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AN EXTENSION OF WHITNEY'S SPECTRAL THEOREM

by J.-Cl. TOUGERON

1. Notations and results.

For $y \in \mathbf{R}^p$ and $Y \subset \mathbf{R}^p$, $|y|$ denotes the euclidean norm of y and $d(y, Y)$ the euclidean distance from y to Y. If Y is empty, we write $d(y, Y) = 1$.

Let Ω_p denote an open set in $\mathbf{R}^p$ and $\mathscr{E}(\Omega_p)$ the $\mathbf{R}$-algebra of all C^∞ real-valued functions in Ω_p. When $y \in \Omega_p$, let $\mathscr{F}_y^m$ denote the $\mathbf{R}$-algebra of Taylor expansions of order m at y of all elements in $\mathscr{E}(\Omega_p)$; if $m < \infty$, $\mathscr{F}_y^m$ is isomorphic to the algebra $\mathscr{F}_p/\mathfrak{m}_p^{m+1}$, where $\mathfrak{m}_p$ denotes the maximal ideal of the formal power series ring $\mathscr{F}_p = \mathbf{R}[[y_1, \ldots, y_p]]$; if $m = +\infty$, $\mathscr{F}_y^m$ (simply written $\mathscr{F}_y$) (¹) is isomorphic to $\mathscr{F}_p$ (by the generalized Borel theorem).

Let $T_y^m : \mathscr{E}(\Omega_p)^q \to (\mathscr{F}_y^m)^q$ denote the projection associating to each function G its Taylor expansion of order m at y. If Y is a compact set in Ω_p, we write $|G|_m^Y = \sup_{\substack{y \in Y \\ |k| \leq m}} |D^k G(y)|$. We provide $\mathscr{E}(\Omega_p)^q$ with its usual structure of a Fréchet space, defined by the family of all semi-norms $G \mapsto |G|_m^Y$, where Y ranges over the set of compacts in Ω_p and $m \in \mathbf{N}$.

Let M be a submodule of $\mathscr{E}(\Omega_p)^q$ and let us write

$$\hat{M} = \{G \in \mathscr{E}(\Omega_p)^q \mid \forall y \in \Omega_p, \exists G' \in M \text{ so that } G - G' \text{ is flat at } y\} = \bigcap_{y \in \Omega_p} (T_y)^{-1}(T_y M).$$

According to a standard result of Whitney (B. Malgrange [1]), $\hat{M}$ is the closure $\overline{M}$ of M in $\mathscr{E}(\Omega_p)^q$: we propose to extend this theorem.

Let Φ denote a C^∞ function from an open set Ω_n in $\mathbf{R}^n$ to Ω_p. The mapping Φ defines a homomorphism of $\mathbf{R}$-algebras $\Phi^* : \mathscr{E}(\Omega_p) \ni g \mapsto g \circ \varphi \in \mathscr{E}(\Omega_n)$. Let Ψ be a Φ^*-homomorphism from $\mathscr{E}(\Omega_p)^q$ to $\mathscr{E}(\Omega_n)^r$, i.e. Ψ is a homomorphism of abelian groups and, $\forall G \in \mathscr{E}(\Omega_p)^q$ and $\forall g \in \mathscr{E}(\Omega_p)$: $\Psi(g.G) = \Phi^*(g).\Psi(G)$. For $y \in \Omega_p$ and $x \in \Phi^{-1}(y)$, the mapping Ψ induces an $\mathbf{R}$-linear mapping $\Psi_x^m : (\mathscr{F}_y^m)^q \to (\mathscr{F}_x^m)^r$, so that $T_x^m \circ \Psi = \Psi_x^m \circ T_y^m$. For $X \subset \Phi^{-1}(y)$, we note Ψ_X^m the $\mathbf{R}$-linear mapping $(\mathscr{F}_y^m)^q \ni V \mapsto (\Psi_x^m(V))_{x \in X} \in \prod_{x \in X} (\mathscr{F}_x^m)^r$. Finally, let T_X^m be the mapping $\mathscr{E}(\Omega_n)^r \ni F \mapsto (T_x^m F)_{x \in X} \in \prod_{x \in X} (\mathscr{F}_x^m)^r$.

We propose to determine the closure $\overline{\Psi(M)}$ of $\Psi(M)$ in $\mathscr{E}(\Omega_n)^r$. Therefore, let us write

$$\widehat{\Psi(M)} = \{F \in \mathscr{E}(\Omega_n)^r \mid \forall y \in \Omega_p, \exists G \in M \text{ such that } \Psi(G) - F \text{ is flat on } \Psi^{-1}(y)\}$$
$$= \bigcap_{y \in \Omega_p} (T_{\Phi^{-1}(y)})^{-1}(\Psi_{\Phi^{-1}(y)} \circ T_y M).$$

(¹) We shall omit afterwards the index m, if $m = +\infty$, and shall write: $T_y, \Psi_x, \ldots$ instead of $T_y^\infty, \Psi_x^\infty, \ldots$

We shall prove the following result:

Theorem **(1.1)**. — *Let us suppose that* Φ *verifies the following condition*:

(H) *For all compact sets* $X \subset \Omega_n$ *and* $Y \subset \Omega_p$, *there exists a constant* $\alpha \geq 0$ *such that*, $\forall y \in Y$:

$$\Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^\alpha / |\Phi(x) - y|) < \infty.$$

Then $\overline{\Psi(M)} = \widehat{\Psi(M)}$.

It is easy to find C^∞ mappings Φ which do not satisfy this condition. Nevertheless, we shall prove the following result:

Theorem **(1.2)**. — *An analytic mapping* Φ *verifies the condition* (H).

Both following paragraphs are devoted to the proofs of these theorems which are independent of each other. In the last paragraph, we give a refinement of the Theorem (1.2), when Φ is a polynomial mapping.

2. Proof of theorem 1.2.

Definition **(2.1)**. — Let $\mathfrak{I}$ be a finitely generated ideal of a subring of the ring of germs at x^0 in $\mathbf{R}^n$ of continuous functions with real values. Let $\varphi_1(x), \ldots, \varphi_s(x)$ denote real valued functions, continuous in a neighborhood of x^0 and such that their germs at x^0 generate $\mathfrak{I}$. Let $V(\mathfrak{I})$ be the set of their zeros.

We say that $\mathfrak{I}$ verifies a *Łojasiewicz inequality of order* $\alpha \geq 0$ (or simply that $\mathfrak{I}$ *verifies* $\mathscr{L}(\alpha)$) if there exist a constant $C > 0$ and a neighborhood V of x^0 such that, $\forall x \in V$, $\sum_{i=1}^{s} |\varphi_i(x)| \geq C . d(x, V(\mathfrak{I}))^\alpha$.

Let Ω_p be an open set in $\mathbf{R}^p$, Ω_n an open set in $\mathbf{R}^n$, $y = (y_1, \ldots, y_p)$ and $x = (x_1, \ldots, x_n)$ coordinate systems in Ω_p and Ω_n respectively. Let $\mathcal{O}$ be the sheaf of germs of analytic functions with real values on $\Omega_n \times \Omega_p$; $\mathscr{I}$ a sheaf of ideals, analytic and coherent on $\Omega_n \times \Omega_p$. For $(x^0, y^0) \in \Omega_n \times \Omega_p$, we denote $\mathscr{I}_{(x^0, y^0)}$ the stalk of $\mathscr{I}$ at the point (x^0, y^0). Let $\varphi_1, \ldots, \varphi_s$ be generators of the ideal $\mathscr{I}_{(x^0, y^0)}$: we denote $\mathscr{I}^n_{(x^0, y^0)}$ the ideal generated by $\varphi_1(x, y^0), \ldots, \varphi_s(x, y^0)$ in the ring $\mathcal{O}^n_{(x^0, y^0)}$ of germs at (x^0, y^0) in $\Omega_n \times \{y^0\}$ of analytic functions with real values. Permuting x and y, we define similarly the ideal $\mathscr{I}^p_{(x^0, y^0)}$ of $\mathcal{O}^p_{(x^0, y^0)}$. Finally, let $V(\mathscr{I})$ be the set of zeros of $\mathscr{I}$.

Theorem (1.2) is an easy consequence of the following one (Łojasiewicz inequality with a parameter):

Theorem **(2.2)**. — *Let* X *be a compact set in* Ω_n, Y *a compact set in* Ω_p. *There exists* $\alpha \geq 0$ *such that the ideal* $\mathscr{I}^n_{(x, y)}$ *verifies* $\mathscr{L}(\alpha)$, $\forall (x, y) \in X \times Y$.

Indeed, let us suppose this theorem is true, and let Φ be an analytic mapping. Let $\mathscr{I}$ denote the analytic and coherent sheaf generated on $\Omega_n \times \Omega_p$ by $\Phi_1(x) - y_1, \ldots, \Phi_p(x) - y_p$. Let X, Y be compact sets in Ω_n, Ω_p respectively. By (2.2) applied to $\mathscr{I}$, $\forall (x^0, y) \in X \times Y$, there exists a constant $C_{(x^0, y)} > 0$ such that for x in a neighborhood of

$x^0 : |\Phi(x)-y| \geq C_{(x^0,y)}.d(x, \Phi^{-1}(y))^\alpha$. Hence, the set X being compact, there exists a constant $C_y>0$ such that, $\forall x \in X$:

$$|\Phi(x)-y| \geq C_y.d(x, \Phi^{-1}(y))^\alpha.$$

Clearly, condition (H) follows.

Proof of (2.2). — Obviously, condition $\mathscr{L}(\alpha)$ is verified, with $\alpha=0$, for $(x,y) \notin V(\mathscr{I})$. The set $X \times Y$ being compact, it suffices to find, for $(x^0, y^0) \in V(\mathscr{I})$, an $\alpha \geq 0$ such that $\mathscr{I}^n_{(x,y)}$ verifies $\mathscr{L}(\alpha)$ for (x,y) in a neighborhood of (x^0,y^0). We shall suppose that (x^0,y^0) is the origin of $\mathbf{R}^n \times \mathbf{R}^p$. Now, it is enough to prove the following result:

(2.3) *There exists an* $\alpha \geq 0$ *such that* $\mathscr{I}^n_{(0,y)}$ *verifies* $\mathscr{L}(\alpha)$ *for* $(0,y) \in V(\mathscr{I})$ *and* $|y|$ *small enough.*

Indeed, let $\varphi_1(x,y), \ldots, \varphi_s(x,y)$ generate $\mathscr{I}$ in a neighborhood of $(0, 0)$, and let us consider the sheaf $\mathscr{I}$ generated on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^p$ by $\varphi_1(x+z,y), \ldots, \varphi_s(x+z,y)$. By (2.3) applied to the sheaf $\mathscr{J}$ (with the parameter (z,y) instead of y), there exists an $\alpha \geq 0$ such that $\mathscr{J}^n_{(0,z,y)} = \mathscr{I}^n_{(z,y)}$ verifies $\mathscr{L}(\alpha)$ for (z,y) in a neighborhood of the origin.

Proof of (2.3). — We proceed by induction on the height k of the ideal $\mathscr{I}_{(0,0)}$. There exist sheafs of ideals $\mathscr{P}^1, \ldots, \mathscr{P}^r$, analytic coherent on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^p$, such that $\mathscr{P}^1_{(0,0)}, \ldots, \mathscr{P}^r_{(0,0)}$ are prime ideals of height $\geq k$, and an integer $\beta \geq 1$, such that:

$$\mathscr{I} \supset (\mathscr{P}^1 \cap \ldots \cap \mathscr{P}^r)^\beta.$$

Clearly, if $\mathscr{P}^i_{(0,y)}$ verifies $\mathscr{L}(\alpha_i)$ for y small enough, $\mathscr{I}_{(0,y)}$ verifies $\mathscr{L}(\beta \sum_{i=1}^{r} \alpha_i)$ for y small enough. Hence, we may suppose that $\mathscr{I}_{(0,0)}$ is prime and its height equals k.

Let $\varphi(y)$ be analytic in a neighborhood of the origin of $0 \times \mathbf{R}^p$ and null in $V(\mathscr{I}) \cap (0 \times \mathbf{R}^p)$ in a neighborhood of the origin. Let $\mathscr{J}$ be the analytic coherent sheaf on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^p$, generated by $\mathscr{I}$ and φ: obviously, $\mathscr{I}^n_{(0,y)} = \mathscr{J}^n_{(0,y)}$ for y small enough. If $\varphi \notin \mathscr{I}_{(0,0)}$, we get ht $\mathscr{J}_{(0,0)} > k$ and hence the result is proved by the induction hypothesis. Therefore, we may suppose that $\varphi \in \mathscr{I}_{(0,0)}$, i.e. $\mathscr{I}_{(0,0)} \supset \mathscr{I}^p_{(0,0)}$ and $\mathscr{I}^p_{(0,0)}$ is the ideal of germs $\varphi(y)$ null in $V(\mathscr{I}) \cap (0 \times \mathbf{R}^p)$.

Lemma **(2.4)**. — *With the preceding hypothesis, let* $k-\ell$ *be the height of the prime ideal* $\mathscr{I}^p_{(0,0)}$. *After an eventual permutation on the coordinates* $x_1, \ldots, x_n$, *there exist* $\varphi_1, \ldots, \varphi_\ell \in \mathscr{I}_{(0,0)}$ *such that* $\xi_1 = \dfrac{D(\varphi_1, \ldots, \varphi_\ell)}{D(x_1, \ldots, x_\ell)} \notin \mathscr{I}_{(0,0)}$.

Proof. — We proceed by induction on the height k of $\mathscr{I}_{(0,0)}$. Let us suppose that $k>\ell$. There is a sequence $(0,y^i) \in V(\mathscr{I})$, $y^i \to 0$, such that for each i: $\mathscr{I}^n_{(0,y^i)} \neq 0$ (otherwise $\mathscr{I}_{(0,0)}$ would be generated by $\mathscr{I}^p_{(0,0)}$). After an eventual linear change of coordinates on the variables $x_1, \ldots, x_n$, we know (following the analytic preparation theorem, Malgrange [1]) that there exists, for each i, a distinguished polynomial $\Psi_i = x_1^{q_i} + a_{1,i}(x',y).x_1^{q_i-1} + \ldots + a_{q_i,i}(x',y) \in \mathscr{I}_{(0,y^i)}$ (we write $x'=(x_2, \ldots, x_n)$ and the $a_{j,i}$

are analytic functions of (x', y) in a neighborhood of $(0, y^i)$). Besides, we may suppose that $\frac{\partial \Psi_i}{\partial x_1} \notin \mathscr{I}_{(0, y^i)}$. (Indeed, there exists a smaller integer $\beta_i \geq 0$ such that $\frac{\partial^{\beta_i+1}\Psi_i}{\partial x_1^{\beta_i+1}} \notin \mathscr{I}_{(0, y^i)}$; we have only to substitute $\frac{\partial^{\beta_i}\Psi_i}{\partial x_1^{\beta_i}}$ for Ψ_i.) Hence, there exists $\varphi_1 \in \mathscr{I}_{(0,0)}$ such that $\frac{\partial \varphi_1}{\partial x_1} \notin \mathscr{I}_{(0,0)}$.

Let $\mathcal{O}'$ be the sheaf of germs of analytic functions with real values on $\mathbf{R}^{n-1} \times \mathbf{R}^p = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^p \mid x_1 = 0\}$ and let us write $\mathscr{I}' = \mathscr{I} \cap \mathcal{O}'$. There exists an integer i_0 such that $\operatorname{ht} \mathscr{I}_{(0, y^i)} = k$ for $i \geq i_0$; besides, $\mathcal{O}_{(0, y^i)} | \mathscr{I}_{(0, y^i)}$ is a finitely generated module over $\mathcal{O}'_{(0, y^i)} | \mathscr{I}'_{(0, y^i)}$ and hence their Krull dimensions are equal (by the Cohen-Seidenberg theorem, Malgrange [1]; th. (5.3), chap. III); therefore $\operatorname{ht} \mathscr{I}'_{(0, y^i)} = k - 1$ for $i \geq i_0$. Since $\mathscr{I}'_{(0,0)}$ is prime, $\operatorname{ht} \mathscr{I}'_{(0,0)} = \operatorname{ht} \mathscr{I}'_{(0, y^i)}$ for i large enough, so that: $\operatorname{ht} \mathscr{I}'_{(0,0)} = k - 1$; finally, $\mathscr{I}'_{(0,0)} \supset \mathscr{I}^p_{(0,0)}$. Applying the induction hypothesis to the sheaf $\mathscr{I}'$ (after an eventual permutation on the variables $x_2, \ldots, x_n$), we see that there exist $\varphi_2, \ldots, \varphi_\ell \in \mathscr{I}'_{(0,0)}$ such that $\frac{D(\varphi_2, \ldots, \varphi_\ell)}{D(x_2, \ldots, x_\ell)} \notin \mathscr{I}'_{(0,0)}$. Hence:

$$\frac{D(\varphi_1, \ldots, \varphi_\ell)}{D(x_1, \ldots, x_\ell)} = \frac{\partial \varphi_1}{\partial x_1} \cdot \frac{D(\varphi_2, \ldots, \varphi_\ell)}{D(x_2, \ldots, x_\ell)} \notin \mathscr{I}_{(0,0)}.$$

Since $\operatorname{ht} \mathscr{I}^p_{(0,0)} = k - \ell$ and $\mathscr{I}^p_{(0,0)}$ is prime, there exist $\varphi_{\ell+1}, \ldots, \varphi_k \in \mathscr{I}^p_{(0,0)}$ such that, after an eventual permutation on the coordinates $y_1, \ldots, y_p$:

$$\xi_2 = \frac{D(\varphi_{\ell+1}, \ldots, \varphi_k)}{D(y_1, \ldots, y_{k-\ell})} \notin \mathscr{I}^p_{(0,0)}; \qquad \text{hence} \qquad \frac{D(\varphi_1, \ldots, \varphi_k)}{D(x_1, \ldots, x_\ell, y_1, \ldots, y_{k-\ell})} = \xi_1 . \xi_2 \notin \mathscr{I}_{(0,0)}.$$

By the jacobian criterion for regular points, the localized ring $(\mathcal{O}_{(0,0)})_{\mathscr{I}_{(0,0)}}$ is regular of dimension k and its maximal ideal is generated by $\varphi_1, \ldots, \varphi_k$. Hence there exists $\xi_3 \in \mathcal{O}_{(0,0)} \setminus \mathscr{I}_{(0,0)}$ such that: $\xi_3 . \mathscr{I}_{(0,0)} \subset (\varphi_1, \ldots, \varphi_k)$.

Let ξ be analytic in a neighborhood of $(0, 0) \in \mathbf{R}^{n+p}$ and inducing the germ $\xi_1 . \xi_2 . \xi_3$ at the origin. Let $\mathscr{J}$ be the sheaf of ideals generated, on a neighborhood of the origin of $\mathbf{R}^{n+p}$, by $\mathscr{I}$ and ξ. For $(0, y) \in V(\mathscr{I})$, y small enough:

— There exists $\alpha \geq 0$ such that $\mathscr{J}^n_{(0,y)} = \mathscr{I}^n_{(0,y)} + \xi . \mathcal{O}^n_{(0,y)}$ verifies $\mathscr{L}(\alpha)$ (because $\operatorname{ht} \mathscr{J}_{(0,0)} > \operatorname{ht} \mathscr{I}_{(0,0)}$ and we apply the induction hypothesis).

— $\xi . \mathscr{I}^n_{(0,y)}$ is contained in the sub-ideal of $\mathscr{I}^n_{(0,y)}$ generated by $\varphi_1, \ldots, \varphi_\ell$.

— Finally, ξ belongs to the ideal generated in $\mathcal{O}^n_{(0,y)}$ by the jacobian $\frac{D(\varphi_1, \ldots, \varphi_\ell)}{D(x_1, \ldots, x_\ell)}$.

So Theorem (2.3) is an immediate consequence of the following lemma (Tougeron and Merrien [2], prop. 3, chap. II):

Lemma **(2.5)**. — *Let $\mathfrak{I}$ be a finitely generated ideal of the ring $\mathscr{E}_n$ of germs at the origin in $\mathbf{R}^n$ of C^∞ functions with real values. Let $\varphi_1, \ldots, \varphi_\ell \in \mathfrak{I}$ and ξ belonging to the ideal generated* *in* $\mathcal{E}_n$ *by* $\varphi_1, \ldots, \varphi_\ell$ *and all the jacobians* $\dfrac{D(\varphi_1, \ldots, \varphi_\ell)}{D(x_{i1}, \ldots, x_{i\ell})}$, *so that* $\xi.\mathfrak{I} \subset (\varphi_1, \ldots, \varphi_\ell)$. *Then if* $\mathfrak{J} = \mathfrak{I} + \xi.\mathcal{E}_n$ *verifies* $\mathscr{L}(\alpha)$, *the ideal* $\mathfrak{I}$ *verifies* $\mathscr{L}(\sup(2\alpha, \alpha+1))$.

Remark **(2.6)**. — Let $\Phi = (\Phi_1, \ldots, \Phi_p)$ be a C^∞ mapping from Ω_n to Ω_p. Let $\mathcal{E}$ be the sheaf of C^∞ functions with real values on Ω_n (or Ω_p); let $\mathscr{I}$ be the sheaf of ideals generated on Ω_n by all the jacobians $\dfrac{D(\Phi_1, \ldots, \Phi_p)}{D(x_{i1}, \ldots, x_{ip})}$: the set $V(\mathscr{I})$ of zeros of $\mathscr{I}$ is the set of singular points of the mapping Φ.

Let us consider the following condition:

(H') $\forall x \in V(\mathscr{I})$, $\mathcal{E}_x/\mathscr{I}_x$ *is* (*by* Φ) *a module of finite type over the ring* $\mathcal{E}_y$ (*we set* $y = \Phi(x)$), i.e. *by the Malgrange preparation theorem* (Malgrange [1]):

$$(\mathcal{E}_x/\mathscr{I}_x) \otimes_{\mathcal{E}_y} (\mathcal{E}_y/\mathfrak{m}_y) = \mathcal{E}_x/(\mathscr{I}_x + \mathfrak{m}_y . \mathcal{E}_x)$$

is a real vector space of finite dimension ($\mathfrak{m}_y$: *maximal ideal of* $\mathcal{E}_y$).

The condition (H') is a very strong one; nevertheless, it is a generic one, i.e. it is verified on an open dense subset of the space of C^∞ mappings from Ω_n to Ω_p, this space being provided with the Whitney topology. Besides, (H') *implies* (H).

Indeed, let X and Y be compact sets in Ω_n and Ω_p respectively. By hypothesis, there exists an $\alpha \geq 0$ such that, $\forall (x^0, y^0) \in X \times Y$, the ideal generated by $\Phi_1(x) - y_1, \ldots, \Phi_p(x) - y_p$ and all the jacobians $\dfrac{D(\Phi_1, \ldots, \Phi_p)}{D(x_{i1}, \ldots, x_{ip})}$ in $\mathcal{E}^n_{(x^0, y^0)}$ (ring of germs at (x^0, y^0) in $\mathbf{R}^n \times \{y^0\}$ of C^∞ functions with real values), verifies $\mathscr{L}(\alpha)$. By Lemma (2.5), the ideal generated by $\Phi_1(x) - y_1, \ldots, \Phi_p(x) - y_p$ in $\mathcal{E}^n_{(x^0, y^0)}$ verifies $\mathscr{L}(\alpha')$, with an α' independent of the point $(x^0, y^0) \in X \times Y$. Clearly, the condition (H) follows.

3. Proof of theorem 1.1.

With the notations of § 1, we must show that: $\overline{\Psi(M)} = \widehat{\Psi(M)}$.

(3.1) *We have*: $\overline{\Psi(M)} \subset \widehat{\Psi(M)}$.

Let $F \in \overline{\Psi(M)}$ and let $y \in \Omega_p$. A finite subset X_m of $\Phi^{-1}(y)$ will be called *m-essential* (m is a positive integer), if $\ker \Psi^m_{\Phi^{-1}(y)} = \ker \Psi^m_{X_m}$; clearly, there always exist m-essential sets X_m such that $\operatorname{card} X_m \leq \operatorname{card}(\mathscr{F}_y^m)^q$.

Let X be a finite subset of $\Phi^{-1}(y)$ containing such an X_m. By hypothesis, $T^m_X F$ is in the closure of the finite dimensional real space $\Psi^m_X(T^m_y M)$, and therefore belongs to it. So there exist G^m with $G^m_X \in T^m_y M$ such that: $T^m_X F = \Psi^m_X(G^m_X)$ and $T^m_{X_m} F = \Psi^m_{X_m}(G^m)$. Obviously, $G^m - G^m_X \in \ker \Psi^m_{X_m} = \ker \Psi^m_X$; thus: $T^m_X F = \Psi^m_X(G^m)$, and X being arbitrary: $T^m_{\Phi^{-1}(y)} F = \Psi^m_{\Phi^{-1}(y)}(G^m)$.

So, $W^m = (\Psi^m_{\Phi^{-1}(y)})^{-1}(T^m_{\Phi^{-1}(y)} F) \cap T^m_y M$ is a finite dimensional and non empty affine space. The inverse limit $W = \varprojlim W^m$ is then non empty and contained in $\varprojlim T^m_y M = T_y M$; besides, $T_{\Phi^{-1}(y)} F = \varprojlim T^m_{\Phi^{-1}(y)} F \in \Psi_{\Phi^{-1}(y)}(W)$; hence, we have (3.1).

(3.2) *We have* $\widehat{\Psi(M)} \subset \overline{\Psi(M)}$.

Let $F \in \widehat{\Psi(M)}$ and let X′ be a compact subset of Ω_n. Let X be a compact neighborhood of X′ in Ω_n and let us put $Y = \Phi(X)$ and $\Phi_0 = \Phi | \mathring{X}$. Finally let ε be a number >0 and μ be a positive integer. We have only to prove the following result:

(3.3) *There exist* $g \in \mathscr{E}(\Omega_p)$ *with* $g = 1$ *in a neighborhood of* Y, *and* $G \in M$, *such that*: $|\Phi^*(g)F - \Psi(G)|_\mu^{X'} < \varepsilon$.

This easily results from two lemmas. We first give a definition:

Definition **(3.4)**. — *A subset* K *of* Y *is* (α, m)-elementary *if the following conditions are verified*:

1) *There exists a constant* $C > 0$ *such that,* $\forall x \in X$ *and* $\forall y \in K$:

$$|\Phi(x) - y| \geq C . d(x, \Phi^{-1}(y))^\alpha.$$

2) *The dimension of the real vector space* $\Psi^m_{\Phi_0^{-1}(y)}(T^m_y M)$ *is constant, for* $y \in K$.

Lemma **(3.5)**. — *Let us suppose that* Φ *verifies the condition* (H) *and let* Z *be a compact and non empty subset of* Y. *Then, there exists a closed set* $E(Z) \subsetneq Z$ *such that each compact set in* $Z - E(Z)$ *is* (α, m)*-elementary* (*m is an arbitrary integer, but* α *is the real number associated to* X *and* Y *by the condition* (H)).

Proof. — With the notations of (1.1), the function: $Y \ni y \mapsto \Gamma(y)$ is lower semi-continuous (because, for a fixed x, the mapping $Y \ni y \mapsto d(x, \Phi^{-1}(y))$ is lower semi-continuous). So there exists an open dense set Z_0 in Z, such that this function is bounded on each compact set in Z_0.

Let $y^0 \in Z_0$: if x^0 belongs to the fiber $\Phi_0^{-1}(y^0)$, we have: $\lim\limits_{\substack{y \to y^0 \\ y \in Z_0}} d(x^0, \Phi_0^{-1}(y)) = 0$. (Indeed, by hypothesis, there exists a constant $C > 0$ such that, for each $y \in Z_0$ in a neighborhood of y^0, we have $|y^0 - y| \geq C . d(x^0, \Phi^{-1}(y))^\alpha$.

Let $X(y^0) = \{x^1(y^0), \ldots, x^s(y^0)\}$ be an m-essential subset of the fiber $\Phi_0^{-1}(y^0)$ for $y^0 \in Z_0$. We can associate to each $y \in Z_0$ a subset $X(y) = \{x^1(y), \ldots, x^s(y)\}$ of $\Phi_0^{-1}(y)$, so that $\lim\limits_{y \to y^0} x^i(y) = x^i(y^0)$ for $i = 1, \ldots, s$. Clearly, we have the following inequalities, for $|y - y^0|$ small enough:

$$\dim_{\mathbf{R}} \Psi^m_{\Phi_0^{-1}(y)}(T^m_y M) \geq \dim_{\mathbf{R}} \Psi^m_{X(y)}(T^m_y M) \geq \dim_{\mathbf{R}} \Psi^m_{X(y^0)}(T^m_{y^0} M) = \dim_{\mathbf{R}} \Psi^m_{\Phi_0^{-1}(y^0)}(T^m_{y^0} M).$$

So the function $Z_0 \ni y \mapsto \dim_{\mathbf{R}} \Psi^m_{\Phi_0^{-1}(y)}(T^m_y M)$ is lower semi-continuous, bounded with integer values. Therefore, there exists an open and non empty subset Z_1 of Z_0 in which this function is constant. Then it suffices to put $E(Z) = Z - Z_1$.

Lemma **(3.6)**. — *Let* K *be a compact and* (α, m)*-elementary subset of* Y, *and let us suppose that* $m \geq \mu\alpha$. *Then we can find* $g \in \mathscr{E}(\Omega_p)$ *with* $g = 1$ *in a neighborhood of* K, *and* $G \in M$, *such that*:

$$|\Phi^*(g)F - \Psi(G)|_\mu^{X'} < \varepsilon.$$

Proof. — The following proof takes inspiration from the proof of the spectral theorem (B. Malgrange [1], lemma (1.4), chap. II).

Let $y^0 \in K$. By hypothesis, there exists a neighborhood V_{y^0} of y^0 and $G_1, \ldots, G_k$ in M such that for $y \in V_{y^0} \cap K$, $\Psi^m_{\Phi_0^{-1}(y)}(T^m_y G_1), \ldots, \Psi^m_{\Phi_0^{-1}(y)}(T^m_y G_k)$ is a basis of the real vector space $\Psi^m_{\Phi_0^{-1}(y)}(T^m_y M)$. Hence there exist continuous functions $\lambda_1, \ldots, \lambda_k$ on $V_{y^0} \cap K$, such that:

$$T^m_{\Phi_0^{-1}(y)} F = \Psi^m_{\Phi_0^{-1}(y)}\left(\sum_{i=1}^{k} \lambda_i(y) . T^m_y G_i\right)$$

for all $y \in V_{y^0} \cap K$. Using a partition of unity, we can find $G_1, \ldots, G_\ell \in M$, continuous functions $\lambda_1, \ldots, \lambda_\ell$ on K, and a constant C, such that, for all $y \in K$:

$$T^m_{\Phi_0^{-1}(y)} F = \Psi^m_{\Phi_0^{-1}(y)}\left(\sum_{i=1}^{\ell} \lambda_i(y) . T^m_y G_i\right)$$

and

$$\sup_{\substack{1 \leq i \leq \ell \\ y \in K}} |\lambda_i(y)| \leq C.$$

Let us put $G_y = \sum_{i=1}^{\ell} \lambda_i(y) G_i$; clearly, $F - \Psi(G_y)$ is m-flat on $\overline{\Phi_0^{-1}(y)}$. Let ω be a modulus of continuity on the compact set X for F, $\Psi(G_1), \ldots, \Psi(G_\ell)$: there exists a constant $C_1 > 0$ such that $C_1 . \omega$ is a modulus of continuity on X for all functions $F - \Psi(G_y)$, $y \in K$.

Let $x \in X'$ and $a \in \overline{\Phi_0^{-1}(y)}$ such that $d(x, \Phi_0^{-1}(y)) = d(x, a)$. The function $F - \Psi(G_y)$ being m-flat at a, we have:

$$|D^k F(x) - D^k \Psi(G_y)(x)| = |(R^m_a(F - \Psi(G_y)))^k(x)| \leq C_1 . d(x, \Phi_0^{-1}(y))^{m-|k|} . \omega(d(x, \Phi_0^{-1}(y))).$$

Clearly, there exists a constant C'_1 such that $d(x, \Phi_0^{-1}(y)) \leq C'_1 . d(x, \Phi^{-1}(y))$ for all $x \in X'$ and $y \in K$. Hence, the compact K being (α, m)-elementary and $m \geq \mu\alpha$, we see that there exist a constant C_2 and a modulus of continuity ω' such that:

$$|D^k F(x) - D^k \Psi(G_y)(x)| \leq C_2 |\Phi(x) - y|^{\mu - |k|} . \omega'(|\Phi(x) - y|) \tag{3.6.1}$$

for all n-integers k such that $|k| \leq \mu$, all $x \in X'$ and all $y \in K$.

Let d be a real number > 0. The open cubes of side $2d$, centered at the points $(j_1 d, \ldots, j_p d)$ ($j_1, \ldots, j_p$ are integers) constitute an open covering $\mathfrak{I}$ of $\mathbf{R}^p$. Let g_i ($i \in \mathfrak{I}$) be a partition of unity subordinate to $\mathfrak{I}$ such that, for $|k| \leq \mu$,

$$\sum_{i \in \mathfrak{I}} |D^k g_i(y)| \leq \frac{C_3}{d^{|k|}} \quad \text{for all } y \in \mathbf{R}^p \tag{3.6.2}$$

(C_3 is a constant only depending on μ and p). Let $\mathfrak{I}'$ be the finite family of those cubes L in $\mathfrak{I}$ which meet K. For $L \in \mathfrak{I}'$, let y_L be a point in $L \cap K$. Let us put:

$$g = \sum_{L \in \mathfrak{I}'} g_L, \qquad G = \sum_{L \in \mathfrak{I}'} g_L . G_{y_L}.$$

Obviously, $g=1$ in a neighborhood of K and:

$$|\Phi^*(g)F-\Psi(G)|_\mu^{X'}\leq \sum_{L\in\mathfrak{I}'} \sup_{\substack{x\in X'\\ |k|\leq\mu}} |D^k(\Phi^*(g_L)(F-\Psi(G_{y_L})))(x)|$$

and so, by Leibniz's formula and (3.6.1), (3.6.2):

$$|\Phi^*(g)F-\Psi(G)|_\mu^{X'}\leq C_4.\omega'(d)$$

where C_4 is independent of d. Hence if we choose d sufficiently small, the lemma follows.

Proof of (3.3). — First let us decompose the compact set Y with the help of Lemma (3.5). Let α be the real number associated to X and Y by the condition (H) and let m be an integer $\geq\mu\alpha$.

Let T be a well ordered set. We construct, by transfinite induction, a mapping $T\ni\tau\mapsto Y_\tau$ with values in the set of compact subsets of Y. If 1 denotes the first element of T, we put $Y_1=Y$. Suppose the mapping is defined in the interval $[1,\tau_1[$: we put $Y_{\tau_1}=\bigcap_{\tau<\tau_1} Y_\tau$, if τ_1 has no predecessor; on the other hand, if $\tau_1=\tau+1$, we put: $Y_{\tau+1}=E(Y_\tau)$ if $Y_\tau\neq\emptyset$ and $Y_{\tau+1}=\emptyset$ if $Y_\tau=\emptyset$.

If the cardinal of T is sufficiently large, there exist some τ such that $Y_\tau=\emptyset$. Let ν_1 be the smallest element τ of T such that $Y_\tau=\emptyset$: we have $\nu_1=\nu+1$ for a $\nu\in T$ (otherwise, we should have $\bigcap_{\tau<\nu_1} Y_\tau=\emptyset$, which is absurd, because the Y_τ, $\tau<\nu_1$, are compact and non empty sets such that $Y_{\tau+1}\subset Y_\tau$ for each τ). Let us consider the following assertion:

(H_τ) *There exist* g_τ *in* $\mathscr{E}(\Omega_p)$ *with* $g_\tau=1$ *in a neighborhood* V_τ *of* Y_τ, *and* G_τ *in* M, *such that* $|\Phi^*(g_\tau)F-\Psi(G_\tau)|_\mu^{X'}<\varepsilon$.

The set of all τ such that (H_τ) is true is non empty: Indeed, by (3.6), it contains ν (because Y_ν is a compact and (α, m)-elementary set). Let τ_1 be the smallest element of this set: we have to show that $\tau_1=1$.

Indeed, suppose that $\tau_1>1$. Necessarily, $\tau_1=\tau+1$ for an element $\tau\in T$ (otherwise, we should have $Y_{\tau_1}=\bigcap_{\tau<\tau_1} Y_\tau$ and therefore $Y_\tau\subset V_{\tau_1}$, hence (H_τ), for a $\tau<\tau_1$, which is absurd).

We have $|\Phi^*(g_{\tau_1})F-\Psi(G_{\tau_1})|_\mu^{X'}\leq\varepsilon'<\varepsilon$, with $g_{\tau_1}=1$ in an open neighborhood V_{τ_1} of Y_{τ_1}. Let us put $K=Y_\tau-V_{\tau_1}$: K is a compact and (α, m)-elementary subset of Ω_p. By (3.6), applied to $\Phi^*(1-g_{\tau_1})F$ instead of F, there exist $h\in\mathscr{E}(\Omega_p)$ with $h=1$ in a neighborhood of K, and $G\in M$, such that:

$$|\Phi^*(h(1-g_{\tau_1})).F-\Psi(G)|_\mu^{X'}<\varepsilon-\varepsilon'.$$

Let us put $g_\tau=g_{\tau_1}+h-h.g_{\tau_1}$ and $G_\tau=G+G_{\tau_1}$. Clearly, $g_\tau\in\mathscr{E}(\Omega_p)$, $g_\tau=1$ in a neighborhood of Y_τ, $G_\tau\in M$ and $|\Phi^*(g_\tau).F-\Psi(G_\tau)|_\mu^{X'}<\varepsilon$. Hence condition ($H_\tau$) is fulfilled, which is absurd.

Remark **(3.7)**. — I do not know if Theorem (1.1) is always true without the hypothesis (H): unfortunately, I have no counter-example.

4. A refinement of theorem 1.2 when Φ is polynomial.

Let us recall the following definition: a set in $\mathbf{R}^n$ is *semi-algebraic* if it is a finite union of subsets X_i, each X_i being defined by a finite number of polynomial equalities or inequalities.

The image of a semi-algebraic set by a polynomial mapping $\Phi : \mathbf{R}^n \to \mathbf{R}^p$ is semi-algebraic (this is a fundamental result of Seidenberg and Tarski, cf. [3]); if X and Y are semi-algebraic sets in $\mathbf{R}^n$ and if $X \subset Y$, the closure of X in Y and $Y \setminus X$ are semi-algebraic. Finally, it is obvious that finite unions or finite intersections of semi-algebraic sets are semi-algebraic.

Let Φ be a polynomial mapping from $\Omega_n = \mathbf{R}^n$ to $\Omega_p = \mathbf{R}^p$ and let X and Y be compact and semi-algebraic sets in $\mathbf{R}^n$ and $\Phi(\mathbf{R}^n)$ respectively. The following theorem improves (1.2):

Theorem **(4.1)**. — *There exists a closed and semi-algebraic set* D(Y) *in* Y, *such that* $Y \setminus D(Y)$ *is dense in* Y, *and constants* $C > 0$, $\alpha \geq 0$, $\beta \geq 0$ *such that, for all* $x \in X$ *and* $y \in Y$:

(4.1.1)
$$|\Phi(x) - y| \geq C . d(x, \Phi^{-1}(y))^{\alpha} . d(y, D(Y))^{\beta}.$$

Proof. — By (1.2), there exists an $\alpha \geq 0$ (we suppose that α is an integer, which is always possible) such that, $\forall y \in Y$:

$$\Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^{\alpha} / |\Phi(x) - y|) < \infty.$$

Let us put

$$D(Y) = \{y \in Y \mid \Gamma \text{ is not bounded in every neighborhood of } y\}.$$

Clearly, D(Y) is closed and $Y \setminus D(Y)$ is dense in Y (because the mapping $Y \ni y \mapsto \Gamma(y)$ is lower semi-continuous). Let us verify that D(Y) is semi-algebraic.

First, the set

$$A_1 = \{(x, y, \tau) \in X \times Y \times \mathbf{R}^+ \mid |\Phi(x) - y| > \tau . d(x, \Phi^{-1}(y))^{\alpha}\}$$

is semi-algebraic. Indeed, A_1 is the image of the semi-algebraic set

$$A_0 = \{(x, x', y, \tau) \in X \times \mathbf{R}^n \times Y \times \mathbf{R}^+ \mid \Phi(x') = y \text{ and } |\Phi(x) - y| > \tau . |x - x'|^{\alpha}\}$$

by the projection: $X \times \mathbf{R}^n \times Y \times \mathbf{R}^+ \to X \times Y \times \mathbf{R}^+$. Now the set

$$A_2 = \{(y, \tau) \in Y \times \mathbf{R}^+ \mid \exists x \in X \text{ such that } |\Phi(x) - y| \leq \tau . d(x, \Phi^{-1}(y))^{\alpha}\}$$

is semi-algebraic, because it is the image of $(X \times Y \times \mathbf{R}^+) \setminus A_1$ by the projection: $X \times Y \times \mathbf{R}^+ \to Y \times \mathbf{R}^+$. Clearly, we have

$$D(Y) \times \{0\} = \bar{A}_2 \cap Y \times \{0\},$$

and therefore D(Y) is semi-algebraic.

Let us prove inequality (4.1.1) (the proof is similar to that of Lemma 1 in [4]). Let us put:

$$B_1 = \{(y, \delta, \tau) \in Y \times \mathbf{R}^+ \times \mathbf{R}^+ \mid d(y, D(Y)) \geq \delta\}$$

$$B_2 = \{(y, \delta, \tau) \in B_1 \mid \forall x \in X, \ |\Phi(x) - y| > \tau d(x, \Phi^{-1}(y))^{\alpha}\}.$$