PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 37 (1969), p. 223-248
<http://www.numdam.org/item?id=PMIHES_1969__37__223_0>

STABILITY OF C^∞ MAPPINGS, IV : CLASSIFICATION OF STABLE GERMS BY R-ALGEBRAS

by John N. MATHER

INTRODUCTION

Let $f:(N, S) \dashrightarrow (P, y)$ be a germ of a C^∞ mapping, where S is a finite subset of N and $y \in P$. (We assume that N and P are finite dimensional manifolds.)

Let the ring homomorphism $f^*: C_y \to C_S$ be defined as in III (i.e., Stability of C^∞ Mappings, III), § 1, (1.11). Thus, C_S is the **R**-algebra of germs at S of C^∞ functions (i.e. C^∞ mappings into **R**) on N, and f^* is given by $f^*(u) = u \circ f$. Let $(\omega f, tf, A, B, \theta(f))$ be as defined in III, § 3, (3.3) et (3.4); that is to say, let A denote the C_y-module of germs at y of C^∞ vector fields on P, let B denote the C_S-module of germs at S of C^∞ vector fields on N, let $\theta(f)$ denote the C_S-module of germs at S of C^∞ vector fields along f, and let $\omega f: A \to \theta(f)$ and $tf: B \to \theta(f)$ be given by $\omega f(\eta) = \eta \circ f$ and $tf(\xi) = Tf \circ \xi$.

We say that f is *infinitesimally stable* if

$$\omega f(A) + tf(B) = \theta(f). \tag{1}$$

This is the notion for germs corresponding to the notion of infinitesimal stability of mappings that we introduced in II. In a later paper, we will define the notion of a stable germ and show that if is equivalent to the notion of an infinitesimally stable germ. However, for the purposes of this paper, we will take " stable " as a shorthand expression for " infinitesimally stable ".

The problem that we consider in this paper is to classify stable germs up to *isomorphism.* If $f':(N', S') \dashrightarrow (P', y')$ is a second C^∞ map-germ, we say f and f' are isomorphic if there exist invertible C^∞ map-germs $h:(N, S) \dashrightarrow (N', S')$ and $h':(P, y) \dashrightarrow (P', y')$ such that $f' = h' f h^{-1}$. The main result reduces the problem of classifying stable germs up to isomorphism to a problem of classifying certain finite dimensional **R**-algebras up to isomorphism.

We define the following **R**-algebras:

$$Q(f) = C_S / f^*(\mathfrak{m}_y) C_S,$$

where $\mathfrak{m}_y$ denotes the unique maximal ideal in C_y:

$$Q_k(f) = Q(f)/\mathfrak{m}^{k+1},$$

where $\mathfrak{m}$ denotes the intersection of the maximal ideals in $Q(f)$ (which is a semi-local ring, i.e. has only finitely many maximal ideals); and

$$\hat{Q}(f) = \varprojlim Q_k(f).$$

Clearly, if f and f' are isomorphic, then $Q(f)$ is isomorphic as an $\mathbf{R}$-algebra to $Q(f')$, and similarly for Q_k and $\hat{Q}$ in place of Q. The main result is that the converse is true for stable map-germs.

Let $x_1, \ldots, x_s$ be the distinct points of S and let $x'_1, \ldots, x'_{s'}$ be the distinct points of S'. Let $f_i = f|(N, x_i) : (N, x_i) \dashrightarrow (P, y)$ and define f'_i similarly. Let n_i (resp. n'_i) denote the dimension of N at x_i (resp. of N' at x'_i), p (resp. p') the dimension of P at y (resp. P' at y').

Theorem A. — *Suppose that f and f' are stable, that $s = s'$, $p = p'$, $n_i = n'_i$, and that (for $1 \leq i \leq s$) there is an isomorphism*

$$Q_{p+1}(f_i) \approx Q_{p+1}(f'_i) \tag{2}$$

of $\mathbf{R}$*-algebras. Then f is isomorphic to f'. Moreover we can choose invertible* C^∞ *map-germs $h : (N, S) \dashrightarrow (N', S')$ and $h' : (P, y) \dashrightarrow (P', y')$ not only so that $f' = h'fh^{-1}$, but also so that $h(x_i) = x'_i$, for $1 \leq i \leq s$.*

The hypothesis that f and f' are stable is essential: consider for example the two map-germs $f, f' : (\mathbf{R}^2, 0) \dashrightarrow (\mathbf{R}^2, 0)$ given by

$$f(x,y) = \langle x, y^3 + xy \rangle$$

$$f'(x,y) = \langle x, y^3 \rangle$$

then

$$Q_3(f) \approx Q_3(f') \approx R[[y]]/(y^3).$$

On the other hand f is stable and f' is not (as we will show in (1.9)); hence f is not isomorphic to f'. (It is trivial to verify that if f and f' are isomorphic and f is stable, then so is f'.)

The rings $Q_k(f)$ and $\hat{Q}(f)$ can be described explicitly. The canonical decomposition of C_S into a Cartesian product

$$C_S = C_{x_1} \times \ldots \times C_{x_s}$$

(where $x_1, \ldots, x_s$ are the distinct points of S) gives rise to a canonical decomposition of $Q(f)$ into a Cartesian product

$$Q(f) \approx Q(f_1) \times \ldots \times Q(f_s), \tag{3}$$

where $f_i = f|(N, x_i) : (N, x_i) \dashrightarrow (P, y)$. This, in turn, gives rise to canonical decompositions of $Q_k(f)$ and $\hat{Q}(f)$ into Cartesian products.

Thus to describe $Q_k(f)$ and $\hat{Q}(f)$ it suffices to consider the case when S is a point, say x. Let $x_1, \ldots, x_n$ be a local system of coordinates for N, null at x and let $y_1, \ldots, y_p$ be a local system of coordinates for P, null at y. The natural isomorphism

$$C_x/\mathfrak{m}_x^{k+1} \approx \mathbf{R}[[x_1, \ldots, x_n]]/\mathfrak{n}^{k+1}$$

(where $\mathfrak{m}_x$ denotes the unique maximal ideal of C_x and $\mathfrak{n}$ denotes the unique maximal ideal of $\mathbf{R}[[x]] = \mathbf{R}[[x_1, \ldots, x_x]]$) gives rise to a natural isomorphism of $Q_k(f)$ with the quotient of $\mathbf{R}[[x]]/\mathfrak{n}^{k+1}$ by the ideal generated by the canonical images of $f^*(y_1), \ldots, f^*(y_p)$. Since the canonical image of $f^*(y_i)$ in $\mathbf{R}[[x]]/\mathfrak{n}^{k+1}$ depends only on the k-jet of f at x, we obtain that $Q_k(f)$ depends only on this k-jet. This still holds in the general case, when S is not necessarily a point. Thus it follows from theorem A that a stable map-germ is $(p+1)$-determined. Actually this result will be a step in the proof of theorem A. Note that $Q_k(f)$ is finite dimensional as an $\mathbf{R}$-algebra.

The natural isomorphism described above gives rise to a natural isomorphism

$$\hat{Q}(f) \approx \mathbf{R}[[x_1, \ldots, x_n]]/(f^*(y_1), \ldots, f^*(y_p)). \tag{4}$$

The proof of Theorem A will be finished in § 6. In the last section (§ 7) we will consider the problem of characterizing the $\mathbf{R}$-algebras that come from stable mappings. We will characterize the $\mathbf{R}$-algebras which are isomorphic to $\hat{Q}(f)$ for some stable $f: (N, S) \to (P, y)$. (By Theorem A, the isomorphism type of $\hat{Q}(f)$ determines that of f (for f stable), since it determines that of the $Q_{p+1}(f_i)$.)

We begin by considering the case when S is a single point x. For any quotient of a formal power series ring over $\mathbf{R}$:

$$A = \mathbf{R}[[x_1, \ldots, x_a]]/\mathfrak{I},$$

we define $\iota(A)$ as $a-b$, where b is the minimum number of elements in a set of generators of $\mathfrak{I}$. Note that $\iota(A)$ depends only on the $\mathbf{R}$-algebra structure on A; it is independent of the particular representation of A as a quotient of a formal power series ring. This may be shown in the same way as we show in § 7 that $\mu_c(A)$ depends only on A and not on the presentation of A as a quotient of a formal power series ring. Clearly

$$\iota(\hat{Q}(f)) \geq n-p \tag{6}$$

where $n = \dim N$, $p = \dim P$, whether $f: (N, x) \to (P, y)$ is stable or not.

Let A be a quotient of a formal power series ring over $\mathbf{R}$ and let c be an integer which is less than or equal to $\iota(A)$. Then there is a unique number $\mu_c(A)$ (which may be a non-negative integer or ∞) with the following property. For any C^∞ map-germ $f: (N, x) \to (P, y)$ such that $c = \dim N - \dim P$ and $\hat{Q}(f) \approx A$, we have

$$\mu_c(A) = d(f, \mathscr{K}), \tag{7}$$

where $d(f, \mathscr{K})$ is as defined in III, (3.5). (See Theorem (7.2)).

Theorem B. — *Let n and p be given positive integers and let* A *be the quotient of a formal power series ring over* $\mathbf{R}$. *Then there exists a stable* C^∞ *map-germ* $f: (N, x) \to (P, y)$, *with*

$\dim N = n$, $\dim P = p$, *and* $A \approx \hat{Q}(f)$ *if and only if* (6) *is satisfied for* A *in place of* $\hat{Q}(f)$, *and*

(8) $$\mu_{n-p}(A) \leq p.$$

More generally, we have:

Addendum. — Let $n_1, \ldots, n_s$ and p be given positive integers. Let $A_1, \ldots, A_s$ be quotients of formal power series rings. Then there exists a stable C^∞ map-germ $f: (N, S) \dashrightarrow (P, y)$ such that S is a set with s points $x_1, \ldots, x_s$, the dimension of N at x_i is n_i, $\dim P = p$, and $A_i \approx \hat{Q}(f_i)$ if and only if

$$\iota(\hat{Q}(f_i)) \geq n_i - p$$

and $$\sum_{i=1}^{s} \mu_{n_i-p}(\hat{Q}(f_i)) \leq p, \qquad 1 \leq i \leq s.$$

The proof of Theorem B and its addendum is relatively easy. It is carried out in § 7.

The number $\mu_{n-p}\hat{Q}(f)$ (defined for a C^∞ map-germ $f: (N, x) \dashrightarrow (P, y)$) can be interpreted in terms of the action of the group $\mathscr{K}^k$ on J^k (defined in III, § 3), for k sufficiently large. First, since $\mu_{n-p}\hat{Q}(f) = d(f, \mathscr{K})$, saying that $\mu_{n-p}\hat{Q}(f)$ is finite is the same as saying that f is finitely determined relative to $\mathscr{K}$ (by III, Theorem (3.5)). Suppose that $\mu_{n-p}\hat{Q}(f)$ is finite and that f is k-determined relative to $\mathscr{K}$. Then we have two results. First, if $f': (N, x) \dashrightarrow (P, y)$ is a second C^∞ map-germ, then $\hat{Q}(f) \approx \hat{Q}(f')$ if and only if $f^{(k)}$ (i.e., the k-jet of f at x) is in the same orbit as $f'^{(k)}$ under the action of $\mathscr{K}^k$ on J^k (by Theorem (2.1) and III, (2.3)). Second, the codimension, relative to J^k, of the orbit of $f^{(k)}$ under the action of $\mathscr{K}^k$ is given by the formula:

(9) $$\operatorname{codim} \mathscr{K}^k . f^{(k)} = \mu_{n-p}\hat{Q}(f) + n - p;$$

except in the trivial case when f is a submersion (by Theorem (2.5) and formula (7) above). This formula shows that (8) is equivalent to saying that the codimension of the $\mathscr{K}^k$-orbit which corresponds to A is $\leq n$.

The theory we develop in this paper permits us to find a " normal form " for a stable C^∞ map-germ; see the remark following Theorem (5.10).

Our results are still valid if " C^∞ " is replaced throughout by " real analytic ", or if " C^∞ " is replaced throughout by " complex analytic " and " **R**-algebra " is replaced throughout by " **C**-algebra ".

1. Conditions for a C^∞ map-germ to be stable.

Let $f: (N, S) \dashrightarrow (P, y)$ be a C^∞ map-germ, where S is a finite subset of N and $y \in P$. (We consider the general case when the dimension of N may vary from point to point of S.)

Proposition (**1.1**). — *f is stable if and only if*

$$(*)\qquad tf(\mathrm{B})+\omega f(\mathrm{A})+(f^*(\mathfrak{m}_y)+\mathfrak{m}_{\mathrm{S}}^{p+1})\theta(f)=\theta(f).$$

Proof. — It suffices to show that $(*)$ implies $\omega f(\mathrm{A})+tf(\mathrm{B})=\theta(f)$. But this is an immediate consequence of III (1.13).

Corollary. — *Whether f is stable depends only on the* $(p+1)$*-jet of f.*

Let $x_1, \ldots, x_s$ denote the distinct points of S and let $f_i=f|(\mathrm{N}, x_i):(\mathrm{N}, x_i)\to(\mathrm{P}, y)$. Clearly if f is stable, then f_i is stable. Conversely, we want to know: assuming $f_1, \ldots, f_s$ are stable, under what conditions is f also stable?

Definition (**1.3**). — *We set*

$$\tau(f_i)=\mathrm{ev}_y((\omega f_i)^{-1}(f_i^*(\mathfrak{m}_y)\theta(f_i)+tf_i(\mathrm{B}_i))),$$

where B_i *denotes the set of germs at* x_i *of* C^∞ *vector fields on* N, *and* $\mathrm{ev}_y:\mathrm{A}\to\mathrm{TP}_y$ *is defined by* $\mathrm{ev}_y(\eta)=\eta(y)$.

This definition makes sense, because ωf_i maps A into $\theta(f_i)$ and tf_i maps B_i into $\theta(f_i)$. Then $\tau(f_i)$ is an **R**-vector subspace of TP_y.

There is a simple geometric interpretation of $\tau(f_i)$, when f_i is stable. We will not use it at all, but it may help to motivate some of the arguments. Let $\tilde{f}_i$ be a representative of f_i, so $\tilde{f}_i$ is a C^∞ mapping of a neighborhood U of x_i into P. Let L be the set of all $x\in\mathrm{U}$ such that the germ of $\tilde{f}_i$ at x is isomorphic to the germ of $\tilde{f}_i$ at x_i. Then L is a submanifold and $t\tilde{f}_i|\mathrm{L}:\mathrm{L}\to\mathrm{P}$ is an immersion (except in the trivial case when $\tilde{f}_i$ is of rank p at x_i). Hence, if U is taken small enough, $t\tilde{f}_i(\mathrm{L})$ is a submanifold of P. Then $\tau(f_i)$ is precisely the tangent space at y to $t\tilde{f}_i(\mathrm{L})$.

Note that $\tau(f_i)\subseteq \mathrm{T}f_i(\mathrm{TN}_{x_i})$, but is not generally equal to it.

Definition (**1.4**). — *Let* $\mathrm{E}_1, \ldots, \mathrm{E}_s$ *be vector subspaces of a finite dimensional vector space* F. *We will say* $\mathrm{E}_1, \ldots, \mathrm{E}_s$ *have* regular intersection (*with respect to* F) *if*

$$\operatorname{codim}(\mathrm{E}_1\cap\ldots\cap\mathrm{E}_s)=\operatorname{codim}\mathrm{E}_1+\ldots+\operatorname{codim}\mathrm{E}_s$$

(*where* codim *denotes the codimension in* F).

Lemma (**1.5**). — $\mathrm{E}_1, \ldots, \mathrm{E}_s$ *have regular intersection if and only if the natural mapping*

$$\mathrm{F}\to(\mathrm{F}/\mathrm{E}_1)\oplus\ldots\oplus(\mathrm{F}/\mathrm{E}_s)$$

is surjective.

Proof. — The kernel of this mapping is $\mathrm{E}_1\cap\ldots\cap\mathrm{E}_s$, so the lemma follows from comparison of dimensions.

Proposition (**1.6**). — *f is stable if and only if each* f_i *is stable and* $\tau(f_1), \ldots, \tau(f_s)$ *have regular intersection with respect to* TP_y.

Proof. — $\omega f:\mathrm{A}\to\theta(f)$ induces a mapping

$$\overline{\omega}f:\ \mathrm{TP}_y=\mathrm{A}/\mathfrak{m}_y\mathrm{A}\to\theta(f)/(f^*(\mathfrak{m}_y)\theta(f)+tf(\mathrm{B}))=\bigoplus_{i=1}^{s}\theta(f_i)/(f_i^*(\mathfrak{m}_y)\theta(f_i)+tf_i(\mathrm{B}_i)).$$

It follows from (1.1) and the definition of stability that f is stable if and only i $\bar{\omega}f$ is surjective. Similarly $\omega f_i : A \to \theta(f_i)$ induces

$$\bar{\omega}f_i: \quad TP_y \to \theta(f_i)/(f_i^*(\mathfrak{m}_y)\theta(f_i) + tf_i(B_i)),$$

and f_i is stable if and only if $\bar{\omega}f_i$ is surjective. Furthermore

a) $$\bar{\omega}f(\eta) = \bar{\omega}f_1(\eta) \oplus \ldots \oplus \bar{\omega}f_p(\eta)$$

for any $\eta \in TP_y$. Hence if $\bar{\omega}f$ is surjective then each $\bar{\omega}f_i$ is surjective, which shows that if f is stable, then each f_i is stable.

Conversely, assume that each f_i is stable. Then $\bar{\omega}f_i$ induces an isomorphism

$$e_i: \quad TP_y/\tau(f_i) \overset{\rightarrow}{\approx} \theta(f_i)/(f_i^*(\mathfrak{m}_y)\theta(f_i) + tf_i(B_i))$$

(since $\bar{\omega}f_i$ is onto and $\tau(f_i)$ is the kernel of $\bar{\omega}f_i$, by definition). Then f is stable if and only if

$$(e_1^{-1} \oplus \ldots \oplus e_p^{-1}) \circ \bar{\omega}f: \quad TP_y \to (TP_y/\tau(f_1)) \oplus \ldots \oplus (TP_y/\tau(f_p))$$

is surjective, since $e_1^{-1} \oplus \ldots \oplus e_p^{-1}$ is an isomorphism. By *a)*, this is the " natural mapping " referred to in Lemma (1.5); hence, by Lemma (1.5), f is stable if and only if $\tau(f_1), \ldots, \tau(f_s)$ have regular intersection.

Throughout the rest of this section, we will assume that S is a point x. By the last proposition, we have practically reduced the problem of determining whether f is stable to this case.

We can choose local coordinates $x_1, \ldots, x_n$ for N, null at x and $y_1, \ldots, y_p$ for P, null at y, such that f has the form

$$\textbf{(1.7)} \qquad \begin{cases} y_i \circ f = x_i, & 1 \leq i \leq r \\ d(y_i \circ f)(x) = 0, & r+1 \leq i \leq p, \end{cases}$$

where d denotes the differential and r is the rank of f at x. For choose the coordinates $y_1, \ldots, y_p$ such that $dy_1|E, \ldots, dy_r|E$ are linearly independent and $dy_{r+1}|E = \ldots = dy_p|E = 0$, where $E = Tf(TN_x)$. Then the second condition is satisfied. Set $x_i = y_i \circ f$, $1 \leq i \leq r$. Then $dx_1, \ldots, dx_r$ are linearly independent at x, so $x_1, \ldots, x_r$ extends to a local system of coordinates $x_1, \ldots, x_n$, null at x. These give the desired systems of coordinates.

For f in the form (1.7), we will set $f_i = y_i \circ f$, $r+1 \leq i \leq p$. The mapping tf is given by

$$\begin{aligned} tf\left(\frac{\partial}{\partial x_i}\right) &= \left(\frac{\partial}{\partial y_i}\right) \circ f + \sum_{j=r+1}^{p} \left(\frac{\partial f_j}{\partial x_i}\right)\left(\frac{\partial}{\partial y_j}\right) \circ f, \quad 1 \leq i \leq r \\ &= \sum_{j=r+1}^{p} \left(\frac{\partial f_j}{\partial x_i}\right)\left(\frac{\partial}{\partial y_j}\right) \circ f, \quad r+1 \leq i \leq p. \end{aligned}$$

We set $\bar{x} = (x_1, \ldots, x_r)$ and $x' = (x_{r+1}, \ldots, x_p)$ and let $\mathscr{E}_{x'}$ denote the ring of germs at 0 of C^∞ functions in the variables $x_{r+1}, \ldots, x_n$. Let $\mathscr{E}_{x'}^{p-r}$ denote the free

$\mathscr{E}_{x'}$-module with basis $\varepsilon_1, \ldots, \varepsilon_{p-r}$. For any $u \in C_x$, let $u' \in \mathscr{E}_{x'}$ be defined by $u'(x_{r+1}, \ldots, x_p) = u(0, \ldots, 0, x_{r+1}, \ldots, x_p)$. Define $\pi : \theta(f) \to \mathscr{E}_{x'}^{p-r}$ by

$$\pi\left(\sum_{i=1}^{r} u_i tf\left(\frac{\partial}{\partial x_i}\right) + \sum_{i=r+1}^{p} u_i\left(\frac{\partial}{\partial y_i}\circ f\right)\right) = \sum_{i=r+1}^{p} u_i' \varepsilon_{i-r}$$

(where the $u_i \in C_x$). This definition makes sense, because $tf(\partial/\partial x_1), \ldots, tf(\partial/\partial x_r)$, $(\partial/\partial y_{r+1})\circ f, \ldots, (\partial/\partial y_p)\circ f$ form a free basis of $\theta(f)$, considered as a C_x-module. We will set $f_\bullet = (f_{r+1}, \ldots, f_p)$ and $f_\bullet' = (f_{r+1}', \ldots, f_p')$. It is easily verified that $\pi(tf(B) + f^*(\mathfrak{m}_y)\theta(f))$ depends only on $f_\bullet'$; we will denote it by $\Omega(f_\bullet')$. In fact $\Omega(f_\bullet')$ is the $\mathscr{E}_{x'}$-submodule of $\mathscr{E}_{x'}^{p-r}$ generated by

$$\left\{\sum_{j=r+1}^{p} \frac{\partial f_j'}{\partial x_i}\varepsilon_{j-r} : i = r+1, \ldots, n\right\}$$

and $$\{f_i'\varepsilon_j : i = r+1, \ldots, n, j = 1, \ldots, p-r\}.$$

We let ∂f denote the r-tuple $\langle \partial_1 f, \ldots, \partial_r f\rangle$, where $\partial_i f \in \mathscr{E}_{x'}^{p-r}$ is defined by

$$\partial_i f = \sum_{j=r+1}^{p} (\partial f_j/\partial x_i)'\varepsilon_{j-r}.$$

For any $v = \langle v_1, \ldots, v_r\rangle$, where $v_i \in \mathscr{E}_{x'}^{p-r}$, we let $[v] = [v_1, \ldots, v_r]$ denote the **R**-vector subspace of $\mathscr{E}_{x'}^{p-r}$ spanned by $v_1, \ldots, v_r$. If V is any subset of $\mathscr{E}_{x'}^{p-r}$, we let $V^{(k)}$ denote the image of V under the projection $\rho_k : \mathscr{E}_{x'}^{p-r} \to \mathscr{E}_{x'}^{p-r}/\mathfrak{m}_{x'}^{k+1}\mathscr{E}_{x'}^{p-r}$. We shall denote the last named module by $\mathscr{E}_{x'}^{p-r,(k)}$.

Proposition (**1.8**). — *Let* $k \geq p$. *If f is of the form* (1.7), *then f is stable if and only if*

(*) $$\Omega(f_\bullet')^{(k)} + [\partial f]^{(k)} = \mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(k)},$$

where $\mathfrak{m}_{x'}$ *denotes the unique maximal ideal of* $\mathscr{E}_{x'}$.

Proof. — In any case:

$$\Omega(f_\bullet')^{(k)} + [\partial f]^{(k)} \subseteq \mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(k)},$$

so (*) holds if and only if

(**) $$\Omega(f_\bullet')^{(k)} + [\partial f]^{(k)} + [\varepsilon_1, \ldots, \varepsilon_{p-r}]^{(k)} = \mathscr{E}_{x'}^{p-r,(k)}.$$

Since

$$(\rho_k\circ\pi)^{-1}(\Omega(f_\bullet')^{(k)} + [\partial f]^{(k)} + [\varepsilon_1, \ldots, \varepsilon_{p-r}]^{(k)}) = tf(B) + \omega f(A) + (f^*(\mathfrak{m}_y) + \mathfrak{m}_x^{k+1})\theta(f),$$

(**) holds if and only if the right hand side of the above equation is equal to $\theta(f)$, which is equivalent to the condition that f be stable, by (1.1) and the definition of stability.

Example (**1.9**). — Let $n \leq p$ and let f be given by

$$f^*(y_i) = x_i, \qquad 1 \leq i \leq n-1$$

$$f^*(y_n) = x_n^{k+1} + \sum_{j=1}^{k-1} x_j x_n^j$$

$$f^*(y_{n+i}) = \sum_{j=1}^{k} x_{ki+j-1}x_n^j, \qquad 1 \leq i \leq p-n,$$

where $1 \leq k \leq n/(p-n+1)$. Then f is stable. This may be checked either by applying (1.8) or by applying (1.1) directly.

In particular the mapping f given (in the case $n=p=2$) by

$$f^*(y_1)=x_1, \quad f^*(y_2)=x_2^3+x_1x_2$$

is stable, whereas the mapping given by

$$f'^*(y_1)=x_1, \quad f'^*(y_2)=x_2^3$$

is not. This is the example mentioned in the introduction, following Theorem A.

Example (**1.10**). — Let $n>p$ and let f be given by

$$f^*(y_i)=x_i, \quad 1 \leq i \leq p-1$$

$$f^*(y_p)=\pm x_p^2 \pm \ldots \pm x_{n-1}^2 \pm x_n^{k+1} + \sum_{j=1}^{k-1} x_j x_n^j,$$

where $1 \leq k \leq p$. Then f is stable. Again this may be checked by applying (1.8) or by applying (1.1) directly.

2. The $\mathscr{K}^k$ orbits-general properties.

Let the Lie group $\mathscr{K}^k$, the C^∞ manifold J^k, and the C^∞ action of $\mathscr{K}^k$ on J^k be defined as in III, § 7. Thus for example, J^k is the set of k-jets of C^∞ map-germs $(N, S) \dashrightarrow (P, y)$. Let $\mathscr{K}_0$ denote the subgroup of $\mathscr{K}$ consisting of those $H : (N \times P, S \times y) \dashrightarrow (N \times P, S \times y)$ in $\mathscr{K}$ such that $H|(S \times y) = \text{identity}$. Let $\mathscr{K}_0^k$ denote the image of $\mathscr{K}_0$ under the projection $\mathscr{K} \to \mathscr{K}^k$. Let $x_1, \ldots, x_s$ be the elements of S. Let J_i^k denote the C^∞ manifold consisting of k-jets of map-germs $(N, x_i) \dashrightarrow (P, y)$. Then J^k is naturally diffeomorphic to the Cartesian product $J_1^k \times \ldots \times J_s^k$. Similarly $\mathscr{K}_0^k$ is naturally equivalent to a Cartesian product $\mathscr{K}_1^k \times \ldots \times \mathscr{K}_s^k$, where $\mathscr{K}_i^k$ is the group which corresponds to $\mathscr{K}^k$ when S is replaced by x_i. The action of $\mathscr{K}_0^k$ on J^k splits up into a direct product:

$$\langle H_1, \ldots, H_k \rangle \langle z_1, \ldots, z_k \rangle = \langle H_1 z_1, \ldots, H_k z_k \rangle.$$

As a result questions concerning the structure of the orbits of $\mathscr{K}_0^k$ on J^k can be reduced to the case when S is a point.

Throughout the rest of this section, we suppose that S is a single point x.

Theorem (**2.1**). — *Let $f, f' : (N, x) \dashrightarrow (P, y)$ be C^∞ map-germs and let k be an integer ≥ 1. Then $f^{(k)}$ and $f'^{(k)}$ are in the same orbit under the action of $\mathscr{K}^k$ if and only if the* $\mathbf{R}$*-algebras $Q_k(f)$ and $Q_k(f')$ are isomorphic.*

Proof. — Suppose $Q_k(f) \approx Q_k(f')$. Then f and f' have the same rank at x, say r. Let $x_1, \ldots, x_n$ be a local system of coordinates for N, null at x, and $y_1, \ldots, y_p$ a local system of coordinates for P, null at y. Without loss of generality, we may suppose that f and f' have the form (1.7). For $r+1 \leq i \leq n$, let $\overline{x}_i$ denote the image of x_i in $Q_k(f)$ under the canonical projection $C(N)_x \to Q_k(f)$ and let $\overline{x}_i'$ denote the image of x_i in $Q_k(f')$

under the projection $C(N)_x \to Q_k(f')$. Let $\varphi : Q_k(f) \approx Q_k(f')$ be an isomorphism of **R**-algebras. For $r+1 \leq i \leq n$, we can express $\varphi(\bar{x}_i)$ as a polynomial $p_i(\bar{x}')$ in $\bar{x}'_{r+1}, \ldots, \bar{x}'_n$. Define $h : (N, x) \dashrightarrow (N, x)$ by

$$\begin{aligned} x_i \circ h &= x_i, \quad 1 \leq i \leq r \\ &= p_i(x), \quad r+1 \leq i \leq n. \end{aligned}$$

Then h is invertible, since the fact that φ is an isomorphism implies that the matrix $\left(\dfrac{\partial p_i}{\partial x_j}\right)_{r+1 \leq i, j \leq n}$ is invertible. Furthermore the following diagram commutes:

$$\begin{array}{ccc} C(N)_x & \overset{h^*}{\approx} & C(N)_x \\ \Big\downarrow{\scriptstyle \text{proj.}} & & \Big\downarrow{\scriptstyle \text{proj.}} \\ Q_k(f) & \overset{\varphi}{\approx} & Q_k(f') \end{array}$$

Hence, replacing f' by $f' \circ h^{-1}$, we may suppose that

$$f^*(\mathfrak{m}_y)C_x + \mathfrak{m}_x^{k+1} = f'^*(\mathfrak{m}_y)C_x + \mathfrak{m}_x^{k+1}.$$

Replacing f' by another map-germ having the same k-jet, we may suppose that

$$f^*(\mathfrak{m}_y)C_x = f'^*(\mathfrak{m}_y)C_x.$$

Then it follows from (III, Proposition (2.3), (ii) $\Rightarrow$ (i)) that f and f' are in the same $\mathscr{C}$-orbit, so that $f^{(k)}$ and $f'^{(k)}$ are in the same $\mathscr{K}^{(k)}$-orbit.

Conversely, suppose $f^{(k)}$ and $f'^{(k)}$ are in the same $\mathscr{K}^{(k)}$-orbit, say $f'^{(k)} = H^{(k)} f^{(k)}$, with $H \in \mathscr{K}$. Then

$$Q_k(f') = Q_k(Hf) \approx Q_k(f)$$

where the last isomorphism is a consequence of (III, Proposition (2.3), (i) $\Rightarrow$ (ii)) and the fact that $\mathscr{K}$ is the semi-direct product of $\mathscr{R}$ and $\mathscr{C}$.

(2.2) It follows from Theorem (2.1) that if $f, f' : (N, x) \dashrightarrow (P, y)$ are two C^∞ map-germs then $f^{(1)}$ and $f'^{(1)}$ are in the same orbit of the action of $\mathscr{K}^1$ on J^1 if and only if they have the same rank at x. For any integer r, $\sup(0, n-p) \leq r \leq n$, we let $\Sigma_r \subseteq \mathscr{F}$ (where $\mathscr{F}$ denotes the set of C^∞ map-germs $f : (N, x) \dashrightarrow (P, y)$, as in III, § 2), denote the set of C^∞ map-germs $f : (N, x) \dashrightarrow (P, y)$ having rank $n-r$ at x. We let $\Sigma_r^{(k)} \subseteq J^k$ denote the image of Σ_r under the canonical projection $\mathscr{F} \to J^k$.

The codimension of $\Sigma_r^{(k)}$ in J^k is the same as the codimension of the set of all matrices of rank $n-r$ in the set of all $n \times p$ matrices. Hence:

(2.3) $$\operatorname{codim} \Sigma_r^{(k)} = (p-n+r)r.$$

By (2.2), the sets $\Sigma_r^{(1)}$ are precisely the orbits of the action of $\mathscr{K}^1$ on J^1. This is true in the complex case (where we consider jets of holomorphic mappings) as well as in the real case (where we consider jets of C^∞ mappings).

Next we show:

Lemma **(2.4)**. — *$f \in \mathscr{F}$ is finitely determined rel. $\mathscr{K}$ if and only if the ideal*

$$f^*(\mathfrak{m}_y)C_x + J(f)$$

in C_x contains a power of the maximal ideal, where $J(f)$ denotes the ideal generated by all $p\times p$ minors of the matrix

$$\left(\frac{\partial(y_i \circ f)}{\partial x_j}\right)_{1\leq i\leq p, 1\leq j\leq n},$$

where $y_1, \ldots, y_p$ is a local system of coordinates for P, null at y, and $x_1, \ldots, x_n$ is a local system of coordinates for N, null at x. (Clearly $J(f)$ is independent of the choices of coordinates.)

Proof. — By (III, (3.5) and (3.6)) the necessary and sufficient condition that f be fin. det. rel. $\mathscr{K}$ is that there exist an integer k such that

$$(*) \qquad f^*(\mathfrak{m}_y)\theta(f) + tf(B) \supseteq \mathfrak{m}_x^k\theta(f).$$

Assume that $(*)$ holds. Let $u = u_1 \ldots u_p$, where each $u_i \in \mathfrak{m}_x^k$. By $(*)$, there exist $a_{ik} \in C_x$ such that

$$\sum_{i=1}^n a_{ik}\frac{\partial(y_j\circ f)}{\partial x_i} = u_k\delta_{jk} \bmod f^*(\mathfrak{m}_y)C_x,$$

for $1\leq j,\ k\leq p$. Since the determinant of the $p\times p$ matrix $(u_k\delta_{jk})$ is u, it follows that u is congruent (mod. $f^*(\mathfrak{m}_y)C_x$) to a linear combination (with coefficients in C_x) of $p\times p$ minors of the matrix $(\partial(y_i\circ f)/\partial x_j)$. Hence

$$f^*(\mathfrak{m}_y)C_x + J(f) \supseteq \mathfrak{m}_x^{kp}.$$

Conversely an application of Cramer's rule shows that

$$f^*(\mathfrak{m}_y)C_x + J(f) \supseteq \mathfrak{m}_x^k$$

implies $(*)$.

Theorem **(2.5)**. — *If $f\in\mathscr{F}$ is k-det. rel. $\mathscr{K}$ and not a submersion, then*

$$(*) \qquad d(f,\mathscr{K}) = \operatorname{codim} \mathscr{K}^k . f^{(k)} - n + p$$

where codim *means the codimension in* J^k.

Proof. — The hypothesis that f is k det. rel. $\mathscr{K}$ implies

$$f^*(\mathfrak{m}_y)\theta(f) + tf(\mathfrak{m}_x B) \supseteq \mathfrak{m}_x^{k+1}\theta(f)$$

by the formula for the tangent space to an orbit of $\mathscr{K}^{k+1}$ (III, (7.4)) and Nakayama's lemma. Hence, the formula for the tangent space at $f^{(k)}$ to $\mathscr{K}^k . f^{(k)}$ (III, (7.4)) yields

$$\operatorname{codim} \mathscr{K}^k . f^{(k)} = \dim_{\mathbf{R}} \frac{\mathfrak{m}_x\theta(f)}{f^*(\mathfrak{m}_y)\theta(f) + tf(\mathfrak{m}_x B)}.$$

Using this and the definition of $d(f, \mathscr{K})$ (III, (3.5)), one see easily that $(*)$ is equivalent to

$$\dim_{\mathbf{R}} \frac{f^*(\mathfrak{m}_y)\theta(f)+tf(\mathrm{B})}{f^*(\mathfrak{m}_y)\theta(f)+tf(\mathfrak{m}_x\mathrm{B})}=n.$$

In other words, if $\xi\in\mathrm{B}$ and

$$(**) \qquad tf(\xi)\in f^*(\mathfrak{m}_y)\theta(f)+tf(\mathfrak{m}_x\mathrm{B}),$$

then $\xi\in\mathfrak{m}_x\mathrm{B}$. Suppose the contrary: there exists $\xi\in\mathrm{B}$ satisfying $(**)$ such that $\xi\notin\mathfrak{m}_x\mathrm{B}$. Then there exists $\xi_0\in\mathfrak{m}_x\mathrm{B}$ such that $tf(\xi-\xi_0)\in f^*(\mathfrak{m}_y)\theta(f)$. Since $\xi-\xi_0\notin\mathfrak{m}_x\mathrm{B}$, we may choose local coordinates $x_1, \ldots, x_n$ for N, null at x, such that $\xi-\xi_0=\partial/\partial x_1$. Let $y_1, \ldots, y_p$ be a local system of coordinates for P, null at y. The equation $tf(\partial/\partial x_1)\in f^*(\mathfrak{m}_y)\theta(f)$ means that there exists $u_{ij}\in\mathfrak{m}_x$ $(1\leq i, j\leq p)$ such that

$$tf\left(\frac{\partial}{\partial x_1}\right)=\sum_i \frac{\partial(y_i\circ f)}{\partial x_1}\left(\frac{\partial}{\partial y_i}\circ f\right)$$
$$=\sum_{i,j} u_{ij}(y_j\circ f)\left(\frac{\partial}{\partial y_i}\circ f\right).$$

Since the $\left(\frac{\partial}{\partial y_i}\circ f\right)$ form a free basis of $\theta(f)$, this means

$$(***) \qquad \frac{\partial(y_i\circ f)}{\partial x_1}=\sum_j u_{ij}(y_j\circ f).$$

Setting $\eta_i(x_1)=(y_i\circ f)(x_1, 0, \ldots, 0)$, we obtain a system of ordinary differential equations

$$\frac{d\eta_i}{dx_1}=\sum_j u_{ij}\eta_j.$$

Since $\eta_i(0)=0$ the uniqueness theorem for solutions of ordinary differential equations implies $\eta_i(x_1)=0$ for all x_1 near 0. In other words, f maps the x_1 axis into 0.

Setting $\eta_{i\alpha}(x_1)=\dfrac{\partial(y_i\circ f)}{\partial x_\alpha}(x_1, 0, \ldots, 0)$, we obtain a system of ordinary differential equations for each α, $1\leq\alpha\leq p$:

$$\frac{d\eta_{i\alpha}}{dx_1}=\sum_j u_{ij}\eta_{j\alpha},$$

by differentiating both sides of $(***)$, by $\partial/\partial x_\alpha$, and using the fact that $\eta_i=0$ in a neighborhood of x_1. It follows from the theory of ordinary differential equations that the rank of the matrix $(\eta_{i\alpha})$ is constant in a neighborhood of 0. But the rank of $(\eta_{i\alpha}(x_1))$ is the same as the rank of the mapping f at the point $(x_1, 0, \ldots, 0)$.

Now suppose f is not a submersion. By what we have shown, f maps the x_1 axis into 0, and the rank of f is $<p$ on the x_1 axis in a neighborhood of 0 (since it is constant

and $<p$ at 0). Hence any element of the ideal $f^*(\mathfrak{m}_y)C_x + J(f)$ vanishes on the x_1 axis, which by Lemma (2.4) contradicts the hypothesis that f is fin. det. rel. $\mathscr{K}$.

This contradiction shows that if $\xi \in B$ and satisfies (**), then $\xi \in \mathfrak{m}_x B$, and thereby proves the theorem.

Corollary **(2.6)**. — *If* $f \in \mathscr{F}$ *and* $g \in \mathscr{K} f$ *then* $d(f, \mathscr{K}) = d(g, \mathscr{K})$.

Proof. — If f is a submersion, then g is also, and $d(f, \mathscr{K}) = d(g, \mathscr{K}) = 0$. In the case f is not a submersion, but $d(f, \mathscr{K})$ is finite, then f is fin. det. rel. $\mathscr{K}$, by (III, (3.5)), say k-det., and g is also k-det. rel. $\mathscr{K}$. Since $f^{(k)}$ and $g^{(k)}$ are in the same orbit of $\mathscr{K}^{(k)}$ the result follows from (2.5). Finally if $d(f, \mathscr{K}) = \infty$, then $d(g, \mathscr{K}) = \infty$; for otherwise g would be finitely determined rel. $\mathscr{K}$, and therefore f would also be finitely determined rel. $\mathscr{K}$, so by (III, (3.5)), we would have $d(f, \mathscr{K}) < \infty$.

3. Stable map-germs are $(p+1)$-determined.

Throughout this section we let $f: (N, S) \dashrightarrow (P, y)$ be a stable C^∞ map-germ. We will show f is $(p+1)$-determined, where $p = \dim P$. More precisely, we will show that f is $(p+1)$-det. rel. $\mathscr{A}^\#$, where $\mathscr{A}^\#$ is defined as $\mathscr{R}^\# \times \mathscr{L}^\#$, where $\mathscr{R}^\#$ is the set of all invertible C^∞ map-germs $h: (N, S) \dashrightarrow (N, S)$ such that $h|S = \text{identity}$ and h is orientation preserving in a neighborhood of each point of S, and $\mathscr{L}^\#$ is the set of all invertible orientation preserving map-germs $h': (P, y) \dashrightarrow (P, y)$. Clearly $\mathscr{A}^\# \subset \mathscr{A}$. Letting $\mathscr{A}^{\# k}$ denote the image of $\mathscr{A}^k$ under the canonical projection $\mathscr{A} \to \mathscr{A}^k$, one sees easily that $\mathscr{A}^{\# k}$ is precisely the connected component of $\mathscr{A}^k$ containing the identity.

To prove that f is $(p+1)$-determined, we need only consider jets of finite order, since we already know that f is finitely determined (III, (3.7)). We need:

Lemma **(3.1)**. — *Let* $\alpha: G \times U \to U$ *be a* C^∞ *action of a Lie group* G *on a* C^∞ *manifold* U, *and let* V *be a connected* C^∞ *submanifold of* U. *Then necessary and sufficient conditions for* V *to be contained in a single orbit of* α *are that*:

a) $T(Gv)_v \supseteq TV_v$, *if* $v \in V$.

b) $\dim T(Gv)_v$ *is independent of choice of* $v \in V$.

Necessity is trivial.

Condition *a)* by itself is not enough for sufficiency. For example, let G be the subgroup of $\mathbf{GL}(2, \mathbf{R})$ consisting of all linear transformations of the x, y plane into itself which leave the x-axis invariant. Let $U = \mathbf{R}^2$ and let α be the canonical action of G on U (given by the inclusion of G in $\mathbf{GL}(2, \mathbf{R})$). Let V be the subset of $\mathbf{R}^2$ defined by $y = x^2$. Then *a)* is satisfied, but V is not contained in an orbit.

Now we prove sufficiency. For each $v \in U$, let $\alpha_v: G \to U$ be defined by $\alpha_v(g) = \alpha(g, v)$. Let TG_1 denote the tangent space to G at the identity. From $T\alpha_v(T_1 G) = T_v(Gv)$, it follows that *a)* and *b)* are equivalent to:

a') $T\alpha_v(T_1 G) \supseteq T_v V$, if $v \in V$.

b') $\dim T\alpha_v(T_1 G)$ is independent of the choice of $v \in V$.

Provide T_1G with a Hilbert norm and for each $v \in V$, let L_v be the orthogonal complement of $\ker(T\alpha_v : T_1G \to T_vV)$ in T_1G. Let $L \subset V \times T_1G$ be $\bigcup_{v \in V}(v \times L_v)$. By *b')*, L is a sub-vector bundle over V of $V \times T_1G$. Let

$$L_0 = \bigcup_{v \in V}((T\alpha_v)^{-1}(T_vV) \cap L_v).$$

By *a')*, L_0 is a sub-vector bundle over V of L and the mapping $\bigcup_{v \in V}(T\alpha_v) : L_0 \to TV$ is an isomorphism of C^∞ vector bundles. Let $\beta : TV \to L_0$ be the inverse of this mapping and let $\pi : V \times T_1G \to T_1G$ denote the projection. Then $\pi \circ \beta : TV \to T_1G$ is a C^∞ mapping and

$$T\alpha_v(\pi \circ \beta(\eta)) = \eta, \qquad \text{for any } \eta \in T_vV.$$

To prove that V is contained in a single orbit of V it is enough to show that any two points v_1, v_2 of V are contained in the same orbit. Since V is connected there is a smooth curve γ in V with v_1 and v_2 as endpoints, i.e. a C^∞ mapping $\gamma : [0, 1] \to V$ such that $\gamma(0) = v_1$ and $\gamma(1) = v_2$. It is enough to show that for any $t_0 \in [0, 1]$, there is an $\varepsilon > 0$ such that if $t_0 - \varepsilon < t < t_0 + \varepsilon$ then $\gamma(t)$ is contained in the same orbit as $\gamma(t_0)$.

Let $\gamma'(t) \in T_{\gamma(t)}V$ denote the derivative of $\gamma(t)$ with respect to t. Let

$$X(t) = \pi \circ \beta(\gamma'(t)) \in T_1G.$$

Clearly $X(t)$ is a C^∞ function of t and

e) $$T_{\alpha_{\gamma(t)}}(X(t)) = \gamma'(t),$$

by *d)*. From the existence theory for ordinary differential equations it follows that there exists a curve $t \mapsto \mu(t)$ in G (defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ for a suitable $\varepsilon > 0$) such that $\mu(t_0) = 1$ and

f) $$\frac{d\mu(t)}{dt} = +\tilde{X}_t(\mu(t)),$$

where $\tilde{X}_t$ is the unique right invariant vector field on G which extends $X(t)$.

To prove the lemma, it suffices to show that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \varepsilon < t < t_0 + \varepsilon$, since this implies that $\gamma(t)$ is in the same orbit as $\gamma(t_0)$ for all t within this range. Using the obvious abbreviations, we have:

$$\frac{d}{dt}(\mu(t)^{-1}\gamma(t)) = \frac{d}{dt}(\mu(t)^{-1})\gamma(t) + \mu(t)^{-1}\frac{d}{dt}\gamma(t) = +\mu(t)^{-1} \cdot \left(-\frac{d\mu(t)}{dt}\mu(t)^{-1}\gamma(t) + \frac{d}{dt}\gamma(t)\right).$$

By *f)* and the fact that $\tilde{X}_t$ is right invariant, the quantity inside the brackets becomes

$$-X(t)\gamma(t) + \gamma'(t).$$

By *e)*, this is 0. Hence $\frac{d}{dt}(\mu(t)^{-1}\gamma(t)) = 0$. Since $\mu(t_0) = 1$, this shows that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \varepsilon < t < t_0 + \varepsilon$, and thereby completes the proof.

In our application of Lemma (3.1), it will be unnecessary to verify condition *b)*, by the following corollary. By a G-*space* (where G is a Lie group) we will mean a C^∞ manifold U, together with a C^∞ action of G on U. By a G-*submersion* $f: U \to U'$ of G-spaces, we mean a C^∞ submersion such that

$$f(gu) = gf(u), \quad \text{for } g \in G, \ u \in U.$$

Corollary **(3.2)**. — *Let* $f: U \to U'$ *be a* G-*submersion, let* $u' \in U'$, *and let* $V = f^{-1}(u')$. *Suppose* V *is connected. Then the necessary and sufficient condition for* V *to be contained in a single orbit of* G *is that* ((3.1), *a)*) *be satisfied.*

Proof. — Since f is a submersion V is a submanifold, so that ((3.1), *a)*) makes sense. Necessity is clear. On the other hand ((3.1), *a)*) implies

$$\dim T(Gv)_v = \dim TV_v + \dim T(Gu')_{u'},$$

for any $v \in V$. The right hand side is clearly independent of the choice of $v \in V$; hence, the conclusion follows from Lemma (3.1).

Now let's see what we have to do to show that f is $(p+1)$-det. rel. $\mathscr{A}^\#$. First, we know that f is fin. det. rel. $\mathscr{A}_1$ (by III, (3.7)); hence it is finitely determined rel. $\mathscr{A}^\#$ (since $\mathscr{A}_1 \subset \mathscr{A}^\#$). Say it is l-det. rel. $\mathscr{A}^\#$. Let $V = \pi^{-1}(f^{(p+1)})$, where $\pi: J^l \to J^{p+1}$ is the projection. Since f is l-determined, it is enough to show that V is in a single orbit of the action of $\mathscr{A}^{\#l}$ on J^l.

By Corollary (3.2), it is enough to show that

$$T(\mathscr{A}^{\#l}.v)_v \supseteq TV_v$$

for all $v \in V$. Using the formula (III, (7.4)) for the tangent space to $\mathscr{A}^l v$ (which is the same as the tangent space to $\mathscr{A}^{\#l} v$, since $\mathscr{A}^{\#l}$ is open in $\mathscr{A}^l$), we see that the above inclusion is equivalent to the inclusion

$$\textbf{(3.3)} \qquad \omega g(\mathfrak{m}_y A) + tg(\mathfrak{m}_S B) + \mathfrak{m}_S^{l+1}\theta(g) \supseteq \mathfrak{m}_S^{p+1}\theta(g),$$

where $g: (N, S) \dashrightarrow (P, y)$ is any representative for v.

The argument that we have just given shows that in order to prove that f is $(p+1)$-det. rel. $\mathscr{A}^\#$ it suffices to show that (3.3) is satisfied for any $g \in \mathscr{F}$ having the same $(p+1)$-jet as f. We now show that this is the case.

First, we remark that such a g is stable by (1.1). Since

$$\omega g(\mathfrak{m}_y A) \subseteq g^*(\mathfrak{m}_y)\theta(g),$$

the fact that g is stable implies

$$\operatorname{codim}(tg(B) + g^*(\mathfrak{m}_y)\theta(g)) \leq \operatorname{codim}(tg(B) + \omega g(\mathfrak{m}_y A)) \leq p,$$

where codim means the codimension in $\theta(g)$. By (III, (1.6)), it follows that

$$\mathfrak{m}_S^p\theta(g) \subseteq tg(B) + g^*(\mathfrak{m}_y)\theta(g).$$

Multiplying both sides of this equation by $\mathfrak{m}_S$, we obtain

$$(*) \qquad \mathfrak{m}_S^{p+1}\theta(g) \subseteq tg(\mathfrak{m}_S B) + g^*(\mathfrak{m}_y)\theta(g).$$

On the other hand, the fact that g is stable implies

$$(3.4) \qquad tg(\mathfrak{m}_S B) + \omega g(\mathfrak{m}_y A) = tg(\mathfrak{m}_S B) + g^*(\mathfrak{m}_y)\theta(g).$$

To show this, it is enough to show that $g^*(\mathfrak{m}_y)\theta(g)$ is in the left hand side. But:

$$g^*(\mathfrak{m}_y)\theta(g) = g^*(\mathfrak{m}_y)(\omega g(A) + tg(B)) = \omega g(\mathfrak{m}_y A) + tg(g^*(\mathfrak{m}_y)B),$$

which gives the desired result.

Now (3.3) follows from (*) and (3.4), which completes the proof of the following:

Proposition **(3.5)**. — *If f is stable then f is $(p+1)$-determined rel.* $\mathscr{A}^\#$.

We conclude this section by remarking that (3.4) and the formulas for the tangent spaces at $f^{(l)}$ of $\mathscr{K}^l f^{(l)}$ and $\mathscr{A}^l f^{(l)}$ (see III, (7.4)) imply that

$$T(\mathscr{K}^l z)_z = T(\mathscr{A}^l z)_z,$$

where $z = f^{(l)}$, in other words, that the orbits of $\mathscr{K}^l$ and $\mathscr{A}^{\# l}$ through $f^{(l)}$ have the same dimension. Since $\mathscr{A}^{\# l} \subseteq \mathscr{K}^l$, this yields:

Lemma **(3.6)**. — $\mathscr{A}^{\# l} f^{(l)}$ *is an open subset of* $\mathscr{K}^l f^{(l)}$ *(where we assume, as always in this section, that f is stable).*

4. Reduction of theorem A to a result about jets.

Let $z \in J^l$, where $l \geq p+1$ and let $f \in \mathscr{F}$ be a representative of z. It follows from proposition (1.1) that whether f is stable depends only on z. We will say z is *stable* if f is. We let St^l denote the set of all stable jets in J^l.

From Proposition (1.1) it follows that $J^l - St^l$ is a closed algebraic subset of J^l. In other words, choosing local coordinates $y^1, \ldots, y^p$ for P, null at y, and for each i, $(1 \leq i \leq |S|)$ choosing local coordinates $x^1_{(i)}, \ldots, x^{n_i}_{(i)}$ for N, null at x_i (where $x_1, \ldots, x_s$ are the points of S), $J^l - St^l$ is the set of zeros of a family of polynomials in $\{y^j_{(i),\omega}\}$, where $\{y^j_{(i),\omega}\}$ is the global system of coordinates for J^l defined by

$$y^j_{(i),\omega}(f) = \frac{\partial^{|\omega|}(y^j \circ f)}{\partial x^\omega_{(i)}}(x_i).$$

Let $\mathscr{K}^l_0$ be defined as in the beginning of § 2. Let $\mathscr{R}_0$ denote the group of invertible C^∞ map-germs $h : (N, S) \dashrightarrow (N, S)$ such that $h|S = \text{identity}$. Let $\mathscr{A}_0 = \mathscr{R}_0 \times \mathscr{L} \subseteq \mathscr{A}$ and let $\mathscr{A}^l_0$ denote the image of $\mathscr{A}_0$ under the projection $\mathscr{A} \to \mathscr{A}^l$.

In proving Theorem A, we may assume, without loss of generality, that $N = N'$, $P = P'$, $y = y'$ and $x_i = x'_i$ for $1 \leq i \leq s$.

In this case it follows from Theorem (2.1) that the hypothesis that

$$Q_{p+1}(f_i) \approx Q_{p+1}(f'_i) \qquad (\text{for } 1 \leq i \leq s)$$

is equivalent to assuming that $f^{(p+1)}$ and $f'^{(p+1)}$ are in the same orbit under the action of $\mathscr{K}^{p+1}_0$. By our remarks above, the hypothesis that f and f' are stable is equivalent to assuming that $f^{(p+1)}$ and $f'^{(p+1)}$ are stable. Finally it follows from Proposition (3.5) that the conclusion of Theorem A is equivalent saying that $f^{(p+1)}$ and $f'^{(p+1)}$ are in the same orbit under the action of $\mathscr{A}_0^{p+1}$. Since St^{p+1} is invariant under the action of $\mathscr{A}_0^{p+1}$, it follows from these remarks that Theorem A is equivalent to:

$$\mathscr{A}_0^{p+1}z = \mathscr{K}_0^{p+1}z \cap \mathrm{St}^{p+1}, \quad \text{for any } z \in \mathrm{St}^{p+1}. \tag{4.1}$$

Thus to prove Theorem A, it suffices to prove (4.1). As a start, we have:

Lemma **(4.2)**. — *If* $z \in \mathrm{St}^{p+1}$, *then* $\mathscr{A}_0^{p+1}z$ *is open and closed in* $\mathscr{K}_0^{p+1}z \cap \mathrm{St}^{p+1}$.

Proof. — Consider the partition of $\mathscr{K}_0^{p+1}z \cap \mathrm{St}^{p+1}$ into orbits under the action of $\mathscr{A}_0^{p+1}$. Since $\mathscr{A}^{\#(p+1)} \subseteq \mathscr{A}_0^{p+1}$, it follows from Lemma (3.6) that each member of this partition is open. Taking complements, one obtains that each member is also closed.

Remark. — Everything that we have done up to now works (with neither more nor less difficulty) in the complex case (where the symbol " C^∞ " is replaced throughout by the word " holomorphic " or " complex analytic ", and the symbol " **R** " is replaced throughout by the symbol " **C** "). In contrast, the proof of Theorem A is much easier in the complex case. For in the complex case, the group $\mathscr{K}_0^{p+1}$ is connected. Since $\mathscr{K}_0^{p+1}$ is a complex analytic group, J^{p+1} is a complex analytic manifold, and the action of $\mathscr{K}_0^{p+1}$ on J^{p+1} is holomorphic, it follows that $\mathscr{K}_0^{p+1}z$ is a connected complex analytic submanifold of J^{p+1}. Since $J^{p+1} - \mathrm{St}^{p+1}$ is a closed algebraic subset of J^{p+1}, it follows that $\mathscr{K}_0^{p+1}z - \mathrm{St}^{p+1}$ is a closed analytic subvariety of $\mathscr{K}_0^{p+1}z$. Furthermore $\mathscr{K}_0^{p+1}z - \mathrm{St}^{p+1}$ is a *proper* subvariety of $\mathscr{K}_0^{p+1}z$, provided z is stable (since in this case z is not contained in this subvariety). Since the complement of any proper closed complex analytic subvariety of a connected complex analytic manifold is connected, it follows that $\mathscr{K}_0^{p+1}z \cap \mathrm{St}^{p+1}$ is connected, if $z \in \mathrm{St}^{p+1}$. Thus (4.1) follows from Lemma (4.2), which completes the proof of Theorem A in the complex analytic case.

5. Proof of theorem A in case S is a point.

Throughout this section, we will suppose that S is a point, say x.

By the remarks in § 4, it is enough to show (4.1). Note that in the case $S = x$, we have $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{K}_0 = \mathscr{K}$.

Let $x_1, \ldots, x_n$ be local coordinates for N, null at x, and $y_1, \ldots, y_p$ local coordinates for P, null at y. Let Λ_r denote the set of all $z \in J^{p+1}$ such that there is a representative $f : (N, x) \dashrightarrow (P, y)$ of z of the form (1.7).

Fix $z \in J^{p+1}$ throughout this section and let $r = \operatorname{rank} z = \operatorname{rank}$ (at x) of the representative f of z. Clearly any $\mathscr{A}^{p+1}$ orbit of z intersects Λ_r; hence to show (4.1), it is enough to consider points of Λ_r.

Let $\lambda : \Lambda_r \to \mathfrak{m}_{x'}^2 \mathscr{E}_{x'}^{p-r,(p+1)}$ be defined by

$$\lambda(f^{(p+1)}) = (f'_{\bullet})^{(p+1)},$$

for any f of the form (1.7). (Here, we are using the notation which we introduced in § 1 following (1.7). We shall continue to use this notation throughout the rest of this proof.) Let $\mathrm{V}=\mathscr{K}^{p+1}z\cap\Lambda_r$ and let $\mathrm{V}'=\lambda(\mathrm{V})$. To prove (4.1) it is enough to show

$$(5.1)\qquad \mathrm{V}\cap \mathrm{St}^{p+1}\subseteq\mathscr{A}^{p+1}v,\quad \text{for any } v\in\mathrm{V}\cap\mathrm{St}^{p+1}.$$

Let D denote the set of all $r\times(p-r)$ matrices with entries in $\mathfrak{m}_{x'}/\mathfrak{m}_{x'}^{p+1}$. If f has the form (1.7), we let $(\partial f)^{(p)}\in D$ denote the $r\times(p-r)$ matrix whose (i,j)-th entry is

$$(\partial f_{j+r}/\partial x_i)'^{(p)}\quad (1\leq i\leq r,\ 1\leq j\leq p-r).$$

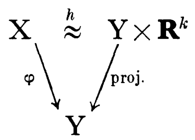
Note that the i-th row of the matrix $(\partial f)^{(p)}$ can be regarded as a member of $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$, since it is a $(p-r)$-tuple of elements of $\mathfrak{m}_{x'}/\mathfrak{m}_{x'}^{p+1}$. As such, it is equal to $(\partial_i f)^{(p)}$ (as defined in § 1, following (1.7)). In particular it follows that

$$(5.2)\qquad [(\partial f)^{(p)}]=[\partial f]^{(p)},$$

where the left hand side denotes the **R**-vector subspace of $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$ spanned by the rows of the matrix $(\partial f)^{(p)}$, and the right hand side is as in Proposition (1.8).

For any $w\in\Lambda_r$, let $\partial w\in D$ be defined as $(\partial f)^{(p)}$, where f is any representative of w of the form (1.7). Clearly ∂w is independent of the choice of representative f.

In this section, we will say that a continuous mapping $\varphi:\mathrm{X}\to\mathrm{Y}$ is *trivial* if there exists a homeomorphism $h:\mathrm{X}\to\mathrm{Y}\times\mathbf{R}^k$, for a suitable non-negative integer k, such that the following diagram commutes:



It is easily seen that the mapping

$$\langle\lambda,\partial\rangle:\Lambda_r\to\mathfrak{m}_{x'}^2\mathscr{E}_{x'}^{p-r,(p+1)}\times D$$

is trivial: in terms of the Taylor series expansion (to order $p+1$) of $w\in\Lambda_r$, $\lambda(w)$ picks out certain coefficients, and $\partial(w)$ picks out others.

It is easily seen that $\mathrm{V}=\lambda^{-1}\mathrm{V}'$. For if $v\in\Lambda_r$ and $\lambda v\in\mathrm{V}'$, then there exists $v'\in\mathrm{V}$ such that $\lambda v'=\lambda v$. It is easily seen that $\lambda v'=\lambda v$ implies $\mathrm{Q}(v)\approx\mathrm{Q}(v')$ (where for any $w\in\mathrm{J}^l$, $\mathrm{Q}(w)$ denotes $\mathrm{Q}_l(g)$ for any representative g of w). Thus v and v' are in the same $\mathscr{K}^{p+1}$-orbit, by Theorem (2.1). Since $v'\in\mathrm{V}$, this implies $v\in\mathrm{V}$.

It follows that the mapping

$$(5.3)\qquad \langle\lambda,\partial\rangle|\mathrm{V}:\mathrm{V}\to\mathrm{V}'\times D$$

is trivial.

Let $v'\in\mathrm{V}'$. Choose $v\in\mathrm{V}$ such that $\lambda(v)=v'$ and f of the form (1.7) such that $v=f^{(p+1)}$. Set

$$\bar{\Omega}(v')=\Omega(f'_{\bullet})^{(p)},$$

where the right hand side is defined as in the paragraphs preceeding Proposition (1.8). Note that $\overline{\Omega}(v')$ depends only on v'. For it follows from $z\in \mathrm{St}^{p+1}$, $v\in \mathscr{K}^{p+1}z$, and Corollary (2.6), that for a suitable representative f of v, we have $d(f, \mathscr{K})\leq p$. Then

$$(*) \qquad \mathfrak{m}_{x'}^{p}\mathscr{E}_{x'}^{p-r}\subseteq \Omega(f'_{\bullet}).$$

Since this inclusion is true for one representative f of v (in the form (1.7)), it is true for every such representative (by Nakayama's lemma). With the aid of this inclusion, one sees that the fact that $\overline{\Omega}(v')$ depends only on v' follows immediately from the definitions.

By definition $\overline{\Omega}(v')$ is an $\mathbf{R}$-vector subspace of $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$. Its dimension is independent of the choice of $v'\in \mathrm{V}'$. For, from the inclusion (*), it follows that the codimension of $\overline{\Omega}(v')$ in $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$ is equal to the codimension of $\Omega(f'_{\bullet})$ in $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r}$, which (by the definitions) is equal to $d(f, \mathscr{K})-(p-r)$. Then the fact that this number is independent of the choices made (including the choice of $v'\in \mathrm{V}'$) follows from Corollary (2.6).

In computing this number, we may take f as a representative of z, since $d(f, \mathscr{K})$ is independent of the choice of f in a given $\mathscr{K}$ orbit. Since $z\in \mathrm{St}^{p+1}$, f is stable; hence f is $(p+1)$-determined (by § 3); hence writing

$$\textbf{(5.4)} \qquad c(z)=\operatorname{codim} \mathscr{K}^{p+1}z-n+r$$

(where codim means codimension in J^{k}) we obtain that $\Omega(v')$ has codimension $c(z)$, by Theorem (2.5). Hence, letting $\mathbf{G}$ denote the Grassmannian of $c(z)$-codimensional vector subspaces of $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$, we see that we have defined a mapping

$$\overline{\Omega} : \mathrm{V}'\rightarrow \mathbf{G}.$$

This mapping is clearly continuous.

Let $v\in \mathrm{V}$. By Proposition (1.8) the necessary and sufficient condition that $v\in \mathrm{St}^{p+1}$ is that

$$\textbf{(5.5)} \qquad \overline{\Omega}(\lambda v)+[\partial v]=\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}.$$

Clearly, the set of $\langle v', d\rangle\in \mathrm{V}'\times D$ such that

$$\overline{\Omega}(v')+[d]=\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$$

(where $[d]$ denotes the subspace of $\mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(p)}$ spanned by the rows of d) is a locally trivial bundle over V'; the fiber has one component if $c(z)<r$, or $r=0$, and two components if $c(z)=r>0$. (Note that $c(z)>r$ is impossible by the hypothesis that z is stable: if $c(z)>r$ the relation (5.5) can never hold for $v\in \mathrm{V}$ (since $\overline{\Omega}(\lambda v)$ has codimension $c(z)$ and $[\partial v]$ is spanned by r elements), but there exists $v\in \mathrm{St}^{p+1}\cap \mathrm{V}$). Applying the fact that the mapping (5.3) is trivial, we obtain:

Lemma **(5.6)**. — *The bundle* $(\mathrm{V}\cap \mathrm{St}^{p+1}, \lambda, \mathrm{V}')$ *is locally trivial; a fiber has one component if* $c(z)<r$ *or* $r=0$, *it has two components if* $c(z)=r>0$. *We never have* $c(z)>r$.

Next, we show:

Lemma (5.7). — *Any fiber of the bundle* $(V \cap St^{p+1}, \lambda, V')$ *is contained in a single orbit of* $\mathscr{A}^{p+1}$.

Proof. — In the case the fiber is connected, this follows from Lemma (4.2). Hence we may suppose $c(z)=r>0$. It follows from Lemma (4.2) that it is enough to show that there is an orbit of $\mathscr{A}^{p+1}$ which meets both components of the fiber. Let $h : (N, x) \to (N, x)$ and $h' : (P, y) \to (P, y)$ be defined by

$$x_1 \circ h = -x_1, \quad x_i \circ h = x_i, \quad \text{for } i>1$$
$$y_1 \circ h' = -y_1, \quad y_i \circ h' = y_i, \quad \text{for } i>1.$$

It is easily verified that if $v \in V \cap St^{p+1}$ and $v' = h'^{(p+1)} \circ v \circ h^{(p+1)}$, then $\lambda(v') = \lambda(v)$ and $\partial v'$ is the matrix whose first row is the negative of the first row of ∂v and whose other rows are the same as the corresponding rows of ∂v. It follows that v and v' are in the two different components of $\lambda^{-1}(\lambda v) \cap St^{p+1}$ (which is the fiber of the bundle $(V \cap St^{p+1}, \lambda, V')$ over λv). This proves Lemma (5.7).

The next step is to analyze the connected components of V'. We do this by showing that V' is an orbit of a certain group $\mathscr{K}'^{(p+1)}$ which acts on $\mathfrak{m}_{x'} \mathscr{E}_{x'}^{p-r,(p+1)}$. We introduce manifolds $N' \subseteq N$ and $P' \subseteq P$ defined by

$$N' = \{x_1 = \ldots = x_r = 0\} \qquad P' = \{y_1 = \ldots = y_r = 0\}.$$

Then we can identify $\mathscr{E}_{x'}$ with the set of germs at x of C^∞ functions on N, since $\mathscr{E}_{x'}$ is the ring of germs at 0 of C^∞ functions in $x_{r+1}, \ldots, x_n$ and $\{x_{r+1}, \ldots, x_n\}$ is a local system of coordinates for N', null at x. We identify $\mathfrak{m}_{x'} \mathscr{E}_{x'}^{p-r}$ with the set $\mathscr{F}'$ of C^∞ map-germs $(N', x) \to (P', y)$ by identifying $(f_1, \ldots, f_{p-r})$ with the map-germ f defined by

$$y_i \circ f = f_{i-r}, \qquad r+1 \leq i \leq p.$$

This gives rise to an identification of $\mathfrak{m}_{x'} \mathscr{E}_{x'}^{p-r,(p+1)}$ with the set $J'^{(p+1)}$ of $(p+1)$-jets of such map-germs. Now the group $\mathscr{K}'^{(p+1)}$ is defined just as $\mathscr{K}^{(p+1)}$, but with N' in place of N and P' in place of P. It is easily verified that a point

$$v' \in \mathfrak{m}_{x'} \mathscr{E}_{x'}^{p-r,(p+1)} = J'^{(p+1)}$$

is in V' if and only if $Q(v') \approx Q(z)$; thus, it follows from Theorem (2.1) that V' is an orbit of the action of $\mathscr{K}'^{(p+1)}$ on $J'^{(p+1)}$.

Since $\mathscr{K}'^{(p+1)}$ has four components, it follows that V' has at most four components. Now consider again the problem of proving (5.1). By Lemmas (4.2) and (5.7), $\lambda^{-1}(V'_0) \cap St^{p+1}$ is contained in a single orbit of $\mathscr{A}^{p+1}$ for any connected component V'_0 of V'.

Define $\mathscr{A}'$ and $\mathscr{A}'^{(p+1)}$ in the same way as $\mathscr{A}$ and $\mathscr{A}^{p+1}$, except with N' in place of N and P' in place of P. Clearly $\mathscr{A}'^{(p+1)}$ meets each component of $\mathscr{K}'^{(p+1)}$. Thus we may prove (5.1) as follows. Take $v, v_1 \in V \cap St^{p+1}$. Since V' is an orbit of $\mathscr{K}'^{(p+1)}$, there exists $\eta \in \mathscr{K}'^{(p+1)}$ such that $\lambda(v_1) = \eta\lambda(v)$. Take $\eta_1 \in \mathscr{A}'^{(p+1)}$ which is in the same component of $\mathscr{K}'^{(p+1)}$ as η. Let $\langle h, h'\rangle \in \mathscr{A}'$ be a representative of η_1 and let $\langle \widetilde{h}, \widetilde{h}'\rangle \in \mathscr{A}$ be an extension of $\langle h, h'\rangle$ such that $x_i \circ \widetilde{h} = x_i$, $y_i \circ \widetilde{h}' = y_i$, for $1 \leq i \leq r$. Let $\widetilde{\eta}_1 = \langle \widetilde{h}, \widetilde{h}'\rangle^{(p+1)} \in \mathscr{A}^{p+1}$. Then $\lambda\widetilde{\eta}_1(v) = \eta_1\lambda(v)$. Hence $\lambda\widetilde{\eta}_1(v)$ is in the same component (say V_0') of V' as $\eta\lambda(v) = \lambda(v_1)$. Since $\lambda^{-1}(V_0') \cap \mathrm{St}^{p+1}$ is contained in a single orbit of $\mathscr{A}^{p+1}$ it follows that $\widetilde{\eta}_1 v$ and v_1 are contained in the same orbit of $\mathscr{A}^{p+1}$; hence v and v_1 are in the same orbit. This completes the proof of (5.1) and therefore of Theorem A in the case S is a point.

As a corollary of what we have just proved and Proposition (1.8), we can obtain a " normal form " for stable map-germs. To describe this normal form we need to introduce some notation.

Let $\mathbf{R}[[x']] = \mathbf{R}[[x_{r+1}, \ldots, x_n]]$ denote the ring of formal power series in indeterminates $x_{r+1}, \ldots, x_n$ and let $\mathfrak{m}$ denote its unique maximal ideal. Let $q = \langle q_1, \ldots, q_{p-r}\rangle$ be a $(p-r)$-tuple of polynomials in $x_{r+1}, \ldots, x_n$ and suppose $q_i \in \mathfrak{m}^2$. Let (q) denote the ideal in $\mathbf{R}[[x']]$ generated by $\{q_1, \ldots, q_{p-r}\}$. For $r+1 \leq i \leq n$, let

$$\partial_i q = \left\langle \frac{\partial q_1}{\partial x_i}, \ldots, \frac{\partial q_{p-r}}{\partial x_i} \right\rangle \in \mathfrak{m}\mathbf{R}[[x]]^{p-r}.$$

Set

$$\Psi(q) = \mathbf{R}[[x']]\{\partial_{r+1}q, \ldots, \partial_n q\} + (q)\mathbf{R}[[x']]^{p-r}.$$

Then $\Psi(q)$ is an $\mathbf{R}[[x']]$-submodule of $\mathfrak{m}\mathbf{R}[[x']]^{p-r}$. Let

$$c = c(q) = \dim\,(\mathfrak{m}\mathbf{R}[[x']]^{p-r}/\Psi(q)).$$

Suppose $c \leq r$. Let $v_1, \ldots, v_c$ be a set of elements of $\mathfrak{m}\mathbf{R}[[x']]^{p-r}$ whose canonical images in $\mathfrak{m}\mathbf{R}[[x']]^{p-r}/\Psi(q)$ form a basis (where $\mathfrak{m}\mathbf{R}[[x']]^{p-r}/\Psi(q)$ is considered as an $\mathbf{R}$-vector space); write $v_i = \langle v_{i1}, \ldots, v_{i,p-r}\rangle$ where $v_{ij} \in \mathfrak{m}\mathbf{R}[[x']]$; and suppose that v_{ij} is a polynomial.

Define $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ as follows:

$$(5.8)\qquad y_i \circ f = \begin{cases} x_i & , \quad 1 \leq i \leq r \\ q_{i-r} + \sum_{j=1}^{c} x_j v_{j,i-r}, & r+1 \leq i \leq p. \end{cases}$$

Lemma (5.9). — *If f is given by* (5.8), *where q and v are as above, then f is stable.*

Proof. — This is a matter of checking that the conditions given in Proposition (1.8) are satisfied. The image of $\Psi(q)$ under the canonical homomorphism $\mathbf{R}[[x']]^{p-r} \to \mathscr{E}_{x'}^{p-r,(k)}$ is $\Omega(f'_{\bullet})^{(k)}$; the image of v_i is $\partial_i f^{(k)}$; hence the hypothesis that the canonical images of $v_1, \ldots, v_p$ in $\mathfrak{m}\mathbf{R}[[x']]^{p-r}/\Psi(w)$ span this $\mathbf{R}$-vector space implies that

$$\Omega(f'_{\bullet})^{(k)} + [\partial f]^{(k)} = \mathfrak{m}_{x'}\mathscr{E}_{x'}^{p-r,(k)},$$

which implies f is stable, by (1.8).

Note that the mapping in examples (1.9) and (1.10) are of the form (5.8).

Theorem **(5.10)** (Normal form for a stable map-germ). — *If* $f:(\mathrm{N},x)\dashrightarrow(\mathrm{P},y)$ *is a stable map-germ whose rank at* x *is* r, *and* $q_1,\ldots,q_{p-r}$ *and* $v_1,\ldots,v_c$ *are as above and furthermore*

$$(*)\qquad \mathbf{R}[[x']]/(\mathfrak{m}^{p+2}+(q))\approx \mathrm{Q}_{p+1}(f)$$

then there exist local coordinates $x_1,\ldots,x_n$ *for* N, *null at* x, *and* $y_1,\ldots,y_p$ *for* P, *null at* y, *such that* f *has the form* (5.8).

Proof. — In any case there exists $f':(\mathbf{R}^n,0)\dashrightarrow(\mathbf{R}^p,0)$ such that f' has the form (5.8). Then the hypothesis $(*)$ amounts to saying that $\mathrm{Q}_{p+1}(f')\approx\mathrm{Q}_{p+1}(f)$. By hypothesis, f is stable; by Lemma (5.9), f' is stable. Then f and f' are isomorphic, by Theorem A, which is another way of stating the conclusion of the theorem.

Remark. — If $f:(\mathrm{N},x)\dashrightarrow(\mathrm{P},y)$ is a stable map-germ, we can find polynomials $q_1,\ldots,q_{p-r}\in\mathfrak{m}^2$ and $v_{ij}\in\mathfrak{m}$ $(1\leq i\leq c,\ 1\leq j\leq p-r)$ such that with respect to suitable local coordinate systems f has the form (5.8)), as follows. Since f has rank r, there exists a surjective **R**-algebra homomorphism

$$\pi:\mathbf{R}[[x']]/\mathfrak{m}^{p+2}\to\mathrm{Q}_{p+1}(f).$$

From the definition of $\mathrm{Q}_{p+1}(f)$, it follows that $\ker\pi$ is generated by $p-r$ or fewer elements; let $q_1,\ldots,q_{p-r}$ be a set of generators of $\ker\pi$. Then the isomorphism $(*)$ holds. Furthermore, $c(q)=d(f,\mathscr{K})-p+r\leq r$, so we can choose $v_1,\ldots,v_c$ as required.

Next, we state and prove a lemma which we will use in the next section. For any $w\in\mathrm{St}^{p+1}$ and any representative f of w, we set

$$\tau(w)=\tau(f)$$

(cf. Definition (1.3)). This is independent of the choice of representative f, since $w\in\mathrm{St}^{p+1}$ implies f is stable, which implies

$$\mathfrak{m}_x^p\theta(f)\subseteq f^*(\mathfrak{m}_y)\theta(f)+tf(\mathrm{B}).$$

Let W denote the set of $w\in\mathrm{St}^{p+1}\cap\mathscr{K}^{p+1}z$ such that $\tau(w)=\tau(z)$.

Lemma **(5.11)**. — *Suppose* $p>\dim\tau(z)>0$. *For any* $w_1,w_2\in\mathrm{W}$, *there exists a number* $\sigma(w_1,w_2)$ *(which is* ±1*) such that the following holds. If* $h'\in\mathscr{L}$ *is such that* $\mathrm{T}h'(\tau(z))=\tau(z)$ *and the automorphism of* $\mathrm{TP}_y/\tau(z)$ *which* h' *induces is orientation preserving (in the case* $\sigma(w_1,w_2)=+1$*) or orientation reversing (in the case* $\sigma(w_1,w_2)=-1$*) then there exists* $h\in\mathscr{R}$ *such that* $\langle h,h'\rangle^{(p+1)}w_1$ *is in the same arcwise connected component of* W *as* w_2.

Proof. — Let $\mathscr{L}^*$ denote the subgroup of $\mathscr{L}$ consisting of those h' for which $\mathrm{T}h'(\tau(z))\subseteq\tau(z)$. Let $\mathscr{L}^{*k}$ denote the image of $\mathscr{L}^*$ under the projection $\mathscr{L}\to\mathscr{L}^k$. Let $\mathscr{A}^{*k}=\mathscr{R}^k\times\mathscr{L}^{*k}$. From (4.1), it follows that $\mathrm{W}\subseteq\mathscr{A}^{p+1}z$, which is easily seen to imply $\mathrm{W}=\mathscr{A}^{*(p+1)}z$. Hence, there exists $h_0'\in\mathscr{L}^*$ and $h_0\in\mathscr{R}$ such that $\langle h_0,h_0'\rangle^{(p+1)}w_1=w_2$. We set $\sigma(w_1,w_2)=+1$ if the automorphism of $\mathrm{TP}_y/\tau(z)$ which h_0' induces is orientation preserving and we set $\sigma(w_1, w_2) = -1$ otherwise. Now let h' be as in the lemma. Choose $h \in \mathscr{R}$ so that hh_0^{-1} is orientation preserving or reversing according to whether

$$T(h'h_0'^{-1}) \mid \tau(z) : \tau(z) \to \tau(z)$$

is orientation preserving or reversing.

To show that $\langle h, h' \rangle^{(p+1)} w_1$ is in the same connected component of W as w_2 it is enough to show that there is $\langle h_1, h_1' \rangle^{(p+1)} \in \mathscr{A}^{\#(p+1)}$ which is in the same connected component of $\mathscr{A}^{\#(p+1)}$ as $\langle hh_0^{-1}, h'h_0'^{-1} \rangle$, and which satisfies $\langle h_1, h_1' \rangle^{(p+1)} w_2 = w_2$, since

$$\langle h, h' \rangle^{(p+1)} w_1 = \langle hh_0^{-1}, h'h_0'^{-1} \rangle^{(p+1)} w_2.$$

In the case $T(h'h_0'^{-1}) \mid \tau(z)$ is orientation preserving, we may take $h_1 = \text{identity}$, $h_1' = \text{identity}$. To see that $\langle hh_0^{-1}, h'h_0'^{-1} \rangle$ is in the same component of $\mathscr{A}^{\#(p+1)}$ as $\langle h_1, h_1' \rangle$, it is enough to observe that the automorphisms of $\tau(z)$ and $TP_y/\tau(z)$ induced by $h'h_0'^{-1}$ are orientation preserving and that hh_0^{-1} is orientation preserving.

In the case $T(h'h_0'^{-1}) \mid \tau(z)$ is orientation reversing, we choose a representative $f : (N, x) \to (P, y)$ of w_2 and local coordinates such that f has the form (5.8) (which we may do by Theorem (5.10) and the remark following it). It is easily seen that $\tau(f) = \tau(z)$ is the subset of TP_y defined by $dy_1 = \ldots = dy_c = dy_{r+1} = \ldots = dy_p = 0$. The hypothesis that $\dim \tau(z) > 0$ implies that $c < r$. If we define h_1 and h_1' by

$$\begin{aligned} x_i \circ h_1 &= x_i, & 1 \leq i \leq n,\ i \neq c+1 \\ &= -x_i, & i = c+1 \\ y_i \circ h_1' &= y_i, & 1 \leq i \leq p,\ i \neq c+1 \\ &= -y_i, & i = c+1 \end{aligned}$$

then it follows from the fact that f has the form (5.8) that $h_1' \circ f \circ h_1^{-1} = f$, and therefore that $\langle h_1, h_1' \rangle^{(p+1)} w_2 = w_2$. To see that $\langle hh_0^{-1}, h'h_0'^{-1} \rangle$ is in the same component of $\mathscr{A}^{\#(p+1)}$ as $\langle h_1, h_1' \rangle$, it is enough to observe that the automorphisms of $\tau(z)$ that h_1' and $h'h_0'^{-1}$ induce are both orientation reversing, that the automorphisms of $TP_y/\tau(z)$ that h_1' and $h'h_0'^{-1}$ induce are both orientation preserving, and that the automorphisms of TN_x that h_1 and hh_0^{-1} induce are both orientation reversing.

6. Proof of theorem A in general.

In this section, we return to the general setting where S is an arbitrary finite set of points of N, say $\{x_1, \ldots, x_s\}$. We recall that $J^k = J_1^k \times \ldots \times J_s^k$, where J_i^k denotes the set of k-jets of C^∞ map-germs $(N, x_i) \to (P, y)$. Similarly $\mathscr{K}_0^k = \mathscr{K}_1^k \times \ldots \times \mathscr{K}_s^k$, $\mathscr{A}_0^k = \mathscr{A}_1^k \times \ldots \times \mathscr{A}_s^k$ and the actions of these groups on J^k are compatible with the product decompositions.

For any $z \in J^k$, we write $z = \langle z_1, \ldots, z_s \rangle$, where $z_i \in J_i^k$. We let $\tau(z_i) \subseteq TP_y$ be defined as in the previous section. By (1.6), if $z \in J^{p+1}$ then $z \in St^{p+1}$ if and only if each $z_i \in St_i^{p+1}$ and $\tau(z_1), \ldots, \tau(z_s)$ have regular intersection in TP_y.

By the remarks in § 4, it is enough to show (4.1) in order to prove Theorem A.

To this end, we consider $w, z \in \mathrm{St}^{p+1}$ and suppose w and z are in the same orbit of $\mathscr{K}_0^{p+1}$. For $1 \leq i \leq s$, we let W_i denote the set of all $w' \in \mathrm{St}_i^{p+1} \cap \mathscr{K}_i^{p+1} z_i$ such that $\tau(w') = \tau(z_i)$.

Lemma **(6.1)**. — *There exists* $\langle h, h' \rangle \in \mathscr{A}_0$ *such that, for* $1 \leq i \leq s$, $\langle h_i, h' \rangle^{(p+1)} w_i$ *is in the arcwise connected component of* W_i *which contains* z_i.

Proof. — Let F be the dual of TP_y (considered as an **R**-vector space); let $E_i \subseteq F$ denote the annihilator of $\tau(z_i)$, and let $E'_i \subseteq F$ denote the annihilator of $\tau(w_i)$. Since $\tau(z_1), \ldots, \tau(z_s)$ have regular intersection (by (1.6)) the sum $E_1 + \ldots + E_s$ is a *direct* sum. Similarly the sum $E'_1 + \ldots + E'_s$ is a direct sum. Furthermore, since z_i and w_i are in the same orbit of $\mathscr{A}_i^{p+1}$ (by Theorem A in the case S is a point), it follows that E_i and E'_i have the same dimension. Hence there exists an automorphism $L : F \to F$ such that $L(E_i) = E'_i$, $1 \leq i \leq s$. Let $h'_1 : (P, y) \dashrightarrow (P, y)$ be such that $Th'_1 : TP_y \to TP_y$ is the dual of L, and let $w' = h_1'^{(p+1)} w$. From $L(E_i) = E'_i$ it follows that $\tau(w'_i) = Th'_1(\tau(w_i)) = \tau(z_i)$.

Note that we may assume that for each i, $1 \leq i \leq s$, $p > \dim \tau(z_i) > 0$. In the case $\dim \tau(z_i) = p$, we have that z_i is the jet of a submersion, so it suffices to prove the lemma for $S_i = \{x_1, \ldots, \hat{x}_i, \ldots, x_s\}$ in place of S. Then we may assume that $p > \tau(z_i)$ for all i. Using this assumption we see that if $\dim \tau(z_i) = 0$ for some i, the hypothesis that $\tau(z_1), \ldots, \tau(z_s)$ have regular intersection implies that x_i is the only point in S; thus, we see that the problem reduces to the case when S is a point.

Now we assume $p > \dim \tau(z_i) > 0$. Clearly $w'_i \in W_i$. Therefore the number $\sigma(w'_i, z_i)$ of Lemma (5.11) is defined; we may choose an automorphism L_0 of F such that $L_0(E_i) = E_i$ and $L_0 | E_i : E_i \to E_i$ is orientation preserving or reversing according to whether $\sigma(w'_i, z_i)$ is $+1$ or -1.

Let $h'_0 : (P, y) \dashrightarrow (P, y)$ be such that $Th'_0 : TP_y \to TP_y$ is the dual of L_0. Then $Th'_0(\tau(z_i)) = \tau(z_i)$ and the automorphism of $TP_y / \tau(z_i)$ which h'_0 induces is orientation preserving or orientation reversing according to whether $\sigma(w'_i, z_i) = +1$ or -1.

Hence, by Lemma (5.11), we may choose $h_i \in \mathscr{R}_i$ such that $\langle h_i, h'_0 \rangle^{p+1} w'_i$ is in the same connected component of W_i as z_i. Let $h = \langle h_1, \ldots, h_s \rangle \in \mathscr{R}_0$ and let $h' = h'_0 \circ h'_1 \in \mathscr{L}$. Then $\langle h, h' \rangle \in \mathscr{A}_0$ and

$$\langle h_i, h' \rangle^{(p+1)} w_i = \langle h_i, h'_0 \rangle^{(p+1)} w'_i$$

is in the same connected component of W_i as z_i, which proves the lemma.

Now we may complete the proof of Theorem A, or rather of (4.1), which as we have seen, implies Theorem A. By Lemma (6.1), we may suppose that for each i, w_i is in the arcwise connected component of W_i which contains z_i. Let $\gamma_i : [0, 1] \to W_i$ be a continuous mapping such that $\gamma_i(0) = w_i$ and $\gamma_i(1) = z_i$, for $1 \leq i \leq s$. Let $\gamma : [0, 1] \to J^{p+1}$ be defined by $\gamma(t)_i = \gamma_i(t) \in J_i^{p+1}$. By Lemma (4.2) it is sufficient to show that $\gamma(t) \in \mathscr{K}_0^{p+1} z \cap \mathrm{St}^{p+1}$ for all $t \in [0, 1]$, in order to show that $w = \gamma(0)$ and $z = \gamma(1)$ are in the same orbit of $\mathscr{A}_0^{p+1}$. But it follows from the definition of W_i that $\gamma(t)_i \in \mathscr{K}_i^{p+1} z_i$; hence $\gamma(t) \in \mathscr{K}_0^{p+1} z$. Also by the definition of W_i, $\gamma(t)_i \in \mathrm{St}_i^{p+1}$ and $\tau(\gamma(t)_1), \ldots, \tau(\gamma(t)_s)$ have regular intersection in TP_y (since $\tau(\gamma(t)_i) = \tau(z_i)$); hence by Proposition (1.6), $\gamma(t) \in \mathrm{St}^{p+1}$.

This shows that w and z are in the same orbit of $\mathscr{A}_0^{p+1}$ and therefore (by (4.1)) completes the proof of Theorem A.

7. Proof of theorem B.

We begin defining the number $\mu_c(\mathrm{A})$, whose existence was asserted in the introduction. Let A denote the quotient of a formal power series ring over $\mathbf{R}$:

$$\text{(7.1)} \qquad \mathrm{A}=\mathbf{R}[[x_1, \ldots, x_a]]/(f_1, \ldots, f_b).$$

Let $c \leq \iota(\mathrm{A})$, where $\iota(\mathrm{A})$ is the number defined in the introduction. We may suppose that the representation (7.1) of A as the quotient of a formal power-series ring is chosen so that $c=a-b$. Let Ψ denote the A-submodule of A^b generated by the canonical image of

$$\left\{\left(\frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_b}{\partial x_i}\right)\right\}_{i=1,\ldots,a} \quad \text{in } \mathrm{A}^b.$$

We set

$$\mu_c(\mathrm{A})=\dim_{\mathbf{R}} \mathrm{A}^b/\Psi.$$

Theorem **(7.2)**. — *The number* $\mu_c(\mathrm{A})$ *depends only on* c *and* A, *not on the particular choice of presentation* (7.1). *Furthermore if* $f:(\mathrm{N}, x) \dashrightarrow (\mathrm{P}, y)$ *is a* C^∞ *map-germ, then*

$$\mu_c(\hat{\mathrm{Q}}(f))=d(f, \mathscr{K}).$$

Proof. — The latter sentence is obvious from the definitions and Nakayama's lemma. To prove the first sentence, we consider a second presentation of A:

$$\text{(7.1)}' \qquad \mathrm{A}=\mathbf{R}[[x_1', \ldots, x_{a'}']]/(f_1', \ldots, f_{b'}'),$$

where we assume that $a'-b'=c$. We carry out the proof in two steps: first, we suppose $a=a'$ and $b=b'$; then we give the proof in general. Let Ψ' be the A-submodule of A^b defined in the same way as Ψ except with f_i' in place of f_i and x_i' in place of x_i. From the assumption that $a=a'$ and (7.1) and (7.1)$'$ it follows that there exists an isomorphism

$$\varphi: \mathbf{R}[[x_1', \ldots, x_a']] \rightarrow \mathbf{R}[[x_1, \ldots, x_a]]$$

mapping the ideal generated by $f_1', \ldots, f_b'$ onto the ideal generated by $f_1, \ldots, f_b$. Such an isomorphism is necessarily given by a " substitution " $x_i'=x_i'(x_1, \ldots, x_s)$; thus, setting $g_i=\varphi(f_i')$, we have $g_i(x_1, \ldots, x_s)=f_i'(x_i', \ldots, x_s')$; by the " chain rule "

$$\frac{\partial g_k}{\partial x_i}=\sum_j \frac{\partial f_k'}{\partial x_j'} \frac{\partial x_j'}{\partial x_i};$$

thus Ψ' is the submodule of A^b generated by the canonical image of

$$\left\{\left(\frac{\partial g_1}{\partial x_i}, \ldots, \frac{\partial g_b}{\partial x_i}\right)\right\}_{i=1,\ldots,a} \quad \text{in } \mathrm{A}^b.$$

From the fact that $f_1, \ldots, f_b$ and $g_1, \ldots, g_b$ generate the same ideal in $\mathbf{R}[[x_1, \ldots, x_a]]$ it follows that there exists an invertible matrix (μ_{ij}) with entries in $\mathbf{R}[[x_1, \ldots, x_a]]$ such that $f_i = \sum_i \mu_{ij} g_j$. Hence letting $\varphi : \mathrm{A}^b \to \mathrm{A}^b$ be the A-module automorphism given by

$$\varphi(\varepsilon_i) = \sum_j \mu_{ij} \varepsilon_j,$$

(where $\varepsilon_i = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$, with the 1 appearing in the i-th place), we see

$$\varphi(\Psi') = \Psi,$$

which completes the proof in case $a = a'$, $b = b'$.

For the proof in general, we may suppose $a \leq a'$. By what we have just shown, it is enough to show that for a given presentation of the form (7.1) there is one presentation of the form (7.1)$'$ which gives the same value of $\mu_c(\mathrm{A})$. Let the presentation of the form (7.1)$'$ be given by $x_i' = x_i$, $1 \leq i \leq a$, $f_i' = f_i$, $1 \leq i \leq b$, $f_i' = x'_{i-b+a}$, $b < i \leq b'$. Then it is trivial to verify that these two presentations give the same value of $\mu_c(\mathrm{A})$.

Proof of Theorem B. — " Only if " is clear. We show " if ". Let $\mathfrak{m}$ denote the maximal ideal of A, and let $a = \dim_{\mathbf{R}}(\mathfrak{m}/\mathfrak{m}^2)$. Our first step is to show $a \leq n$. There is a representation of A in the form (7.1); since $a = \dim_{\mathbf{R}}(\mathfrak{m}/\mathfrak{m}^2)$, we necessarily have $f_1, \ldots, f_b \in \mathfrak{m}^2$. The hypothesis that $\iota(\mathrm{A}) \geq n-p$ implies that we can choose b so that $a - b = n - p$. Clearly $\dim_{\mathbf{R}}(\mathfrak{m}\mathrm{A}^b/\mathfrak{m}^2\mathrm{A}^b) = ab$; on the other hand, it follows from the definition of Ψ that $\dim_{\mathbf{R}}((\Psi + \mathfrak{m}^2\mathrm{A}^b)/\mathfrak{m}^2\mathrm{A}^b) \leq b$. It follows that

$$(a-1)\tau \leq \pi_{a-b}(\mathrm{A}) = \mu_{n-p}(\mathrm{A}) \leq p.$$

Now suppose $a \leq n$ is not satisfied. Then $a \geq n+1$, $b \geq p+1$, so we have

$$n(p+1) \leq (a-1)b \leq p,$$

which is evidently impossible. Hence $a \leq n$.

Now we set $a = n - r$, where $r \geq 0$. Then $b = p - r$. We write the presentation (7.1) of A in the form

$$\mathrm{A} \approx \mathbf{R}[[x_{r+1}, \ldots, x_n]]/(q_1, \ldots, q_{p-r}). \tag{7.3}$$

It follows from (III, (3.5)) (applied in the case $\mathscr{S} = \mathscr{K}$) that we can take $q_1, \ldots, q_{p-r}$ to be polynomials. It is then easily seen that $q = \langle q_1, \ldots, q_{p-r} \rangle$ satisfies the hypotheses we imposed on q in order to obtain the canonical form (5.8); moreover:

$$c(q) = \mu_{n-p}(\mathrm{A}) - (p-r),$$

so the hypothesis that $\mu_{n-p}(\mathrm{A}) \leq p$ implies that $c(q) \leq r$. Hence we can find $v_1, \ldots, v_c$ as required to obtain a stable f in the form (5.8) (cf. Lemma (5.9)). For f in this form, we clearly have $\hat{\mathrm{Q}}(q) \approx \mathrm{A}$, which completes the proof of Theorem B.

It is clear that if $f : (\mathrm{N}, x) \dashrightarrow (\mathrm{P}, y)$ is stable, then

$$d(f, \mathscr{K}) = \dim_{\mathbf{R}}(\mathrm{TP}_y/\tau(f)). \tag{*}$$

With this remark, Theorem B, and Proposition (1.6), it is easy to show the addendum. " Only if " is immediate. To show " if " we observe first that we can choose stable $f_i : (N, x_i) \dashrightarrow (P, y)$ such that $A_i \approx \hat{Q}(f_i)$ by Theorem B. By the second inequality that we assumed in the proposition and (*), we can choose the f_i so that $\tau(f_1), \ldots, \tau(f_s)$ have regular intersection in TP_y. By Proposition (1.6) this implies that $f: (N, S) \dashrightarrow (P, y)$ (defined by $f \mid (N, x_i) = f_i$) is stable.

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Manuscrit reçu le 1er juin 1968.

N. d. l. R. : Par suite d'une erreur, le manuscrit du travail [1] a été indiqué comme reçu le 2 septembre 1968 ; il faut remplacer cette date par « 1er janvier 1968 ».

1970. — Imprimerie des Presses Universitaires de France. — Vendôme (France)
ÉDIT. N° 31 282 IMPRIMÉ EN FRANCE IMP. N° 21 918