.) It is hubris for any approach to foundations to assume that it has completely determined the bounds of mathematical imagination.

Intuition comes from looking at familiar objects from many viewpoints. The admonition to think deeply of simple things (which I heard often from Arnold Ross in the SSTP program at Ohio State in the 1960's) can be joined with a suggestion to consider simple things broadly as well.

**Why foundations need to be connected to each other** Necessary truths are those which hold when the foundations shift– we need to look at what is preserved to add a modality. If we are to allow a multitude of possible foundations, each giving a different viewpoint on the subject, we will need some discipline and organization to prevent mathematics from becoming chaotic. A means for tying different foundations together is needed in order to integrate what we learn of the truth from different viewpoints.

Our means of connecting different foundations together must be capable of solving the problem of transworld identity– we need to be able to recognize that we are looking at the same object in different contexts. In the semantics of modal logic this problem is solved by requiring consistency in the definition of frames.

**How categorical foundations can provide such connections** Categorical foundations allow for context to be made explicit, but they need to be expanded to allow for transworld identity so that objects can have different context, i.e. be objects in different categories.

Different categories allow us to see how varied the possibilities of foundation for mathematics are. Functors between those categories are needed to relate what we learn about mathematical reality form one viewpoint to what we lear from another.

Early on in topos theory the examination of the properties of change of base functors led to consideration of classifying topoi. This introduced a modality into foundations: properties of the generic object which were preserved under change of base can be thought of as those properties which are necessarily true. Specific examples can then be thought of as illustrating what is possible.

# 6 Recovering a theory of sets using categories with inclusions

Because set theory is part of mathematics, however, any competing foundation needs to be able to provide a foundation for set theory. Early attempts by Lawvere were not completely successful (though they were part of the inspiration for elementary topos theory). Several categorical constructions have been suggested for capturing finite sets.

Mac Lane suggested an approach at the Association for Symbolic Logic meeting in Urbana in May 2000: a foundation based on categories with specified inclusions. Such a setting allows for the specification of certain kinds of objects completely (not just up to isomorphism). We can define what it means to be a set in such a category with inclusions and with sufficient properties imposed on the setting recover much of axiomatic set theory.

Awodey presented a paper [Awodey 2002] at the 2002 summer ASL meeting at which he used Joyal and Moerdijk's approach to set theory based on the subset relation (rather than elementhood) from [Joyal and Moerdijk 1995]together with Mac Lane's idea to produce a model of Zermalo Set theory.

My approach sketched here is somewhat different from his and, for that matter, from the proposal made by Mac Lane in 2000.

**Definition 6.1.** *An inclusion structure on a category is a designation of some of the monomorphisms in the category as inclusions. These must satisfy the following axioms:*

1. *An isomorphism is an inclusion if and only if it is an identity.*

2. *Compositions of inclusions are inclusions.*

3. *If* $A' \overset{m_1}{\hookrightarrow} A$ *and* $A' \overset{m_2}{\hookrightarrow} A$ *are inclusions then* $m_1 = m_2$. *(This says that there are given objects as domain and codomain there is at most one inclusion between them).*

4. *If* $B' \hookrightarrow B$ *is an inclusion and* $A \xrightarrow{f} B$ *is any morphism then there is a unique inclusion* $A' \hookrightarrow A$ *making the square*

$$\begin{array}{ccc} A' & \hookrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \hookrightarrow & B \end{array}$$

*a pullback. (Inclusions lift uniquely along any morphism.) Furthermore, if* $f$ *is an inclusion, so is the morphism induced from* $A'$ *to* $B'$.

5. *If* $A \xrightarrow{f} B$ *is any morphism then there is a unique* $B' \hookrightarrow B$ *which is the smallest inclusion through which* $f$ *factors. (This is a weak form of factorization–no assumptions are made about the first map in the factorization). The resulting* $B'$ *is called the image of* $A$ *under* $f$.

Axioms 1 and 2 can be taken to say that there are enough inclusions; axioms 1 and 3 say there aren't too many. The last two axioms let us transport inclusions along morphisms in such a way that we get a pair of adjoint functors.

Inclusions behave a lot like subsets and provide us with a ready notion of predicate:

**Proposition 6.1.** *If* $A \overset{m}{\hookrightarrow} B$ *and* $B \overset{m'}{\hookrightarrow} A$ *are inclusions, then* $A$ *and* $B$ *are the same and the inclusion is the identity.*

PROOF:

Since the composition of inclusions is an inclusion we can conclude that $mm' : A \hookrightarrow A$ is an inclusion and thus is equal to the inclusion $\mathrm{id}_A$ by axioms 1 and 3. Similarly $m'm = \mathrm{id}_B$, so $m$ is an isomorphism, hence an identity by axiom 1. ∎

**Definition 6.2.** *If* $\mathcal{C}$ *is a category with specified inclusions and* $A$ *is an object of* $\mathcal{C}$ *then the category of propositions about* $A$*, called* $\mathcal{P}(A)$*, is the category of inclusions into* $A$ *with inclusions over* $A$ *as maps.*

The axioms giving unique pullbacks and best images of inclusions give rise to adjoint functors which give change of type and existential quantification:

**Proposition 6.2.** *If* $A \xrightarrow{f} B$ *is any morphism then the functors* $f^\star : \mathcal{P}(B) \to \mathcal{P}(A)$ *obtained by pulling back and* $\exists_f : \mathcal{P}(A) \to \mathcal{P}(B)$ *obtained by taking image have* $\exists_f \dashv f^\star$*.*

When the category $\mathcal{C}$ has further structure we can ask for the inclusion structure to respect it:

**Definition 6.3.** *An inclusion structure has a unique empty object if* $\mathcal{C}$ *has an initial object* $\emptyset$ *such that the unique map* $\emptyset \overset{!}{\hookrightarrow} A$ *is an inclusion for every object* $A$ *in the category.*

Asking for the unique map from an initial object to be an inclusion and for isomorphic inclusions to be identities forces the initial object to be unique. Nothing similar restricts the terminal object. This fits with the intuition that there is only one empty set, but there are a multitude of one element sets, each of which is a terminal object in **Sets**.

**Definition 6.4.** *A singleton inclusion into an object* $A$ *in a category* $\mathcal{C}$ *with inclusions and a terminal object is an inclusion whose domain is a terminal object.*

Note that we cannot say *the* terminal object here since terminals are only defined up to isomorphism. Singletons will play the role of elements in set theory where $\{x\} \subseteq A$ if and only if $x \in A$. We do not get an element relation between objects from singleton inclusions. In general singleton inclusions are not sufficient to specify an object.

**Definition 6.5.** *A* **set** *in a category with inclusions* $\mathcal{C}$ *is an object which is the colimit of its singleton inclusions.*

What this definition captures is the essential role that extensionality plays for sets. What distinguishes sets from other mathematical structures is that they are completely determined by the singletons which are subsets, since those singleton inclusions correspond exactly to the elements.

Given a category $\mathcal{C}$ with specified inclusions, the sets and functions in $\mathcal{C}$ form a category, $\mathbf{Set}_{\mathcal{C}}$.

To get the other properties usually associated with sets we need to insist on more from our category. Since we know that the category **Sets** forms a topos we can start there. We will ask that the inclusion structure respect limits. In addition we will ask that the generic subobject $t : 1 \to \Omega$ in the subobject classification axiom be a singleton inclusion. Notice that if a topos has an inclusion for the generic subobject $t : 1 \hookrightarrow \Omega$ then the topos has canonical subobjects in the sense of [Lambek and Scott 1986] . The canonical subobjects induced by the monomorphisms from $A^C \to B^C$ and $P(A) \to P(B)$ then give the inclusions needed for the structure to respect function space and powerobject formation.

**Theorem 6.3.** *Let $\mathcal{E}$ be a topos with an inclusion structure which respects coproducts and products and has a generic inclusion $t : \{t\} \hookrightarrow \Omega$ as subobject representor. Then $\mathcal{E}$ satisfies the following axioms of Set theory:*

1. *emptyset*
2. *singleton*
3. *bounded comprehension with type specification*
4. *powerset*
5. *properly typed union*

*If we posit the existence of a natural numbers object in which $\ulcorner 0 \urcorner : 1 \hookrightarrow \mathbb{N}$ is given as a singleton inclusion, then we get the axiom of infinity.*

To get the axiom of choice we will need to ask for choice functions or splitting of epimorphisms.

The axiom of regularity has no obvious analog in this setting. It would appear to require a chain condition on types.

If we require that all objects in our topos be sets (as defined in a category with inclusions) we will gain extensionality and force the logic to be two valued (since a terminal can have only one nonempty singleton inclusion, namely the identity, and thus has only two subsets).

# References

Awodey, Steve, Butz, Carsten, Simpson, A. and Streicher, Th..
2002 Mac Lane set theory. *Slides from ASL presentation*, personal communication, 2002.

Bell, J.L.
1988 *Toposes and Local Set Theories*, Oxford: Oxford U. Press, 1988.

Cohen, Paul J.
1966 *Set Theory and the Continuum Hypothesis*, New York and Amsterdam: Benjamin, 1966.

Freyd, Peter 1972 Aspects of topoi, *Bulletin of the Australian Mathematical Society*, 1972.

Friedman, Harvey
2002 Re: Fom: {n: n notin f(n)}, e-mail to FOM list, August 30 2002, Archived at:
http://www.cs.nyu.edu/pipermail/fom/2002-August/005787.html

Goldblatt, R.
1979 *Topoi: the Categorial Analysis of Logic*, Amsterdam, New York, and Oxford: North Holland, 1979.

Höhle, Ulrich
1991 Monoidal closed categories, weak topoi and generalized logics, *Fuzzy Sets and Systems*, 1991.

Jacobs, Bart
1999 *Categorical Logic and Type Theory*, *Studies in Logic and the Foundations of Mathematics*, 141, Amsterdam: Elsevier, 1999.

Johnstone, Peter T.
1977 *Topos Theory*, London, New York, and San Francisco: Academic Press, 1977.

Joyal, Andre and Moerdijk, Ieke
1995 *Algebraic Set Theory*, *London Mathematical Society Lecture Notes*, Number 220, Cambridge: Cambridge University Press, 1995.

Kock, Anders
1981 *Synthetic Differential Geometry*, *London Mathematical Society Lecture Notes Series*, Number 51 . Cambridge U. Press, 1981.

Kock, Anders and Wraith, Gavin C.
1971 *Elementary Toposes*, in Lecture Notes, Number 30, Aarhus: Aarhus Universitet Matematisk Institut, 1971.

Lakov, George and Núñez, Raphael
2000 *Where Mathematics Comes From: How the embodied mind brings mathematics into being*, New York: Basic Books, 2000.

Lambek, Joachim and Scott, P.J.
1986 *Higher Order Categorical Logic*, *Cambridge studies in advanced Mathematics*, Number 7, Cambridge: Cambridge U. Press, 1986.

Lawvere, F.W.
1972 Introduction, *Toposes, Algebraic Geometry and Logic*, *Lecture Notes in Mathematics*, Number 274, Berlin, Heidelberg and New York: Springer Verlag, 1972.

Mac Lane, Saunders
1990 *Mathematics: Form and Function*, New York, Berlin, Heidelberg, Tokyo: Springer Verlag, 1990.

Makkai, Michael
1998 Towards a categorical foundation of mathematics, *in* Johann A. Makowsky and Elena V. Ravve, (eds.), *Logic Colloquium '95*, *Lecture Notes in Logic*, 11 153–190. Association for Symbolic Logic, Springer Verlag, 1998.

Marquis, Jean-Pierre
1995 Category theory and the foundations of mathematics: philosophical excavations, *Synthese*, 103(3):421–447, 1995.

Mayberry, John
1994 What is required of a foundation for mathematics, *Philosophia Mathematica (3)*, 2, 16–35, 1994.

Monro, G. P.
1986 Quasitopoi, logic and heyting-valued models, *Journal of Pure and Applied Algebra*, 42, 141–164, 1986.

Penon, Jacques
1977 Sur les quasitopos, *Cahiers de Topologie et Géométrie Différentielle*, 18, 181–218, 1977.

Robinson, Abraham
1996 *Non-standard Analysis*, revised edition, Princeton Landmarks in Mathematics, Princeton: Princeton U. Press, 1996.

Simpson, Stephen G.
1996 What is foundations of mathematics? On web page at
http://www.math.psu.edu/simpson/hierarchy.html,
1996, still active Sept. 9, 2002.

Stout, Lawrence N.
1992 The logic of unbalanced subobjects in a category with two closed structures, *in* U. Höhle S.E. Rodabaugh, E.P. Klement,

(eds.), *Applications of Category Theory to Fuzzy Subsets*, Dordrecht, Boston, London: Kluwer, 1991.

Wyler, Oswald

1991 *Notes on Topoi and Quasitopoi*, Singapore: World Scientific, 1991.