Dogmas and the Changing Images of Foundations

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Abstract: I offer a critical review of several different conceptions of the activity of foundational research, from the time of Gauss to the present. These are (1) the traditional image, guiding Gauss, Dedekind, Frege and others, that sees in the search for more adequate basic systems a logical excavation of *a priori* structures, (2) the program to find sound formal systems for so-called classical mathematics that can be proved consistent, usually associated with the name of Hilbert, and (3) the historicist alternative, guiding Riemann, Poincaré, Weyl and others, that seeks to perfect available conceptual systems with the aim to avoid conceptual limitations and expand the range of theoretical options. I shall contend that, at times, assumptions about the foundational enterprise emerge from certain dogmas that are frequently inherited from previous, outdated images. To round the discussion, I mention some traits of an alternative program that investigates the epistemology of mathematical knowledge.

Resumen: Ofrecemos una revisión crítica de varias concepciones de la investigación sobre los fundamentos de la matemática, desde los tiempos de Gauss hasta el presente. Se trata de (1) la imagen tradicional, que guió a Gauss, Dedekind, Frege y otros, y que ve en la búsqueda de sistemas básicos más adecuados una excavación lógica de estructuras *a priori*, (2) el programa de encontrar sistemas formales correctos para la llamada matemática clásica que puedan demostrarse consistentes, habitualmente asociado al nombre de Hilbert, y (3) la alternativa historicista, que guió a Riemann, Poincaré, Weyl y otros, la cual busca perfeccionar los sistemas conceptuales disponibles a fin de evitar limitaciones conceptuales y ampliar el abanico de opciones teóricas.

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Defenderé que, en ocasiones, se encuentran supuestos acerca del trabajo sobre fundamentos que emergen de ciertos dogmas, frecuentemente heredados de imágenes previas ya superadas. Para completar la discusión, menciono algunos rasgos de un programa alternativo, que investiga la epistemología del conocimiento matemático.

Confronted with my title, the reader may have thought that it is an unhappy idea to put together the notion of foundations and the word "dogma." After all, foundational research consists of logical and/or mathematical results formulated and proved in the most rigorous possible way. Thus we are talking of a domain of objective results, unaffected by beliefs or vogues. To put it simply: we are talking about logic, not about any aspect of culture that may be affected by dogmas or by historical shifts. Only the fact that we have been living in this intellectual atmosphere of postmodernism — you may have reflected — can explain why such titles are taken even a bit seriously.

What can I reply? The difficult thing for me would be to convince you that those points include some subtleties that are far from being "evident," and, at the same time, that granting this does not throw us in the arms of postmodern thinking. That, as I say, would be a difficult argument, and I will not try to make it here. So let me begin in a different way. Let me underscore that the word "images" is in my title for some reason.

My aim is to reflect on the *activity* of foundations research. Once we conceive of it as an activity, a practice, it is automatic that foundations research does not come down to a bunch of theorems — not even a multilayered set of problems, methods and theories. Following Leo Corry [1997],[1] let us call those theories, methods and results the *body* of foundational knowledge. As all practitioners know, the practice of foundations research is also guided by certain *images* of this enterprise, images which may vary from researcher to researcher, and certainly from time to time. These images tell us about the goals one pursues when doing foundational research, about important and irrelevant problems, acceptable and unacceptable, promising and unpromising ways of approaching these problems, and so on.

What my title suggests is not that the *body* of foundational results is

[1] Corry talks about the modern structural approach in the images of mathematics and in the body of mathematics (e.g., with Bourbaki, or with category theory).

affected by dogmas, just that some *images* of foundations are. Put this way, I am sure more than one reader will be relieved.

In fact, examples of dogmatic attitudes abound. A noteworthy one can be found in Quine's writings on set theory. I do not mean his distrust of the Zermelo–Fraenkel system for axiomatic set theory (ZFC) and his desire to consider alternative systems. What seems dogmatic to me is his philosophical evaluation of the paradoxes, the so-called "bankruptcy theory." Here, Quine was fully in agreement with Russell: the paradoxes had shown commonsense logic to be contradictory. This view is put forward not only in his *Mathematical Logic* [Quine 1940, § 29], but also in the much later *Set Theory and its Logic* [Quine 1963, Introduction]. Here he writes that the "only natural attitude" towards the notion of class, which is so fundamental to thought, is the Comprehension Principle: that every open sentence in one variable determines a class. The effect of the paradoxes was to discredit this natural attitude, to show that "commonsense is bankrupt," and "intuition" is not to be trusted.

Quine was dismayed to find that most logicians were "retraining their intuition" by immersing themselves in the system of ZFC set theory. In his view, one ought to consider the whole variety of possible systems (type theory, ZFC, von Neumann–Bernays–Gödel set theory, Quine's own systems known as NF and ML,[2] and so on), treating none of them as standard. For it would be imprudent to consider one single system as the standard, natural one [Quine 1963, pp. viii, 1, 5].

In my opinion, Quine's views on what he calls "commonsense" and "bankruptcy" are worthless as a philosophy. There is no rationale for thinking that the Comprehension Principle is more "natural" as a hypothesis than any other, say Zermelo's Axiom of Separation. As a piece of history, however, Quine's views are very interesting and revealing — they display before us the picture of a generation of logicians whose "intuitions," to use his word, had been trained in the logic of classes based on Comprehension.[3] One is tempted to apply here Kuhn's terminology, speaking of a paradigm shift (from the Comprehension logic of classes to axiomatic systems of set theory), of Quine as a member of the old generation, unable to absorb the impact of the revolution, and so on. But this is not our topic today.

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[2]The abbreviations come from the titles of the works in which he proposed those systems: 'New Foundations' (1937) and *Mathematical Logic* [Quine 1940].

[3]The notion of intuition that emerges here is very far from the Kantian one, but perhaps close to the teachings of Felix Klein.

1. Generalities about dogma.

In what follows I intend to discuss several different images of foundations, and a few dogmas that are in fact interrelated. Let me open my mind and mention at least one of these right away. We all have some image of what the *relations* are *between foundations, logic, and human reason* (a noteworthy triangle). This frequently belongs to our cherised intimate convictions, and it may well happen that we do not want to discuss them with strangers. I beg your pardon, but as a philosopher it is my business to thematize such intimate convictions and reflect about them, eventually trying to see if they depend too much on traditional beliefs.

Just in case somebody doubts that the connections between logic, reason and foundations can really be found in noteworthy contributions to this field, let me offer an example. Recall the well-known Hilbert program. It is interesting to reflect on some of the more philosophical opinions that Hilbert expressed (and actually he was generous in offering such opinions). For instance, in his address *'Axiomatisches Denken'* of 1918 we can read:

> [...] the most important mathematical thinkers [...] have always cultivated the relations to the neighboring sciences, especially the great empires of physics and epistemology, above all for the benefit of mathematics itself. The essence of these relations and the reasons for their fruitfulness will be most clear [... by] describing [...] the *axiomatic method*. [Hilbert 1935, 146]

In recent years, historians such as Volker Peckhaus [1991] and Leo Corry [1997] have made clear the extent to which this was not propaganda, but the expression of a serious concern of Hilbert's with physics (especially its axiomatization) and with epistemology.

In the celebrated 1900 lecture on *'Mathematische Probleme'*, Hilbert already expresses his concern with deductive finitism, and says as follows:

> This requirement of logical deduction by means of a finite number of processes is simply the requirement of rigour in reasoning. Indeed the requirement of rigour, which has become proverbial in mathematics, corresponds to a universal philosophical necessity of our understanding (*Verstand*). [Ewald 1996, vol. 2, 1099]

As Michael Hallett [1994] has made clear in a paper devoted to connections between Hilbert's axiomatic method and the "laws of thought," his finitism was thus meant to express a key (though weak) principle about the workings of the mind. In 1928, Hilbert made the point by saying nothing less than the following:

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. [van Heijenoort 1967, 475]

In these sentences, the word *Verstand* seems to have been chosen to comply with Kant's terminology,[4] but, being a bit less literal, we could perfectly well translate it by "reason."

In sum, according to Hilbert there is a very close link between foundational research and the study of thinking and reason, i.e., epistemology. To be sure, Hilbert did not pretend that his *Beweisstheorie* would exhaust that study and supplant epistemology. Many important epistemological matters would remain open even after the satisfactory completion of his program (see Peckhaus [1991]). But the foundational results of *Beweisstheorie* certainly had, in his view, quite direct epistemological implications.

Is this a conception that one must share? Or, does it depend on certain assumptions that are open to questioning? A short general reflection may suggest the latter. In both scientific and mathematical research, ten years are a very long time, but, as you know, in the realm of human beliefs and intellectual traditions a century may not be a very long time. It is only in the last century and a half that we have started to think about human beings and minds in terms of evolution.[5] We belong to a culture that has almost always regarded men and women as having an eternal essence, and most of us have been raised in this belief. Different names for that essence, obviously with different connotations, have been "the soul" and "reason."

Kant's philosophical ideas, to be sure, were formulated long before any need could be felt to even faint that such a belief should be questioned. So one might well employ the following rule of thumb: whenever you see a foundational researcher that is deeply influenced by Kant, beware and consider the possibility that he may be equating epistemology with foundations in a way that is unwarranted, i.e., that is based on questionable assumptions.

Intimately entangled with the idea of a special link between logical foundations and human reason, one can find other noteworthy beliefs. When we do research on the foundations of mathematics, we look for systematic foundations. Of course it is very healthy and interesting to

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[4]His distinction of the more dry faculty of the *Verstand* (understanding) from the idealizing faculty of the *Vernunft* (reason).

[5]I mean here the *fact* of biological evolution on earth, and leave open the thorny question of the right *theory* of evolution.

search for *unification and system*, and I could have nothing against it. But, when coupled with the above-mentioned belief in a transcendent essence, one can easily be led into thinking that the systematic foundations one obtains tell us something about human reason. The very idea of systematicity may thus, coupled with other beliefs, give rise to a second dogma.

We might go on trying to locate and reconsider beliefs that could be dogmatic. Some come to my mind rather quickly. The idea that mathematics and logic are marked out by enjoying *absolute certainty*, by consisting of results built on the most solid rock (not "on sand," as Weyl once wrote). Or perhaps our habit of taking logic to be *a* (even *the*) kernel of human knowledge, and our associated tendency to assume that any act of thinking ought to correspond to some kind of logic (say, e.g., inductive logic). Also the high expectations we place on the capacity of formal systems to capture all kinds of subtle concepts and relations.

But enough. At this point we shall stop talking generalities and begin to examine some particular, historically given images of foundations. This will allow me to review some key stages in the evolution of foundational ideas, and also to analyze the emergence and persistence of some dogmatic beliefs. We shall concentrate on the first two mentioned above: the idea of a special link between foundations and epistemology, and the epistemological interpretation of systematicity.

2. Two traditional images: Gauss to Hilbert.

Let me begin with Gauss, just 200 years ago. As a young man, Gauss was interested in philosophy and read Kant and other philosophers with some care. We are fortunate to have a few statements of his, expressing thoughts about the philosophical foundations of mathematics — what he called, using now obsolete terminology, the "metaphysics of mathematics."[6] Occasionally, Gauss felt the need to enter into a discussion of such philosophical topics in order to clear the way to novel mathematical developments. Thus he ended up touching on the "metaphysics" of number, of magnitudes, and of space. Particularly important were his parallel reflections on the epistemological foundations of geometry and of arithmetic. These reflections were rich in mathematical content, since they related to Gauss's evolving views on non-Euclidean geometry and differential geometry, on number theory and function theory. Their final

[6] Fragments from 1800, 1816–17, 1825, 1831. See [Ferreirós, *forthcoming*].

outcome was synthesized by the learned man in the form of a Greek motto:

Ὁ θεος ἀριθμητίζει

"God does arithmetic," his thoughts consist in numbers and number-relations, even when we cannot follow them. The significance of this statement can only be understood if we take into account that Plato is reported to have said, "God does geometry." The Gaussian motto documents the end of the millenary domination of geometry in Western images of mathematical knowledge, and starts a historical period in which pure mathematics would be "under the sign of number" [Hilbert 1897]. The idea was spelled out in letters of 1817 and 1830; I quote the 1830 letter to Bessel:

> According to my most intimate conviction, the theory of space has a completely different position with regards to our knowledge *a priori*, than the pure theory of magnitudes. Our knowledge of the former lacks completely *that* absolute conviction of its necessity (and therefore of its absolute truth) which is characteristic of the latter. We must humbly acknowledge that, if number is *just* a product of our minds, space also has a reality outside our minds, and that we cannot prescribe its laws *a priori*. [Gauss 1900, 201]

The language is strikingly Kantian, certainly more so than the language we find in Poincaré's allusions to Kantian epistemology.

Gauss was not doing "metaphysics" in our current sense of the word. What he did was to search for new systematic accounts of several mathematical topics, to elaborate new mathematical theories, hoping to find thereby the philosophical and epistemological basis of mathematical knowledge. But his results and theories did not possess philosophical meaning in and off themselves. They acquired epistemological significance because Gauss viewed them against the background of Kantian (and Leibnizian) epistemology. He was guided by the old image of human knowledge as a combination of elements derived from two sources: Reason and the senses, the rational and the empirical. In Gauss's view, agreeing with Leibniz and Kant, mathematical knowledge has a strictly rational core, which is an *a priori* product of pure Reason. Geometry did not belong in that core, which he now identified with "arithmetic" in a broad sense — the theory of the complex number system in all its aspects.

Though some of these reflections became available to the public as of 1831, for the most part they remained in the hands of a few friends until they finally were published in the 1860s. From this time on, it became customary among relevant German authors to conceive of "arithmetic" as another name for pure mathematics, and to exclude geometry from this domain.[7] To give just an example, when Dedekind gave expression to the view that pure mathematics is logic, 4 years after Frege but independently of him, he wrote: "arithmetic (algebra, analysis) is just a part of logic." Interestingly, Dedekind chose to synthesize his new view with a motto that prolonged the Plato–Gauss tradition:

Ἀεὶ ὁ ἄνθρωπος ἀριθμετίζει

i.e., "man ever arithmetizes." With this new move, the pure theory of numbers and their relations ceases to be a godly matter, to become human, very human.

In the preface to his epoch-making *Was sind und was sollen die Zahlen?*, Dedekind made clear his opinion that arithmetic and pure mathematics are a matter of logic, an immediate product of the *reine Denkgesetze*, the pure laws of thought. They rest solely on the notions of set and mapping, and therefore on primitive abilities of the mind without which no thinking at all is possible. Thus they are available to anyone in possession of a *gesunder Menschenverstand*, a sane common understanding. Dedekind went as far as to propose the view that the chain of proofs that he presented in detail takes place actually in the mind of his readers as soon as they employ numbers; only it takes place unconsciously, and so our consciousness extends merely to some byproducts, complex arithmetical truths that we usually mistake for simple, intuitive evidences.

To me, the situation with Dedekind is reminiscent of what we have seen apropos of Quine. I admire very much Dedekind's work, but his belief that the notions of set and mapping are primitive ones, that they are engraved in our minds from the time of conception — this I can only regard as an unwarranted belief, and in all likelihood a false one. It is surprising to find that so many authors, including good philosophers of mathematics in recent years, still consider the notion of set as an intuitive one, as epistemologically primitive.[8] Of course this point is easier for us to grasp with hindsight, thanks to our knowledge of issues like the

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[7]There is an early exception, Martin Ohm, who identified *Zahlenlehre* with pure mathematics from as early as 1819. He is certainly relevant because his textbooks and his views enjoyed wide diffusion in the *Gymnasien*.

[8]Choosing only among first-rate authors, an example can be found in Maddy [1992].

polemics surrounding the axiom of choice, the possibility of predicative and other deviant conceptions of sets, and so on.

The work of Weierstrass, Dedekind and others on the "arithmetization" of pure mathematics led to modern systems of logic and set theory, which started to become explicit in work of Dedekind, Peano, and Frege during the 1880s. But these men were still immerse in the old conception of human knowledge, and through them it influenced other authors like Hilbert himself. As we have seen with Dedekind [1888, iv–v], they believed that actual human knowledge, as historically given, is (partly at least) the product of unconscious rational, logical thinking activities.[9] Their search for deeper systems of mathematics was for them a logical excavation in the hidden structure of Reason. This constellation of ideas we shall call *Image 1.*

Image 1 was also in Frege's mind when he compared arithmetic "with a tree that unfolds upwards in a multitude of techniques whilst the root drives into the depths" [Frege 1893, xiii]. Frege's main goal was epistemological: he wanted to prove most strictly that the laws of arithmetic are *a priori*, indeed that they are purely logical laws. It was only as a *means* to obtain full control of his assumptions and developments, to check systematically that his goal had been attained, that he became interested in formal systems of logic. As I have said before, under the assumptions of Image 1, it was natural to expect that the systematic search for sounder and broader logical bases would amount to a search for epistemological foundations. But images of foundations and foundational research have changed greatly since the nineteenth century.

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The search for logical foundations culminated in the new axiom systems proposed during the decade 1899–1908 for geometry, for the arithmetic of both the natural and the real numbers, for set theory. The gain in terms of unification, systematization, and freedom to work in modern mathematics, was undeniable and certainly wonderful.

These new systems were perfected and made fully precise some twenty years later with their strict formalization (due in good measure to Weyl, Skolem, Hilbert and Bernays)[10]. Around 1920, Hilbert conceived of a

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[9] Frege is a different matter, for his fight against psychologistic logic distanced him from this standpoint, but he had little success until well into the 20^{th} century.

[10] Of course, here one must also remember Frege, Peano, Russell and Whitehead, but when I say "strict" formalization I must refrain from citing them (especially the last three). We are talking about first-order formal axiomatizations.

change in perspective that would give rise to a new foundational program and a new image. The new Image 2 inherited many of the traits of 1, unreflectedly to a good extent. Hilbert was still attracted by Image 1, but he was aware of the difficulties involved in trying to show the *a priori* (indeed, the logical) nature of any of the above-mentioned systems, and particularly aware of the need to refine and develop logic to suit the needs of modern mathematics.

Hilbert and Bernays started the new conception when it finally became clear that the former's hopes for a revival of logicism, in the wake of *Principia Mathematica*,[11] had foundered. This had a neat effect on the goals of the whole foundational program, for now it was no longer a matter of establishing the *truth* of the propositions belonging to pure mathematics, or the *sources* for such truths, but merely a question of establishing the *acceptability* of classical mathematical systems by a strict proof of consistency. The shift was not voiced very much aloud, but it is very noteworthy — it involved acceptance of the *hypothetical character* of pure mathematics (what some call, ambiguously, its 'quasi-empirical' status), and thus a deep reform of received images of mathematical knowledge.[12]

Within *Image 2*, the main goal was to find sound *formal* systems, sufficiently powerful to derive all of classical mathematics within them (a requirement that was at first simply equated with completeness), but such that they could be *finitarily* proven to be *consistent*. Recourse to formal languages, which in Frege had been merely a means to check the sufficiency of the proposed axioms/principles,[13] now became an essential trait of the foundational program.

But many authors continued to believe that the formal systems would somehow uncover the hidden logical structure of mathematical Reason, which supposedly had always acted behind the course of historical events. Some even hoped that the chosen system would be all-embracing, in such a way that new mathematical developments would remain within its bounds. As if our historical experience did not show mathematics to be a creative human activity, and mathematical theories the temporary outcomes of an open-ended process of development.

[11] I am referring to the 1917 Zürich address *Axiomatisches Denken* (in [Hilbert 1935]) and to some of Hilbert's courses at Göttingen. On this topic, see Sieg [1999].

[12] From Plato to Kant, from Descartes (perhaps even Euclid?) to Frege, the propositions of mathematics were taken to be truths *simpliciter*.

[13] "thus we obtain a basis for judging the epistemological nature of the law we have proven" (preface to Frege [1893]).

When Gödel's incompleteness results forced foundational research to shift its main target, from consistency to goals such as relative consistency (and others), a very interesting branch of mathematics came to maturity — indeed, a handful of branches. We might go on here and attempt a finer analysis of several new and different *Images 2.1, ... 2.n* of foundational research, among which programs like those of proof theory, model theory, or reverse mathematics, are particularly noteworthy. These programs have made available very interesting results, such as those of Gentzen and his followers on the consistency of arithmetic and other systems, the results of Cohen on the independence of the Continuum Hypothesis, etcetera, and more recently the results obtained in predicative mathematics and reverse mathematics.

From the standpoint of my present review of the connections between foundations and epistemology, however, we must emphasize that one can no longer see in any of these projects and results a full-fledged program to establish the foundations of mathematical knowledge. Their epistemological relevance is not obvious, which contrasts strongly with Hilbert's desire to "eliminate from the world once and for all the question of the foundations of mathematics" by establishing the absolute "freedom from contradiction" of the classical theories with his *Beweisstheorie*. (To be sure, the idea I am now presenting is not new, it has been emphasized, e.g., by Feferman and Sieg; but too many others are not yet aware of it.)

I would not wish to go on without one further comment. To say that foundational studies, in their present shape, are not immediately relevant to the epistemology of mathematics, is by no means the same as saying that they are (or even worse: that they must be) irrelevant. Many of the classical results in foundational studies are highly illuminating as to the nature and characteristic traits of classical mathematics. A paradigmatic example was the increasingly clearer realization of the contrast between (process-oriented) constructive mathematics, and classical mathematics (object-oriented, "platonistic"). Another was, of course, the discoveries about possibilities and limits of the formalization of mathematical theories.

While many present-day results on foundational matters will probably be of no consequence to future epistemological debates, some are of a different kind. From this standpoint, I would like to warn against a wrong interpretation of my words, and I would like to make a call for renewed interactions between foundational studies and the philosophy of mathematics (to which I must add the history of mathematics).

3. A historical alternative: Riemann to Kitcher.

Let us come back to the presumed convergence of systematic foundations with epistemological roots. I have suggested (very sketchily) some ideas about its origins and development, and how it became increasingly dissolved within the transformations suffered by foundational research in the twentieth century. Today, the old static image of knowledge is deeply outdated, judging it against the framework of present-day philosophy or, for that matter, of contemporary scientific knowledge. Once we discard the *a priori* belief in the existence of a transcendent human faculty that goes under the name of Reason; once we take into account scientific discoveries about the biological and cultural evolution of humanity; once we consider historical studies of past mathematical theories and practices; it becomes quite doubtful that a hidden structure may have been present throughout.

For reasons like these, in recent decades new kinds of historicistic and naturalistic conceptions of mathematical knowledge have emerged. It is interesting to realize that these recent trends are akin to a very different approach to the foundations that existed since the mid-nineteenth century, coexisting with Images 1 and 2. We may speak of a tradition starting already with Riemann, and continuing through a good number of twentieth-century authors. Interestingly, Poincaré counts among them.

Due to his philosophical beliefs, Riemann consciously avoided the image of Reason as the *a priori* source of knowledge. In his view, all knowledge arises from the interplay of "experience" broadly conceived (*Erfahrung*) and "reflection" (*Nachdenken*) in the sense of reconceiving and rethinking. Human knowledge begins in everyday experiences and proceeds to propose conceptual systems which aim to clarify experience by going beyond the surface of appearances. Reason in the old sense is found nowhere, there is no hidden *a priori* structure — those elements in our theories which do not simply arise from sense-data are just of a conjectural nature, hypotheses like the axioms of geometry [Riemann 1854].[14] To give you at least a superficial impression of Riemann's turn of mind, let me quote a fragment from the last page of his famous lecture *On the hypotheses upon which geometry is founded*:

> A decision regarding these questions [about the validity of geometrical as-

[14] It may be convenient to remind the reader that Riemann's differential-geometric axioms or "hypotheses" are quite different from, and deeper than, Euclid's.

> sumptions at different levels of physical reality] can only be taken by starting from the previous conception of phenomena, whose foundations were laid by Newton and which has been confirmed by experience, and by reforming it gradually, considering facts that cannot be explained from it. Investigations which start from general concepts, like the one developed here, can only serve to avoid that such work may be hindered by conceptual limitations, and that the progress in our knowledge of the connections among things may be limited by prejudices handed down by tradition. [Riemann 1854, 286]

For *Image 3*, the point of foundational studies is to perfect available conceptual systems by spotting conceptual or theoretical inadequacies, to expand the range of available concepts and avoid conceptual limitations, to strive for greater generality, and to eliminate traditional prejudices. Thus, Image 3 is strongly diachronistic. It is no longer a matter of excavating hidden structures, but of going beyond traditional ideas in order to gain a deeper grasp of reality (though perhaps not a perfectly realistic one).

New forms of developmental understanding of mathematical knowledge would later be found in Weyl, in Piaget, in French authors like Cavaillès, even in one of Hilbert's closer collaborators, Paul Bernays [1976]. More recently, philosophers such as Lakatos and Kitcher joined this group and tried to analyze the fine structure of historical processes of mathematical development.

But, should we come to the conclusion that mathematical knowledge is absolutely undetermined, except for the constraints imposed by tradition and history? In my opinion, not at all.

4. Towards a new image of the roots of mathematics.

Human history, including mathematical history, is an expression of human activities. Its diversity and degrees of freedom will therefore be limited by any strong constraints acting upon human activity. Here, I must limit myself to a rough sketch of what seem to be key constraints (though I am obviously aware of the controversial nature of these issues, and the resulting need to discuss carefully and substantiate each of the following points).

Human beings are members of the human species, "linguistic animals" as Aristotle said, at once biological and social. A world of physical objects, biological abilities and needs, sense-perception and motor action, the use of language within a web of social life and common activities — these are some of the constraints bounding human activity and therefore history.

That constitutes an invitation to explore and formulate varieties of so-called naturalism that may fit with biology and history at a time. To avoid confusions generated by the trendy and equivocal term "naturalism," it might be preferable to describe the kind of viewpoints I am thinking about through some other label. Perhaps one might use the phrase *genetic epistemology*, but in a sense divergent from orthodox Piagetianism. Indeed the adjective "genetic" suggests the biological genesis of human knowledge, its emergence from our natural abilities, but also the historical genesis of human knowledge.

At this point, we are not talking about an "image 4" of foundations, because this kind of viewpoint incorporates a deep shift, which we could try to make explicit by distinguishing *roots* from *foundations*. Now we are not analyzing the foundations of mathematics, looking for a purely rational or (at least) a perfectly systematic framework within which to develop current mathematics. Instead, we aim to explore the epistemological roots of mathematical knowledge. This is what I mean to connote by changing the biological metaphor of roots for the architectural metaphor of foundations.

But, again, this is not the place to attempt an original theoretical development. So let me close with some remarks on a most noteworthy proponent of naturalism, none other than Quine [1969]. The writings of this great philosopher and logician radiate with the joy of engaging in the search for systems. Quine was a great system builder both as a logician, as a philosopher of language, and as a naturalist. But when he transferred this trend of mind from logic to epistemology, I believe he was making a characteristic mistake. In line with the old epistemological tradition, Quine believed that the search for systematic foundations converges with the quest for epistemological roots. In this latter context, however, systematicity may well be a trap rather than a virtue.

Let me give a telegraphic example. There is not the least reason to believe that arithmetic as practiced by a 10-year-old child "must converge" with the foundations of our set of natural numbers, or that fractions as used by different cultures must be systematically explained together with the rational number system [Benoit *et al.*, 1992]. Elementary, commonsense arithmetic can be understood from a purely constructive standpoint, while the step to N as a set, harmless as it may seem to present-day mathematicians, involves of course the introduction of actual infinity.

Peano arithmetic involves quantification over infinite domains, and thus what Hilbert and his followers called a "transfinite axiom" [Hilbert 1925, 382].

What a genetic epistemology (in the above sense) should do, is to analyze the epistemological roots of these *different* practices and theories, and to understand the links between them (be they cognitive, historical, social, or what not). It should also emphasize the shifts and displacements which distance them, and search for the factors that help explain those shifts. In doing so, one must carefully avoid the temptation of being over-systematic. So let me take exception to Quine's dogma, and invite you again, in the spirit of Riemann, to reconsider received ideas on all of these issues.

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