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ON ANALYTICALLY EQUIVALENT IDEALS

by A. SEIDENBERG [1]

Two ideals $\mathfrak{a}$, $\mathfrak{a}'$ in a ring of formal power series $k[[X_1, \ldots, X_n]]$ over a field k are said to be *analytically equivalent over* k if there is an automorphism of $k[[X]]$ over k mapping $\mathfrak{a}$ onto $\mathfrak{a}'$; and they are said to be *analytically equivalent over an extension field* Ω *of* k if $\Omega[[X]].\mathfrak{a}$, $\Omega[[X]].\mathfrak{a}'$ are analytically equivalent over Ω. In [S_1, p. 41], the following question was posed. In the case that k is algebraically closed, if the ideals $\mathfrak{a}$, $\mathfrak{a}'$ are analytically equivalent over an extension field Ω of k, then does it follow that they are already analytically equivalent over k? The answer, as we show below, is yes. Without loss of generality, we take Ω to be a universal domain over k [2].

We shall need the following:

Lying-over theorem for algebraic groups. — *Let* G, G′ *be algebraic groups* [3] *lying in affine spaces* A, A′ *with* $A' = A \times B$ (B *affine*) *and let* π *be the projection of* $A \times B$ *on its first factor. Assume that the set-theoretic projection of* G′ *is contained in* G *and that* π *is compatible with the operations of* G, G′ (i.e., *that* $\pi(a.b) = \pi(a).\pi(b)$). *Then the set-theoretic projection of* G′ *is closed in* G.

Proof. — This follows from [R, p. 409], which says that " if τ is a rational homomorphism of *any* algebraic group G_1 into an algebraic group G_2, then the kernel and image of τ are algebraic subgroups of G_1, G_2 respectively ".

Corollary. — *Let* $G_1, G_2, \ldots$ *be a sequence of algebraic groups such that for each* i, G_i, G_{i+1} *stand in the relation of* G, G′ *in the theorem*; *and let, for* $i<j$, π_{ij} *be the projection from* G_j *to* G_i. *Then for each* m *there is an* $N = N(m)$ *such that* $\pi_{mN}(G_N) = \pi_{m,N+1}(G_{N+1}) = \ldots = H_m$; *and over every point of* H_m *there lies a point of* H_{m+1}.

Proof. — The $\pi_{mj}(G_j)$ form a decreasing sequence of closed sets in G_m and so must become stationary after a finite number of steps. The second point follows because the projections are set-theoretic projections.

We define a subset of an abstract variety V to be *constructable* if it is the finite union of sets each of which is the intersection of a closed set and an open set; the *k-constructable* subsets of a variety V defined over k are similarly defined. The complement in V of a k-constructable set and the finite union and finite intersection of k-constructable sets are k-constructable: these are rather immediate assertions. Defining the dimension of a constructable set C to be the dimension of its closure F_1, one also easily sees that $C = F_1 - C_1$ with C_1 constructable and, unless C is empty, $\dim C_1 < \dim C$. (We write $A - B$ only if B is contained in A, so C_1 is uniquely determined.) A similar statement holds for k-constructable sets, and is also easily seen.

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(2) For the definitions of: *universal domain, variety, field of definition of a variety, k-generic point of a variety, geometric projection of a variety, (k-)component of a (k-)closed set, k-independent points, rational map*, see [W].

(3) An *algebraic group* is a group G which is a finite union of disjoint (abstract) varieties, called its components, such that for any pair of components G_i, G_j the product $g_i g_j$ for $g_i \in G_i$, $g_j \in G_j$ is given by an everywhere defined rational map of $G_i \times G_j$ into one of the components G_k; and similarly for g_i^{-1}.

Remark. — *a)* A constructable subset of a variety defined over k is k-constructable if and only if it is invariant under every automorphism of the universal domain Ω over k. *b)* The closure of a k-constructable set is the same as its k-closure. *c)* Besides the properties already mentioned, the main property of the class of constructable sets is that the set theoretic projection of a constructable subset of a product $V \times W$ on a factor is constructable. This is known; see [C, p. 38, Th. 3, Cor.]. A similar property holds for k-constructable sets, as one can see by using *a)*. We omit the proofs, as we do not employ the assertions.

Lemma. — *Let* G *be an algebraic group defined over a field k and let* G *be connected* (i.e., *assume* G *is a variety*). *Let* H *be a subgroup of* G *(not given to be algebraic). Then if* H *is k-constructable, it is k-closed (and hence is an algebraic subgroup).*

Proof. — Let $\bar{H}$ be the k-closure of H. Then $\bar{H}$ is a subgroup of G. In fact, let $\bar{x}, \bar{y} \in \bar{H}$. Since H is k-constructable, its k-closure is the set of k-specializations of its points. Hence there exist $x, y \in H$ with $x \to \bar{x}$, $y \to \bar{y}$ over k. Since the product in G is everywhere defined over k, we have $xy \to \bar{x}\bar{y}$ over k; similarly, $x^{-1} \to \bar{x}^{-1}$ over k. Hence $\bar{x}\bar{y}, \bar{x}^{-1} \in \bar{H}$ and $\bar{H}$ is a subgroup. Hence it is an algebraic group (see [R, p. 408]). Let $\bar{H} = H_1 \cup \ldots \cup H_s$, where the H_i are the components of $\bar{H}$. Then $H = (H_1 - C_1) \cup \ldots \cup (H_s - C_s)$, where C_i is constructable, $C_i \subset H_i$, and $\dim C_i < \dim H_i$. We claim that the C_i are empty. In fact, let H_1 be the component of the identity and suppose $C_1 \neq \emptyset$. Let $x \in C_1$. Let k_1 be a field of definition of H_1 containing k and such that C_1 is k_1-constructable. Let P be a $k_1(x)$-generic point of H_1. Then xP and P^{-1} are also $k_1(x)$-generic points of H_1. Hence $xP, P^{-1} \in H$, as they are not in C_1. Then $x = xP \cdot P^{-1} \in H$, contradiction. Hence $C_1 = \emptyset$; and as the H_i are the cosets of H_1, also the other C_i are empty. Hence $H = \bar{H}$. Q.E.D.

Let k be an arbitrary field. Every automorphism τ of $k[[X_1, \ldots, X_n]]$ over k is a substitution of the form $X_i \mapsto \bar{X}_i = \lambda_{i1} X_1 + \ldots + \lambda_{in} X_n + \ldots$, where the coefficients are in k and $\det \tau = \det(\lambda_{ij}) \neq 0$; and conversely. In particular this is so for the universal domain Ω. For every $F \in \Omega[[X]]$, let F_N be the sum of the terms of degree $\leq N$ in F. By *the* N-th *approximant* τ_N *of an automorphism* $\tau : X_i \mapsto \bar{X}_i$ of $\Omega[[X]]/\Omega$ $(N \geq 1)$, we mean the substitution $\tau_N : X_i \mapsto \bar{X}_{iN}$. Let $\sigma : X_i \mapsto \mu_{i1} X_1 + \ldots + \mu_{in} X_n + \ldots$ be another automorphism. The N-th approximants are also automorphisms, but we compose them anew according to the definition: $\sigma_N \circ \tau_N = (\sigma_N \cdot \tau_N)_N$; this is the same as $(\sigma \cdot \tau)_N$. Then the set of N-th approximants of the set of all automorphisms of $\Omega[[X]]/\Omega$ forms a group $\mathcal{J}_N$; the inverse of τ_N is $(\tau^{-1})_N$ for any automorphism τ having τ_N as N-th approximant. $\mathcal{J}_N$ is an algebraic group, defined over the prime field (1).

Let k be a subfield of Ω and let $\mathfrak{m} = k[[X]] \cdot (X_1, \ldots, X_n)$. Note that $\tau_N(F) \equiv \tau(F) \bmod \Omega[[X]] \cdot \mathfrak{m}^{N+1}$ for any $F \in \Omega[[X]]$ and automorphism τ of $\Omega[[X]]/\Omega$.

Let $\mathfrak{a}$, $\mathfrak{a}'$ be two ideals in $k[[X]]$ and consider the automorphisms of $\Omega[[X]]/\Omega$ which map $\Omega[[X]] \cdot \mathfrak{a}$ onto $\Omega[[X]] \cdot \mathfrak{a}'$. These *a fortiori* map $\Omega[[X]] \cdot (\mathfrak{a}, \mathfrak{m}^{N+1})$ onto

(1) Here we identify a τ_N via its coefficients with a point of an affine space. The notions are relative to $X_1, \ldots, X_n$.

$\Omega[[X]].(\mathfrak{a}', \mathfrak{m}^{N+1})$; and so do their N-th approximants. We will also consider the set of N-th approximants τ_N such that τ_N maps the first of these ideals onto the second.

To get the conditions on the coefficients of an automorphism τ in order that it map $\Omega[[X]]\mathfrak{a}$ onto $\Omega[[X]]\mathfrak{a}'$, write down an automorphism

$$\tau : X_i \mapsto \overline{X}_i = \Lambda_{i1} X_1 + \ldots + \Lambda_{in} X_n + \ldots$$

with indeterminate coefficients Λ. The coefficients of τ^{-1} are rational functions of Λ which can be written with powers of $\det(\tau)$ as denominators. Let $(F_1, \ldots, F_s)$, $(F'_1, \ldots, F'_{s'})$ be bases of $\mathfrak{a}$ and $\mathfrak{a}'$. Let $A_1(C, X), \ldots, A'_{s'}(C, X)$ be $s+s'$ power-series with indeterminate coefficients C. Now write $F_i(\tau(X)) = A'_1(C, X)F'_1 + \ldots + A'_{s'}(C, X)F'_{s'}$. Comparing coefficients, we get a system of equations in C, Λ which, together with the inequality $\det \tau \neq 0$, determines the automorphisms mapping $\Omega[[X]].\mathfrak{a}$ into $\Omega[[X]].\mathfrak{a}'$. Writing $F'_i(\tau^{-1}(X)) = A_1(C, X)F_1 + \ldots + A_s(C, X)F_s$, comparing coefficients, and clearing with powers of $\det(\tau)$, we similarly get the conditions that τ^{-1} map $\Omega[[X]].\mathfrak{a}'$ into $\Omega[[X]].\mathfrak{a}$. Together, we get a system $\mathscr{S}$, consisting of polynomial equations over k in C, Λ and the inequality $\det(\Lambda_{ij}) \neq 0$, which determines the automorphisms mapping $\Omega[[X]].\mathfrak{a}$ onto $\Omega[[X]].\mathfrak{a}'$. Replacing τ by τ_N and the two equalities by congruences mod $\Omega[[X]].\mathfrak{m}^{N+1}$, we get the conditions for τ_N to map $\Omega[[X]].(\mathfrak{a}, \mathfrak{m}^{N+1})$ onto $\Omega[[X]].(\mathfrak{a}', \mathfrak{m}^{N+1})$; but this time we get a *finite system* $\mathscr{S}_N$. Moreover, $\mathscr{S}_N$ is a subsystem of $\mathscr{S}$. By [S_2, p. 237, Th. 1] the C can be eliminated from $\mathscr{S}_N$, yielding a finite disjunction of finite conjunctions of polynomial equations and inequalities over k in (a finite subset of the) Λ. Thus the τ_N form a *k-constructable set* [1], and we have the following:

Theorem 1. — Let $\mathfrak{a}$, $\mathfrak{a}'$ *be ideals in* $k[[X]]$. *Then the* N-th *approximants* τ_N *which map* $\Omega[[X]].(\mathfrak{a}, \mathfrak{m}^{N+1})$ *onto* $\Omega[[X]].(\mathfrak{a}', \mathfrak{m}^{N+1})$ *form a (possibly empty) set* C_N *k-closed in the set of all* N-th *approximants.*

Proof. — The lemma gives the case $\mathfrak{a} = \mathfrak{a}'$; and the case $\mathfrak{a} \neq \mathfrak{a}'$ now also follows.

Corollary. — If $\mathfrak{a}$, $\mathfrak{a}'$ *are analytically equivalent over* Ω *and if k is algebraically closed, then for every* N, $(\mathfrak{a}, \mathfrak{m}^{N+1})$, $(\mathfrak{a}', \mathfrak{m}^{N+1})$ *are analytically equivalent over k.*

Proof. — We are given that $\mathscr{S}$ has a solution c, λ in Ω. The system $\mathscr{S}_N$ is a subsystem of $\mathscr{S}$. Thus a finite subset of c, λ yields a solution of $\mathscr{S}_N$. This finite subset has an algebraic (hence k-rational) specialization over k. This k-rational solution of $\mathscr{S}_N$ shows that $(\mathfrak{a}, \mathfrak{m}^{N+1})$, $(\mathfrak{a}', \mathfrak{m}^{N+1})$ are analytically equivalent over k. Q.E.D.

Consider the case $\mathfrak{a} = \mathfrak{a}'$. Then C_N is an algebraic group G_N defined over k. We have a sequence $G_1, G_2, \ldots$ such that for any i, j with $0 < i < j$, the lying-over theorem may be applied to G_i, G_j. Then for every m there is an $N = N(m)$ such that $\pi_{mN}(G_N) = \pi_{m,N+1}(G_{N+1}) = \ldots$ Let $H_m = \pi_{mN}(G_N)$.

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[1] Instead of [S_2, p. 237, Th. 1] we may use Remark *c)*, above. [S_2, p. 237, Th. 1] is a special case (of the assertion that the set-theoretic projection of a k-constructable set is k-constructable), from which, however, the extension to abstract varieties is immediate. For further properties of k-constructable sets, see [S_3, p. 370 and p. 373, Remark *c)*] and a forthcoming work of ours (in Crelle's J.) entitled *On k-constructable sets, k-elementary formulae, and elimination theory.*

Theorem 2. — *Let τ be an automorphism of $\Omega[[X]]/\Omega$ mapping $\Omega[[X]].\mathfrak{a}$ onto itself. Then $\tau_m \in H_m$ for every m. Conversely, if $\tau_m \in H_m$, then τ_m is the m-th approximant of an automorphism mapping $\Omega[[X]].\mathfrak{a}$ onto itself.*

Proof. — Let $N=N(m)$ as in the preliminary remark. Then τ_N projects into τ_m. Hence $\tau_m \in H_m$. Conversely, let $\tau_m \in H_m$. Then there exists a τ_{m+1} in H_{m+1} lying over τ_m; and a τ_{m+2} in H_{m+2} lying over τ_{m+1}; etc. The sequence $\tau_m, \tau_{m+1}, \ldots$ by an obvious limit process yields an automorphism such that for every N, τ maps $\Omega[[X]].(\mathfrak{a}, \mathfrak{m}^{N+1})$ onto itself, whence [1] τ maps $\Omega[[X]].\mathfrak{a}$ onto itself. Q.E.D.

Let $\tau : X_i \mapsto \overline{X}_i = \lambda_{i1}X_1 + \ldots + \lambda_{in}X_n + \ldots$ be an automorphism of $\Omega[[X]]/\Omega$. By *a specialization $\overline{\tau}$ of τ over k* we mean a substitution obtained by specializing its coefficients over k; we also assume $\det \overline{\tau} = \det(\overline{\lambda}_{ij}) \neq 0$, so that $\overline{\tau}$ is an automorphism. Assume k is algebraically closed. Then every τ_N has a k-rational specialization over k. By Theorem 1, *if τ_N maps $\Omega[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$ onto $\Omega[[X]](\mathfrak{a}', \mathfrak{m}^{N+1})$, then it has a k-rational specialization τ'_N over k which maps $\Omega[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$ onto $\Omega[[X]](\mathfrak{a}', \mathfrak{m}^{N+1})$.* Moreover, *such a τ'_N maps $k[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$ onto $k[[X]](\mathfrak{a}', \mathfrak{m}^{N+1})$*; for it maps $k[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$ *into*

$$\Omega[[X]](\mathfrak{a}', \mathfrak{m}^{N+1}) \cap k[[X]] = k[[X]](\mathfrak{a}', \mathfrak{m}^{N+1}),$$

and its inverse $(\tau'_N)_N^{-1}$, similarly, maps $k[[X]](\mathfrak{a}', \mathfrak{m}^{N+1})$ into $k[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$.

Theorem 3. — *Let k be algebraically closed, $\mathfrak{a}$, $\mathfrak{a}'$ ideals in $k[[X]]$ and assume there is an automorphism β which maps $\Omega[[X]]\mathfrak{a}$ onto $\Omega[[X]]\mathfrak{a}'$. Let m be a given integer ≥ 1. Then there exists an automorphism β' having a k-rational specialization of β_m over k as m-th approximant and mapping $\Omega[[X]]\mathfrak{a}$ onto $\Omega[[X]]\mathfrak{a}'$.*

Proof. — Let $G_1, G_2, \ldots$ be the groups previously described in connection with the automorphisms leaving $\Omega[[X]]\mathfrak{a}$ fixed; and let $N=N(m)$ be the integer there described. Let β'_N be a k-rational specialization over k of β_N mapping $\Omega[[X]](\mathfrak{a}, \mathfrak{m}^{N+1})$ onto $\Omega[[X]](\mathfrak{a}', \mathfrak{m}^{N+1})$. Then $(\beta^{-1}\beta'_N)_N \in G_N$, whence $\beta_m^{-1} \circ \beta'_m = \sigma_m \in H_m$ and $\beta'_m = \beta_m \circ \sigma_m$; here $\beta'_m = (\beta'_N)_m$. By the previous theorem, there is an automorphism σ of $\Omega[[X]]/\Omega$ mapping $\Omega[[X]]\mathfrak{a}$ onto itself and having σ_m as m-th approximant. Then $\beta.\sigma$ is an automorphism mapping $\Omega[[X]]\mathfrak{a}$ onto $\Omega[[X]]\mathfrak{a}'$ and having $\beta'_m = \beta_m \circ \sigma_m$ as its m-th approximant. Q.E.D.

Theorem 4. — *Let k, $\mathfrak{a}$, $\mathfrak{a}'$, β be as in the last theorem. Then there is an automorphism of $k[[X]]/k$ mapping $\mathfrak{a}$ onto $\mathfrak{a}'$.*

Proof. — Starting from β and any positive integer m, we construct $\beta' = \beta^{(m)}$ as in the last theorem. Then starting from $\beta^{(m)}$ and the integer $m+1$ we construct $\beta^{(m+1)}$ as in the last theorem; etc. As the $(m+i)$-th approximants of $\beta^{(m+i)}$ and $\beta^{(m+i+1)}$ are the same, for every i, an obvious limit process applied to the $\beta^{(j)}$ yields a desired β'. Q.E.D.

Remark. — By a theorem of Krull [$K._2$, p. 365], Hilbert's Nullstellensatz continues to hold also for infinitely many quantities $(\lambda_i)_{i \in I}$ (i.e., any such system has a k-rational specialization over the algebraically closed field k) provided that the cardinal number

(1) By Krull's theorem [K_1, p. 207, Th. 2], which says that if $\mathfrak{a}$ is an ideal in a local ring with maximal ideal $\mathfrak{m}$, then $\bigcap_n (\mathfrak{a}, \mathfrak{m}^n) = \mathfrak{a}$.

of k is greater than the cardinal number of I. Thus in the case card k is greater than aleph null, a simple specialization argument yields a desired β'; and, indeed, a β' that is a k-specialization of β (one specializes the coefficients of β along with the quantity $1/\det\beta$). Note, though, that Theorem 4 does not (and could not) give β' as a k-specialization of β. Thus Theorem 3, though it perhaps has no independent interest, is not to be subsumed under Theorem 4.

We can reformulate and extend our results somewhat as follows. First observe that if the ideals $\mathfrak{a}$, $\mathfrak{a}'$ in $k[[X]]$ are analytically equivalent over k (an arbitrary field), then the rings $k[[X]]/\mathfrak{a}$, $k[[X]]/\mathfrak{a}'$ are isomorphic over k. The converse is also true: *if the rings* $\mathfrak{o}=k[[x_1, \ldots, x_m]]=k[[X_1, \ldots, X_m]]/\mathfrak{a}$ *and* $\mathfrak{o}'=k[[x'_1, \ldots, x'_m]]=k[[X_1, \ldots, X_m]]/\mathfrak{a}'$, *are isomorphic over* k, *then* $\mathfrak{a}$, $\mathfrak{a}'$ *are analytically equivalent over* k. In fact, let us identify the rings $\mathfrak{o}$, $\mathfrak{o}'$ via the given isomorphism. From $(x_1, \ldots, x_m)$ we select a minimal basis $x_1, \ldots, x_r$ (say) for the maximal ideal $\mathfrak{m}=(x_1, \ldots, x_m)$; and similarly, let $x'_1, \ldots, x'_{r'}$ be a minimal basis of $\mathfrak{m}$. By [K_1, p. 208] or [Co, p. 56], $r=\dim_k \mathfrak{m}/\mathfrak{m}^2=r'$; here $r=\mathrm{Emb}(\mathfrak{o})$, the embedding dimension of $\mathfrak{o}$. Expanding $x_1, \ldots, x_r$ in terms of $x'_1, \ldots, x'_r$ over k, one finds $x_i=\sum_j c_{ij}x'_j+\ldots, i, j=1, \ldots, r$. Here $\det(c_{ij})\neq 0$ because of the minimality conditions. Now expand the other x_i in terms of $x_1, \ldots, x_r$ and the other x'_j in terms of $x'_1, \ldots, x'_r$ (so that one has

$$d_{r+i,1}x_1+\ldots+d_{r+i,r}x_r+x_{r+1}+\text{higher degree terms}=$$
$$=e_{r+i,1}x'_1+\ldots+e_{r+i,r}x'_r+x'_{r+1}+\text{higher degree terms } (=0)).$$

From this one will get $x_i=\sum_j c_{ij}x'_j+\ldots, i, j=1, \ldots, m$, with $\det(c_{ij})\neq 0$. Then the substitution $X_i\mapsto\sum_j c_{ij}X_j+\ldots$ maps $\mathfrak{a}$ onto $\mathfrak{a}'$. Q.E.D.

Now we extend our previous definition of analytically equivalent ideals by saying that ideals $\mathfrak{a}$, $\mathfrak{a}'$ in $k[[X_1, \ldots, X_m]]$, $k[[X'_1, \ldots, X'_{m'}]]$ are *analytically equivalent over* k if and only if $k[[X]]/\mathfrak{a}$, $k[[X']]/\mathfrak{a}'$ are isomorphic over k (cf. [S_1, p. 31, n° 9]); and we say they are *analytically equivalent over* Ω if and only if $\Omega[[X]]/\Omega[[X]]\mathfrak{a}$, $\Omega[[X']]/\Omega[[X']]\mathfrak{a}'$ are isomorphic over Ω.

Let k be a field and $\mathfrak{o}$ a ring of the form $k[[X_1, \ldots, X_m]]/\mathfrak{a}$. By Cohen's theorem [Co, p. 72, Th. 9], originally conjectured by Krull [K_1, p. 219], every complete local ring containing a field is of this form; and k is uniquely determined up to an isomorphism from $k\simeq\mathfrak{o}/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{o}$.

Let Ω be an extension field of k. We write $k[[x_1, \ldots, x_m]]$ for $k[[X_1, \ldots, X_m]]/\mathfrak{a}$ and $\Omega[[x]]$ for $\Omega[[X]]/\Omega[[X]]\mathfrak{a}$. The ring $\Omega[[x]]$ is the completion of the tensor product $\Omega\otimes_k k[[x]]$ with respect to $\mathfrak{M}=(\Omega\otimes_k k[[x]]).\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $k[[x]]$. Therefore $\Omega[[x]]$ depends, up to an isomorphism over k, only on Ω and $k[[x]]$, but not on x. Formally, we define *the extension of* $k[[x]]$ *by* Ω as the completion of $\Omega\otimes_k k[[x]]$. In statements on the ring $\mathfrak{o}=k[[x_1, \ldots, x_m]]$, by replacing $x_1, \ldots, x_m$ by a minimal basis of the maximal ideal, we may suppose $m=\mathrm{Emb}(\mathfrak{o})$. The embedding dimension does not change upon extension of the base field.

Let $\mathfrak{o}=k[[x_1, \ldots, x_m]]=k[[X]]/\mathfrak{a}$ and $\mathfrak{o}'=k[[x'_1, \ldots, x'_{m'}]]=k[[X'_1, \ldots, X'_{m'}]]/\mathfrak{a}'$.

We say that $\mathfrak{o}$, $\mathfrak{o}'$ are *analytically equivalent over* k if they are isomorphic over k. Thus $\mathfrak{o}$, $\mathfrak{o}'$ are analytically equivalent over k if and only if $\mathfrak{a}$, $\mathfrak{a}'$ are analytically equivalent over k. We say $\mathfrak{o}$, $\mathfrak{o}'$ are *analytically equivalent over* Ω if their extensions by Ω are analytically equivalent over Ω. Thus $\mathfrak{o}$, $\mathfrak{o}'$ are analytically equivalent over Ω if and only if $\mathfrak{a}$, $\mathfrak{a}'$ are so. In studying these equivalences, if $\mathrm{Emb}(\mathfrak{o}) = \mathrm{Emb}(\mathfrak{o}')$, which will be the case if and only if the extensions by Ω have equal embedding dimensions, we may without loss of generality add the assumption $m = m'$. Hence we get the following:

Theorem 5. — Let $\mathfrak{a}$, $\mathfrak{a}'$ *be ideals in* $k[[X_1, \ldots, X_m]]$, $k[[X'_1, \ldots, X'_{m'}]]$ *with* k *algebraically closed. If* $\mathfrak{a}$, $\mathfrak{a}'$ *are analytically equivalent over an extension field* Ω *of* k, *then they are already analytically equivalent over* k. *A similar theorem holds for complete local rings.*

We now make a connection with a result of Hironaka [H].

Let $\mathfrak{m}$, $\mathfrak{m}'$ *denote the maximal ideals in* $k[[x]]$, $k[[x']]$. *Assume* k *is algebraically closed. If the completions of* $\Omega \otimes_k k[[x]]$ *and* $\Omega \otimes_k k[[x']]$ *are isomorphic over* Ω, *then for any positive integer* e, $k[[x]]/\mathfrak{m}^e \simeq k[[x']]/\mathfrak{m}'^e$ *over* k.

This is just a reformulation of the corollary to Theorem 1.

Now according to a result of Hironaka, if the ideals $\mathfrak{m}$, $\mathfrak{m}'$ represent isolated singularities of the configurations given by $k[[x]]$, $k[[x']]$ and if k is algebraically closed of characteristic 0, then the isomorphism over k of $k[[x]]/\mathfrak{m}^e$ and $k[[x']]/\mathfrak{m}'^e$ for some sufficiently large e already implies the isomorphism of $k[[x]]$, $k[[x']]$ over k. Thus our result in a special case is a corollary of Hironaka's. Still, although Hironaka's is a much deeper result than ours, it does not yield ours in the most general case.

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