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A CLASSIFICATION OF THE TOPOLOGICAL TYPES OF $\mathbf{R}^2$-ACTIONS ON CLOSED ORIENTABLE 3-MANIFOLDS

by G. CHATELET, H. ROSENBERG, D. WEIL

In this paper we shall classify the topological type of non singular actions of $\mathbf{R}^2$ on closed orientable 3-manifolds. If φ is a non singular action of $\mathbf{R}^2$ on V then we denote by $\mathscr{F}(\varphi)$ the foliation of V defined by the orbits of φ; φ non singular means the orbits are of dimension two, therefore $\mathscr{F}(\varphi)$ is a 2-dimensional foliation of V whose leaves are planes, cylinders and tori. V is assumed orientable, therefore $\mathscr{F}(\varphi)$ is a transversally orientable foliation. We consider two non singular actions φ and ψ to be equivalent if there is a homeomorphism $h : V \to V$ which sends leaves of $\mathscr{F}(\varphi)$ to leaves of $\mathscr{F}(\psi)$. We assume throughout this paper that the actions are at least of class C^2.

In [7], it is shown that if V admits a non singular action of $\mathbf{R}^2$ and if V is a closed orientable 3-manifold, then V is a fibre bundle over the circle $\mathbf{S}^1$ with fibre the 2-torus $\mathbf{T}^2$. Therefore V is diffeomorphic to $(\mathbf{T}^2 \times I)/F$ where F is a diffeomorphism $\mathbf{T}^2 \to \mathbf{T}^2$ induced by an element of $\mathbf{GL}(2, \mathbf{Z})$; $(\mathbf{T}^2 \times I)/F$ denotes the quotient space of $\mathbf{T}^2 \times I$ where $(x, 1)$ is identified with $(F(x), 0)$ for $x \in \mathbf{T}^2$. Since V is orientable, we have $\det F = +1$. We can now annonce the main results; naturally we assume φ is a non singular action on the closed orientable 3-manifold $V \approx (\mathbf{T}^2 \times I)/F$:

Theorem 1. — If all the orbits of φ are planes, then V *is diffeomorphic to $\mathbf{T}^3$ and $\mathscr{F}(\varphi)$ is equivalent to a linear action.*

Theorem 2. — If φ has no compact orbits and not all the orbits of φ are planes, then all the orbits of φ are cylinders, F *has eigenvalues equal to $+1$ and φ is equivalent to the suspension of a non singular action of the circle on $\mathbf{T}^2$.*

Theorem 3. — If φ has a compact orbit T, *then the manifold obtained by cutting* V *along* T *is diffeomorphic to $\mathbf{T}^2 \times I$. All the compact orbits of φ are isotopic in* V, *and if T_1 and T_2 are compact orbits of φ which bound a submanifold* W *of* V *whose interior contains no compact orbits, then $W \approx \mathbf{T}^2 \times I$ and all the orbits of φ in* W *are either planes or cylinders (but there is no mixture of the two) which spiral in a precise manner towards T_1 and T_2 (this will be made precise in the sequel).*

Theorem 1 is not new: in [4] it is shown that a closed orientable 3-manifold foliated by planes is diffeomorphic to $\mathbf{T}^3$, and in [6] it is shown that such foliations of $\mathbf{T}^3$ are equivalent to linear foliations. Part of the interest of theorem 2 is the existence of compact orbits when F has no eigenvalue equal to $+1$.

Some notation. — Let $p: \mathbf{T}^2 \times \mathrm{I} \to \mathrm{V}$ be the natural projection and $\mathrm{T}_0 = p(\mathbf{T}^2 \times \{0\})$. If $\mathrm{T} \subset \mathrm{V}$ is an embedded surface, we say T is incompressible if the inclusion $i: \mathrm{T} \subset \mathrm{V}$ induces a monomorphism $i_*: \pi_1(\mathrm{T}) \to \pi_1(\mathrm{V})$. We denote by M(T) the 3-manifold with boundary obtained by cutting V along T. Notice that $\mathrm{M}(\mathrm{T}_0)$ is diffeomorphic to $\mathbf{T}^2 \times \mathrm{I}$; when there is no fear of confusion, we shall identify these two manifolds and call the components of the boundary of $\mathrm{M}(\mathrm{T}_0)$, T_0 and T_1. We note $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ and if $p \in \mathbf{R}^2$, $[p]$ denotes the coset of p in $\mathbf{T}^2$. Let $* = p([0, 0], 0)$ be the base point in V; we write $\pi_1(\mathrm{V})$ and $\pi_1(\mathrm{T}_0)$ to mean $\pi_1(\mathrm{V}, *)$ and $\pi_1(\mathrm{T}_0, *)$ respectively. Let

$$\mu(t) = p([0, 0], t) \quad \text{for} \quad t \in \mathrm{I},$$

and define ε to be the homotopy class of μ in $\pi_1(\mathrm{V})$. Let a and b be a basis of $\pi_1(\mathrm{T}_0)$. Then $\pi_1(\mathrm{V})$ is the free group on a, b and c with the relations:

$$ab = ba$$
$$cac^{-1} = \mathrm{F}_*(a)$$
$$cbc^{-1} = \mathrm{F}_*(b).$$

1. In this section we shall study the manner in which the compact orbits of φ are embedded in V. We prove that $\mathrm{M}(\mathrm{T}) = \mathbf{T}^2 \times \mathrm{I}$ for any compact orbit T, and if F has an eigenvalue equal to -1, then there exist compact orbits and they are isotopic to T_0.

(1.1) *Let* T *be a compact orbit of* φ. *Then* T *does not separate* V *and* T *is incompressible.*

Proof. — First we remark the foliation $\mathscr{F}(\varphi)$ contains no Reeb components, i.e. invariant submanifolds homeomorphic to $\mathbf{D}^2 \times \mathbf{S}^1$ such that $\partial(\mathbf{D}^2 \times \mathbf{S}^1)$ is a leaf; this is proved in [3]. Also, it is known that if $\mathscr{F}$ is a transversally oriented foliation of a closed 3-manifold W which contains no Reeb components, then each leaf of $\mathscr{F}$ is incompressible [5]. Therefore, if T is a compact orbit of φ, T is incompressible.

Now suppose that T does separate V; let W be one of the connected components of V—T; W is a closed 3-manifold and φ acts on W so that $\partial \mathrm{W} = \mathrm{T}$ is an orbit. If there are no compact orbits of φ in Int W then the proof of theorem (5.3) of [5] shows that all the orbits of φ in Int W are $\mathbf{R}^2$. But then W is diffeomorphic to $\mathbf{D}^2 \times \mathbf{S}^1$ by theorem 1 of [5], which is impossible since an action has no Reeb components. Thus there exist compact orbits of φ in Int W. By lemma (5.3) of [7], there exist K compact orbits of φ in Int W, $\mathrm{T}_1, \ldots, \mathrm{T}_\mathrm{K}$, such that $\mathrm{A} = \bigcup_{i=1}^{\mathrm{K}} \mathrm{T}_i$ does not separate W but for every other compact orbit T′ of φ, $\mathrm{T}' \cup \mathrm{A}$ does separate W. We remark that in order to apply (5.3), one must know that not every compact orbit of φ in Int W separates W. This is indeed the case (cf. remark at end of the proof of theorem 3 of [5]). Let W_1 be the manifold obtained by cutting W along $\mathrm{T}_1, \ldots, \mathrm{T}_\mathrm{K}$; W_1 has $2\mathrm{K}+1$ tori in its boundary, each an orbit of φ, and every other compact orbit of φ in W_1 separates W_1. But it is proved in [7] (page 462) that a compact orientable 3-manifold with non empty boundary, that admits a non singular action of $\mathbf{R}^2$ such that every compact orbit in the interior separates, is necessarily $\mathbf{T}^2 \times I$. Thus $W_1 \approx \mathbf{T}^2 \times I$ which contradicts the fact that W_1 has an odd number of boundary components. Therefore no compact orbit of the action φ on V can separate V.

(1.2) *Let* T *be a torus embedded in* V *which is incompressible and does not separate* V. *Then* $M(T) \approx \mathbf{T}^2 \times I$.

Before proving (1.2), we need:

Lemma **(1.3)**. — *Let* T *be a torus embedded in* $\mathrm{Int}(\mathbf{T}^2 \times I)$ *such that* T *is incompressible and separates* $\mathbf{T}^2 \times I$ *into two components* A *and* B *such that* $\mathbf{T}^2 \times \{0\} \subset A$ *and* $\mathbf{T}^2 \times \{1\} \subset B$. *Then* $A \approx \mathbf{T}^2 \times I$ *and* $B \approx \mathbf{T}^2 \times I$ (*in fact,* T *is necessarily incompressible if the other hypotheses are satisfied*).

Proof. — Let $\mathcal{F}$ be a Reeb foliation of $\mathbf{T}^2 \times I$, i.e. a C^2-foliation such that each leaf of $\mathcal{F}$ in $\mathrm{Int}(\mathbf{T}^2 \times I)$ is $\mathbf{R}^2$ and the boundary components of $\mathbf{T}^2 \times I$ are leaves [cf. 5]. Since T is incompressible, T is isotopic to a torus $T' \subset \mathrm{Int}(\mathbf{T}^2 \times I)$ such that T′ is transverse to $\mathcal{F}$ and the foliation of T′ defined by the intersection of the leaves of $\mathcal{F}$ with T′ is an irrational flow (Theorem (1.1) of [6]). Therefore we can assume T is transverse to $\mathcal{F}$ and $\mathcal{F} \cap T$ is an irrational flow. Let T_0 be a torus embedded in int A such that $T_0 + (\mathbf{T}^2 \times \{0\})$ bound a product cobordism in A and T_0 is transverse to $\mathcal{F}$ with $\mathcal{F} \cap T_0$ an irrational flow. Such a torus T_0 is constructed in exemple 3 of [5]. Let A_0 be the manifold with boundary $T_0 + T$; clearly $A_0 \cong A$. Now each leaf of $\mathcal{F}$ in the interior of A_0 is homeomorphic to $\mathbf{R}^2$ since every closed submanifold of $\mathbf{R}^2$ diffeomorphic to $\mathbf{R}$ separates $\mathbf{R}^2$ into two components, each homeomorphic to $\mathbf{R}^2$. Now the proof of theorem (3.5) of [5] shows that $A_0 \approx \mathbf{T}^2 \times I$, hence A as well. Clearly the same reasoning applies to B.

Proof of **(1.2)**. — Let $T \subset V$ be an incompressible torus which does not separate V. Suppose that $T \subset \mathrm{Int}\, M(T_0)$. Clearly T then separates $M(T_0)$ into two connected components A and B, each of which contains one of the boundary components of $M(T_0)$. Thus A and B are both homeomorphic to $\mathbf{T}^2 \times I$ by lemma (1.3). Since M(T) is obtained by glueing one end of A to an end of B, it follows easily that $M(T) \approx \mathbf{T}^2 \times I$.

In general we proceed by putting T into general position with respect to T_0 and mimic the argument which proves that a simple closed curve C on $\mathbf{T}^2$ which is incompressible in $\mathbf{T}^2$ has the property that $M(C) \approx \mathbf{S}^1 \times I$.

To be precise, let T intersect T_0 transversally so that $T \cap T_0 = \varnothing$ or $T \cap T_0$ is a 1-manifold. We have just considered the case $T \cap T_0 = \varnothing$, therefore we may assume

$$T \cap T_0 = C_1 \cup \ldots \cup C_n,$$

where each $C_i \approx \mathbf{S}^1$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. First we modify T by an isotopy, to remove those C_i which are null homotopic. Suppose C_i is null homotopic on T_0. Then $C_i = \partial D_i$ where $D_i \subset T_0$ and $D_i \approx \mathbf{D}^2$. By choosing C_i minimal, we can suppose $\operatorname{Int} D_i$ contains no C_j, for $j = 1, \ldots, n$. Since $C_i \subset T$ and T is incompressible we know that C_i is null homotopic on T. Let $D \subset T$ satisfy $\partial D = C_i$ and $D \approx \mathbf{D}^2$. Then $S = D \cup D_i$ is a 2-sphere embedded in V which is smooth except along the corner C_i. Since V is covered by $\mathbf{R}^3$, V is irreducible (cf. [4]), therefore S bounds a ball $B \subset V$. Now by an isotopy of D to D_i across the ball B, one removes the intersection curve C_i from $T \cap T_0$; this isotopy is described in detail in [10].

Thus we can assume $T \cap T_0 = C_1 \cup \ldots \cup C_n$, where each C_i is a generator of $\pi_1(T)$ and $\pi_1(T_0)$. Two simple closed curves on a torus, which are disjoint and not null homotopic, separate the torus into two cylinders which have the curves as their common boundary. Therefore, we can label the C_i so that, for each i, C_i and C_{i+1} bound a cylinder A_i on T, whose interior contains no C_j. Choose a simple closed curve b on T which meets each C_i in exactly one point x_i. We fix an orientation of b and an orientation of the normal bundle of $T_0 \subset V$, and to each x_i we associate a + or — depending on whether the orientation of b at x_i coincides with the orientation of the normal bundle of T_0 at x_i.

Now suppose x_i and x_{i+1} have opposite signs. Then A_i can be considered as a cylinder embedded in $M(T_0) \approx \mathbf{T}^2 \times I$, which intersects $\partial(\mathbf{T}^2 \times I)$ in $C_i + C_{i+1}$, both of which are contained in $T^2 \times \{0\}$. Let B_1, B_2 be the cylinders in $\mathbf{T}^2 \times \{0\}$, satisfying $\partial B_1 = \partial B_2 = C_i + C_{i+1}$, $B_1 \cap B_2 = C_i + C_{i+1}$. One of the B_i, B_1 say, has the property that $A_i \cup B_1$ bounds a solid torus in $\mathbf{T}^2 \times I$ and is isotopic to B_1 across this solid torus, relative to $C_i + C_{i+1}$. This is proved explicitly in [10], or one can apply theorem (5.5) of [9]. Using this isotopy one removes C_i and C_{i+1} from $T \cap T_0$. Therefore we may suppose all the x_i have the same sign, and each A_i can be considered as embedded in $\mathbf{T}^2 \times I$, having one boundary in $\mathbf{T}^2 \times \{0\}$ and the other in $\mathbf{T}^2 \times \{1\}$. Here we regard $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ as the two boundary components of a tubular neighborhood of T_0 in V.

Let $a_1, \ldots, a_n$ denote the circles of intersection of T with $\mathbf{T}^2 \times \{0\}$, labelled so that $a_i \cup a_{i+1}$ bound a cylinder E_i on $\mathbf{T}^2 \times \{0\}$ whose interior is disjoint from each a_j, and $a_{n+1} = a_1$. Similarly, let $b_1, \ldots, b_n$ be the circles of $T \cap (\mathbf{T}^2 \times \{1\})$, labelled so that $a_i + b_i$ bound a cylinder A_i on T such that $\operatorname{Int} A_i \subset \mathbf{T}^2 \times (0, 1)$. Let H_i be the cylinder on $\mathbf{T}^2 \times \{1\}$ with boundary $b_i + b_{i+1}$ whose interior contains no b_j.

Now $E_i \cup H_i \cup A_i \cup A_{i+1}$ separates V into two connected components; let $M(i)$ be the component whose interior is disjoint from T_0. It is not hard to see that $M(i)$ is homeomorphic to $\mathbf{S}^1 \times I \times I$ by a map sending $\mathbf{S}^1 \times I \times \{0\}$ to E_i and $\mathbf{S}^1 \times I \times \{1\}$ to H_i. This can be proved directly (e.g. by using the theory of Reeb foliations) or one can apply [9].

Now V is the quotient space of $\mathbf{T}^2 \times I$ where $(x, 1)$ is identified with $(F(x), 0)$, for each $x \in \mathbf{T}^2$. T is embedded in V, therefore for each i there exists $\psi(i) \in I$ such that H_i is identified with $E_{\psi(i)}$ (via F).

Now suppose $n=1$, so that $\psi(1)=1$. Then $M(T)$ is the quotient space of $\mathbf{S}^1\times I\times I$ where $(\theta, t, 1)$ is identified with $(F_1(\theta, t), 0)$, for each $(\theta, t)\in\mathbf{S}^1\times I$;

$$F_1 : \mathbf{S}^1\times I \to \mathbf{S}^1\times I,$$

the diffeomorphism induced by F. Since T has a trivial normal bundle in V, $\partial M(T)$ has two connected components; therefore $F_1(\mathbf{S}^1\times 0)=\mathbf{S}^1\times 0$ and $F_1(\mathbf{S}^1\times 1)=\mathbf{S}^1\times 1$. V is orientable so F_1 is orientation preserving. Thus F_1 is homotopic to the identity map $\mathbf{S}^1\times I\to\mathbf{S}^1\times I$, therefore, F_1 is isotopic to the identity map. Hence

$$M(T)\approx\mathbf{S}^1\times I\times\mathbf{S}^1\approx\mathbf{T}^2\times I.$$

Now suppose $n>1$. Then $\psi(1)\neq 1$, since if $\psi(1)=1$, $M(T)$ would have two connected components, contradicting the hypothesis that T does not separate V. Then $M(1)\bigcup_F M(\psi(1))$ is homeomorphic to $\mathbf{S}^1\times I\times I$ since it is obtained from

$$(\mathbf{S}^1\times I\times I)+(\mathbf{S}^1\times I\times I)$$

where a point $(x, 1)$ in the first factor is identified with $(F(x), 0)$ in the second factor, for $x\in\mathbf{S}^1\times I$. We observe that the numbers $1, \psi(1), \psi^2(1), \ldots, \psi^{n-1}(1)$, are distinct and $\psi^n(1)=1$, since T does not separate V. Therefore

$$M(1)\bigcup_F M(\psi(1))\bigcup_F\ldots\bigcup_F M(\psi^{n-1}(1))$$

is homeomorphic to $\mathbf{S}^1\times I\times I$ and $M(T)$ is homeomorphic to the quotient space of $\mathbf{S}^1\times I\times I$ where a point $(x, 1)$ is identified with $(h(x), 0)$, for $x\in\mathbf{S}^1\times I$; $h:\mathbf{S}^1\times I\to\mathbf{S}^1\times I$ a diffeomorphism. Just as in the case $n=1$, we have $h(\mathbf{S}^1\times 0)=\mathbf{S}^1\times 0$ and $h(\mathbf{S}^1\times 1)=\mathbf{S}^1\times 1$ since $\partial M(T)$ has two components. Also h preserves orientation since $M(T)$ is orientable, therefore h is isotopic to the identity map and $M(T)\approx\mathbf{T}^2\times I$.

(1.4) *Let* T *be an incompressible torus in* V *which does not separate* V. *If* F *has no eigenvalue equal to* $+1$ *or* -1, *then* T *is isotopic to* T_0.

Proof. — Suppose T is not isotopic to T_0. As in the proof of (1.2), we put T into general position with respect to T_0. Clearly T is not disjoint from T_0, since we proved in (1.3) that this implies T is isotopic to T_0. As before, we remove all the circles of intersection from $T\cap T_0$ which are null homotopic, and then we remove the circles C_i and C_{i+1} which have opposite sign. Thus $T\cap(\mathbf{T}^2\times\{0\})=a_1\cup\ldots\cup a_n$ and $T\cap(\mathbf{T}^2\times\{1\})=b_1\cup\ldots\cup b_n$ where a_i and b_i bound a cylinder A_i on T whose interior is contained in $\operatorname{Int} M(T_0)$. By construction, we have $F(b_1)=a_j$ for some j, $1\leq j\leq n$.

The cylinder A_1 in $\mathbf{T}^2\times I$ is isotopic to $a_1\times I$ in $\mathbf{T}^2\times I$; one can prove this using Reeb foliation theory or [9]. Therefore, on $\mathbf{T}^2$, a_1 is isotopic to b_1 and since a_j is isotopic to a_1 we have a_1 isotopic to $F(a_1)$. Let C be a (linear) simple closed curve through the base point $(0, 0)$ of $\mathbf{T}^2$ which is isotopic to a_1. We have $F(C)$ isotopic to C. Let $f:\mathbf{T}^2\to\mathbf{T}^2$ be a diffeomorphism such that $f(F(C))=C$, $f(0, 0)=(0, 0)$, and f isotopic to the identity. Then $(f\circ F)_*=F_*$ and $(f\circ F)_*[C]=\pm[C]$ where $[C]$ denotes the homotopy class of C in $\pi_1(\mathbf{T}^2)$. Therefore F_* has an eigenvalue equal to $+1$ or -1.

(1.5) *If* F *has an eigenvalue equal to* -1 *and* T *is an incompressible torus in* V *which does not separate* V, *then* T *is isotopic to* T_0.

Proof. — Suppose, on the contrary, that T is not isotopic to T_0. As in (1.4), we put T into general position with respect to T_0 so that $T \cap T_0 = a_1 \cup \ldots \cup a_n$. Let a be the homotopy class of a_1 in $\pi_1(T_0)$ and choose $b \in \pi_1(T_0)$ so that a and b form a basis of $\pi_1(T_0)$. Let c be the third generator of $\pi_1(V)$ as defined in the introduction. We know $\pi_1(V)$ is the group generated by a, b and c with the relations:

$$ab = ba$$
$$cac^{-1} = a^{-1}$$
$$cbc^{-1} = a^{K} b^{-1}.$$

This follows from the fact that $F_*(a) = a^{\pm 1}$ and since $\det F = +1$ both eigenvalues of F must be -1; therefore $F_*(a) = a^{-1}$. Choose a basis of $\pi_1(T)$ of the form a, $b^m c^\gamma$. We know that $M(T) \approx \mathbf{T}^2 \times I$ by (1.2), so T is a fibre of a fibration of V over $\mathbf{S}^1$. Hence $\pi_1(T)$ is an invariant subgroup of $\pi_1(V)$ with quotient $\mathbf{Z}$.

First we remark that γ is even since a and $b^m c^\gamma$ commute. Next observe that $b^{2m} \in \pi_1(T)$, since $\pi_1(T)$ is invariant, for:

$$cb^m c^\gamma c^{-1} \in \pi_1(T),$$
$$cb^m c^\gamma c^{-1} = a^{mk} b^{-m} c^\gamma \quad \text{hence} \quad b^{-m} c^\gamma \in \pi_1(T),$$
$$b^{2m} = b^m c^\gamma (b^{-m} c^\gamma)^{-1}.$$

Also $c^{2\gamma} \in \pi_1(T)$:

$$(b^m c^\gamma)(b^{-m} c^\gamma) \in \pi_1(T)$$

$b^m c^\gamma b^{-m} c^\gamma = a^{Km} c^{2\gamma}$ since γ is even.

Now a, b^{2m} and $c^{2\gamma}$ belong to $\pi_1(T)$. We know that $\pi_1(V)/\pi_1(T)$ is isomorphic to $\mathbf{Z}$. The case $\gamma \neq 0$, $m \neq 0$ is therefore impossible. If $\gamma \neq 0$ and $m = 0$ then $\gamma = 1$ which is impossible (a and c do not commute).

The only remaining possibility is $\gamma = 0$ and $m = 1$, hence $\pi_1(T)$ is generated by (a, b) and T is isotopic to T_0.

(1.6) *Suppose* F *has an eigenvalue equal to* -1 *and* φ *is a non singular action of* $\mathbf{R}^2$ *on* V. *Then* φ *has a compact orbit, and all the compact orbits are isotopic to* T_0.

Proof. — Assume, on the contrary, that φ has no compact orbits. Then by theorem 9 of [8], all the orbits of φ are cylinders and each orbit is dense in V; the orbits cannot all be planes since this would imply $V \approx \mathbf{T}^3$. Let X and Y be commuting, linearly independent vector fields on V which are tangent to the orbits of φ and such that all the orbits of Y are closed, of the same period [7]. Let C be a Y-orbit and L the φ-orbit which contains C. Let A be a cylinder transverse to $\mathscr{F}(\varphi)$ which is the union of Y-orbits and such that $C \subset \operatorname{Int} A$ [cf. 7]. It is proved in [7] that $(L - C) \cap A \neq \emptyset$.

Let D be a first circle of return of $L \cap A$; i.e. $D \subset L \cap A$ and $D + C$ bound a cylinder $E \subset L$ such that $(\text{Int } E) \cap A = \emptyset$. Let B be the cylinder on A bounded by $C + D$. Then the topological torus $E \cup B$ can be smoothed in a neighborhood of A to obtain a torus T which is an orbit of a non singular $\mathbf{R}^2$ action φ_1 on V (theorem (3.1) of [7]). By (1.1) and (1.5), we know that T is isotopic to T_0. Now T is isotopic to a torus T′ such that X is transverse to T′ and Y is tangent to T′. This is a slight modification of the construction of lemma (4.3) of [7]; lemma (4.3) gives a T′ isotopic to T such that X is transverse to T′. To ensure that Y is tangent to T′, we define T′ to be the $M(\theta_0)$ of lemma (4.3), saturated by the orbits of Y, union the annulus in A(C) bounded by $(\mathbf{S}^1 \times I \times \{0\}) + (\mathbf{S}^1 \times I \times \{1\})$ (cf. (4.3) of [7]). Thus we can suppose X is transverse to T_0 and Y is tangent to T_0.

Now consider the torus T which is a smoothing of $E \cup B$, where $C \subset T_0$ is a Y-orbit and E and B are the cylinders defined above. Each orbit of φ in $M(T_0)$ is a cylinder with one boundary in $\mathbf{T}^2 \times \{0\}$ and the other in $\mathbf{T}^2 \times \{1\}$. Therefore $\pi_1(T)$ contains an element of the form $b^m c^\gamma$ where $\gamma =$ the number of circles in $E \cap T_0$, and $\gamma > 0$. Consequently $\pi_1(T) \neq \pi_1(T_0)$. But T is an orbit of a non singular $\mathbf{R}^2$ action φ_1 on V, so by (1.1) and (1.5), T is isotopic to T_0. This is a contradiction, therefore φ has at least one compact orbit.

Proof of Theorem 2. — Suppose φ is an action of $\mathbf{R}^2$ on V with all the orbits cylinders. In the proof of (1.6), we showed that φ can be approximated by an $\mathbf{R}^2$ action φ_1 such that φ_1 has a compact orbit T and T is not isotopic to T_0. By (1.4), we know that F has an eigenvalue equal to $+1$ or -1. Since the eigenvalues of F are of the same sign, we know from (1.6) that both eigenvalues of F are $+1$. Therefore, if F has no eigenvalue equal to $+1$, every $\mathbf{R}^2$ action on V has at least one compact orbit.

Now consider the action φ with all orbits cylinders. After composing φ with a diffeomorphism of V we may assume φ is transverse to T_0 and the orbits of φ in $M(T_0)$ are homeomorphic to $\mathbf{S}^1 \times I$, with one component of the boundary in T_0 and the other in T_1 (see the proof of (1.6)). Let $\mathscr{F}_0$ be the foliation of $M(T_0) \cong \mathbf{T}^2 \times I$ induced by the orbits of φ. The foliation $\mathscr{F}_0$ has no holonomy since $\mathscr{F}_0 \cap (\mathbf{T}^2 \times \{0\})$ is topologically equivalent to the foliation of $\mathbf{T}^2$ given by $\mathbf{S}^1 \times \{\theta\}$, $\theta \in \mathbf{S}^1$. Thus, by the Reeb Stability theorem, $\mathscr{F}_0$ is topologically equivalent to the foliation $\mathbf{S}^1 \times \{\theta\} \times I$, $\theta \in \mathbf{S}^1$, of $\mathbf{T}^2 \times I$. Clearly V is then homeomorphic to $(\mathbf{T}^2 \times I)/H$ where $H : \mathbf{T}^2 \to \mathbf{T}^2$ is a diffeomorphism which leaves the foliation $\mathbf{S}^1 \times \{0\}$, of $\mathbf{T}^2$ invariant. The manifold $(\mathbf{T}^2 \times I)/H$ is foliated by the cylinders $p(\mathbf{S}^1 \times \{0\} \times I)$ where $p : \mathbf{T}^2 \times I \to (\mathbf{T}^2 \times I)/H$ is the projection. Thus, the foliation of V defined by φ is topologically equivalent to this suspension.

2. The models.

In this section we shall explain theorem 3. We start with a non singular action φ of $\mathbf{R}^2$ on V which has a compact orbit T. We know that cutting V along T we obtain $\mathbf{T}^2 \times I$; therefore we shall classify the foliations of $\mathbf{T}^2 \times I$ induced by actions tangent to the boundary. We denote by $\mathscr{F}$ the foliation of $\mathbf{T}^2 \times I$ induced by φ. The classification is analogous to the classification of foliations of $\mathbf{S}^1 \times I$ which are tangent to the boundary: each compact leaf is a circle isotopic to $\mathbf{S}^1 \times \{0\}$, and the complement of the set of compact leaves is the union of a countable family of open sets W_i with $\overline{W}_i \cong \mathbf{S}^1 \times I$ and the foliation of $\overline{W}_i$ is of type 0 or 1 of figure 1.

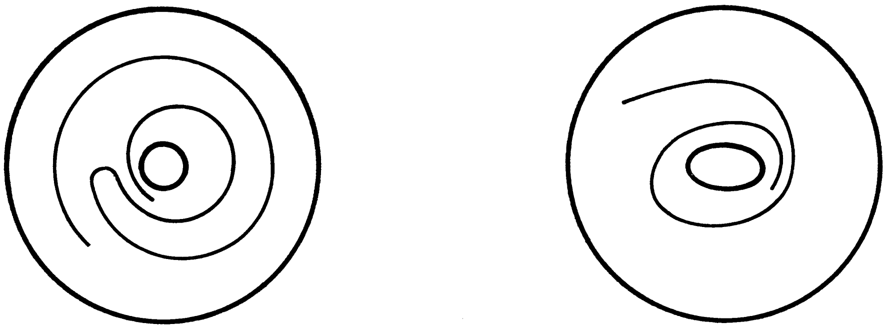

FIG. 1

(2.1) *Definition of* $\mathscr{F}(\alpha, 0)$ *and* $\mathscr{F}(C, 0)$.

Let X, Y and Z be the vector fields on $\mathbf{R}^2 \times I$;

$$X = (\cos \pi x, 0, \sin 2\pi x(1-x))$$
$$Y = (1, \alpha, 0)$$
$$Z = (0, 1, 0),$$

(the foliation of figure 1, type 0, are the orbits of X), where $0 \leqslant x \leqslant 1$ and α is irrational. These vector fields are linearly independent and pairwise commute. Moreover the fields are invariant by the translations $(x_1, x_2) \mapsto (x_1+1, x_2)$ and $(x_1, x_2) \mapsto (x_1, x_2+1)$. Therefore (X, Y) and (X, Z) induce actions of $\mathbf{R}^2$ on $\mathbf{T}^2 \times I$. It is easy to check that $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ are the compact orbits of these actions; the other orbits of the (X, Y) action are planes and the other orbits of the (X, Z) action are cylinders. We denote the corresponding foliations by $\mathscr{F}(\alpha, 0)$ and $\mathscr{F}(C, 0)$ respectively. Notice that no transversal arc joins $\mathbf{T}^2 \times \{0\}$ to $\mathbf{T}^2 \times \{1\}$ for these foliations.

(2.2) *Definition of* $\mathscr{F}(\chi)$.

Let $\mathscr{G}$ be the group of diffeomorphisms of the interval [0, 1] which leave 0 and 1 fixed. Let χ be a representation of $\pi_1(\mathbf{T}^2)$ in $\mathscr{G}$. We associate an action of $\mathbf{R}^2$ to χ as follows. Let $f, g \in \mathscr{G}$ be the images of the standard basis of $\mathbf{T}^2$ by χ. Then $\mathbf{T}^2 \times I$ is diffeomorphic to the quotient of $I \times I \times I$ where $(x, 0, \lambda) \sim (x, 1, g(\lambda))$ and $(0, y, \lambda) \sim (1, y, f(\lambda))$. Since f and g commute, the vector fields (1, 0, 0) and (0, 1, 0) on I^3 project to commuting vector fields X and Y on $\mathbf{T}^2 \times I$. We denote the foliation induced by this $\mathbf{R}^2$-action on $\mathbf{T}^2\times I$ by $\mathscr{F}(\chi)$. The holonomy of this foliation on $\mathbf{T}^2\times\{0\}$ is precisely χ. $\mathscr{F}(\chi)$ is transverse to the segments $\{\Theta\}\times\{\Theta'\}\times I$ and can have compact leaves in $\operatorname{int}\mathbf{T}^2\times I$. One can consider $\mathscr{F}(\chi)$ is the foliation canonically associated to the fibration $(\mathbf{T}^2\times I, I, \mathbf{T}^2, \mathscr{G})$, I the fibre, $\mathbf{T}^2$ the base and $\mathscr{G}$ with the discrete topology [6]. Two such foliations $\mathscr{F}(\chi_1)$ and $\mathscr{F}(\chi_2)$ are equivalent if and only if χ_1 is conjugate to χ_2.

(2.3) *Definition of* $\mathscr{F}((1, i_1), (2, i_2), \ldots, (n, i_n))$.

This is a foliation of $\mathbf{T}^2\times I$ obtained by gluing together the preceding models (for each K, $1\leqslant K\leqslant n$, we have $i_K=0$ or 1). For $i_K=1$, and $\chi_K:\pi_1(\mathbf{T}^2)\to\mathscr{G}$ a homomorphism, we define $\mathscr{F}(K, i_K)=\mathscr{F}(\chi_K)$, the foliation defined in (2.2). For $i_K=0$, we define $\mathscr{F}(K, i_K)$ to be $\mathscr{F}(\alpha, 0)$ or $\mathscr{F}(C, 0)$, the foliations defined in (2.1). Then $\mathscr{F}((1, i_1), \ldots, (n, i_n))$ is the foliation of $\mathbf{T}^2\times I$ obtained by gluing the leaf $\mathbf{T}^2\times\{1\}$ of $\mathscr{F}(K, i_K)$ to the leaf $\mathbf{T}^2\times\{0\}$ of $\mathscr{F}(K+1, i_{K+1})$, for each K, $1\leqslant K\leqslant n-1$. Notice that for $i_K=0$, no transversal of the foliation $\mathscr{F}(K, i_K)$ goes from $\mathbf{T}^2\times\{0\}$ to $\mathbf{T}^2\times\{1\}$; whereas, for $i_K=1$, the segments $\{(\Theta, \Theta')\}\times I$ are transversal to $\mathscr{F}(K, i_K)$.

Theorem 3. — Let φ *be a non singular action of* $\mathbf{R}^2$ *on* $\mathbf{T}^2\times I$, *with* $\mathbf{T}^2\times\{0\}$ *and* $\mathbf{T}^2\times\{1\}$ *orbits of* φ. *Then* $\mathscr{F}(\varphi)$ *is equivalent to* $\mathscr{F}((1, i_1), \ldots, (n, i_n))$, *for some choice of* (K, i_K), $1\leqslant K\leqslant n$.

The proof will be proceeded by several lemmas.

(2.4) (*Nancy Kopell* [2]). *Let f and g be germs of commuting* C^2-*diffeomorphisms of* $\mathbf{R}^+=\{x\geqslant 0\}$, *such that* $f(0)=g(0)=0$. *If f is a contraction* (i.e. $f(x)<x$ *for* $x>0$), *and* $g\neq\mathrm{id}$ *then* 0 *is the only fixed point of g.*

(2.5) *Let* φ *be a non singular action of* $\mathbf{R}^2$ *on* $\mathbf{T}^2\times I$ *such that* $\mathbf{T}^2\times\{0\}$ *and* $\mathbf{T}^2\times\{1\}$ *are the only compact orbits. There exist embedded tori* T′ *and* T″ *satisfying*:

a) T′ *and* T″ *can be chosen transverse to* $\mathscr{F}(\varphi)$.

b) T′ *is isotopic to* $\mathbf{T}^2\times\{0\}$ *and can be chosen inside any tubular neighborhood of* $\mathbf{T}^2\times\{0\}$; *in particular, one can suppose the segments* $\{(\Theta, \Theta')\}\times I$ *are transverse to* $\mathscr{F}(\varphi)$ *inside the region* $\mathscr{U}'$ *bounded by* $\mathbf{T}^2\times\{0\}$ *and* T′. *The same property holds for* T″, $\mathbf{T}^2\times\{1\}$ *and* $\mathscr{U}''$.

c) *If* L *is an orbit of* φ, *then* $L\cap T'$(resp. $L\cap T''$) *is a circle if* $L\cong\mathbf{S}^1\times\mathbf{R}$ *and is the union of copies of* $\mathbf{R}$ *if* $L\cong\mathbf{R}^2$.

d) *There exists a vector field* Y *on* $\mathbf{T}^2\times(0, 1)$, *tangent to the (open)* φ *orbits, such that* $Y(T', (-\infty, 0))\subset\mathscr{U}'$, $Y(T'', (0, \infty))\subset\mathscr{U}''$, *and* $Y(T', 1)=T''$ (*hence the foliations of* T′ *and* T″, *induced by* $\mathscr{F}(\varphi)$, *are conjugate by the orbits of* Y). *By* $Y(x, t)$ *we mean the integral curve of the vector field* Y *at time t, which passes by x at* $t=0$.

Proof of **(2.5)**. — If φ has a cylindrical orbit then (2.5) follows from (4.3), (4.5) and (4.6) of [7]. If all open φ orbits are planes, then (2.5) follows from the classification of Reeb foliations of $\mathbf{T}^2\times\mathrm{I}$ given in [1].

Corollary **(2.6)**. — *If φ is an action of $\mathbf{R}^2$ on $\mathbf{T}^2\times\mathrm{I}$ such that $\mathbf{T}^2\times\{0\}$ and $\mathbf{T}^2\times\{1\}$ are the only compact leaves, then the open leaves are planes or cylinders but there is no mixture of the two types.*

Proof. — This follows from (2.4) and (2.5) where (2.4) is applied to the germs obtained by the representation $\pi_1(\mathbf{T}^2\times\{0\})\to g$, given by the holonomy of the foliation $\mathscr{F}(\varphi)$. Since there are no compact leaves in a neighborhood of $\mathbf{T}^2\times\{0\}$ (other than $\mathbf{T}^2\times\{0\}$), the generators of $\pi_1(\mathbf{T}^2\times\{0\})$ can be chosen so that the associated germs are contractions or the identity and a contraction.

Proof of theorem 3. — Now consider the foliation $\mathscr{F}=\mathscr{F}(\varphi)$ of $\mathbf{T}^2\times\mathrm{I}$, tangent to the boundary. We know each compact orbit of $\mathscr{F}$ is isotopic to $\mathbf{T}^2\times\{0\}$. Let K be the union of the set of compact orbits. We have $\overline{(\mathbf{T}^2\times\mathrm{I})-\mathrm{K}}=\bigcup_{i=1}^{\infty}\mathrm{W}_i$ where each $\mathrm{W}_i\cong\mathbf{T}^2\times\mathrm{I}$, W_i is invariant by φ and the open leaves of W_i are all planes or cylinders. We fix once and for all an orientation of $\mathscr{F}$. Let $\mathrm{W}_1^0, \ldots, \mathrm{W}_r^0$ denote those W_i such that the orientations induced on the boundary of W_i are opposite, i.e. if on one component of $\partial\mathrm{W}_i$, the normal field points to the interior of W_i (respectively the exterior) the normal field points to the interior (the exterior) on the other component. By continuity, there are at most a finite number of such W_i. Let $\mathrm{C}_1, \ldots, \mathrm{C}_s$ be the connected components of the closure of the complement of $\mathrm{W}_1^0\cup\ldots\cup\mathrm{W}_r^0$ in $\mathbf{T}^2\times\mathrm{I}$. Let $p_{\mathrm{K}}^{i_{\mathrm{K}}}$ be a family of embeddings of $\mathbf{T}^2\times\mathrm{I}$ into $\mathbf{T}^2\times\mathrm{I}$, $1\leqslant\mathrm{K}\leqslant n$ satisfying:

1) if $i_{\mathrm{K}}=0$, $p_{\mathrm{K}}^{i_{\mathrm{K}}}(\mathbf{T}^2\times\mathrm{I})$ is some W_j^0, for $1\leqslant j\leqslant r$;
2) if $i_{\mathrm{K}}=1$, $p_{\mathrm{K}}^{i_{\mathrm{K}}}(\mathbf{T}^2\times\mathrm{I})$ is some C_j, for $1\leqslant j\leqslant s$, and
3) $p_{\mathrm{K}}^{i_1}(\mathbf{T}^2\times\{0\})=\mathbf{T}^2\times\{0\}$,
$p_{\mathrm{K}}^{i_{\mathrm{K}}}(\mathbf{T}^2\times\{1\})=p_{\mathrm{K}+1}^{i_{\mathrm{K}+1}}(\mathbf{T}^2\times\{0\})$ for $1\leqslant\mathrm{K}\leqslant n-1$;
$p_n^{i_n}(\mathbf{T}^2\times\{1\})=\mathbf{T}^2\times\{1\}$.

We have sketched a cross section of this indexation in figure

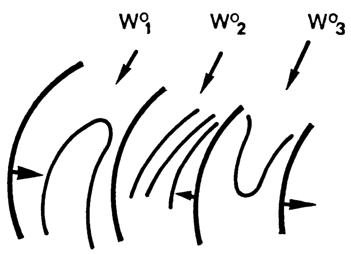

FIG. 2

We shall first construct the conjugation on the C_j and then on the W_k^0; the C_j are conjugate to the models of type $\mathscr{F}(\chi)$ for some representation χ; and the W_K^0 to the models of type $\mathscr{F}(\alpha, 0)$ or $\mathscr{F}(C, 0)$.

(2.7) *Let C_j be one of the manifolds defined above and denote by* N *the normal vector field to $\mathscr{F}$. Let* K *be the integer such that $p_K^1(\mathbf{T}^2 \times I) = C_j$. There exists a vector field X_j on C_j which is transverse to $\mathscr{F}$ satisfying:*

1) $X_j = N$ *on the compact orbits of* C_j, *and*
2) *each orbit of* X_j *starting at a point of* $p_K^1(\mathbf{T}^2 \times \{0\})$ *goes to a point of* $p_K^1(\mathbf{T}^2 \times \{1\})$.

Proof of **(2.7)**. — We may suppose N points into C_j on $p_K^1(\mathbf{T}^2 \times \{0\})$. As before, we write the complement of the compact leaves in C_j as $\bigcup_{n=1}^{\infty} W_{j,n}$ where the $W_{j,n}$ are diffeomorphic to $\mathbf{T}^2 \times I$, invariant by φ, and φ has no compact orbits in the interior of $W_{j,n}$.

We construct a vector field $X_{j,n}$ in each $W_{j,n}$ which is equal to N in a neighborhood of $\partial W_{j,n}$ as follows. Let T′ and T″ be transverse tori embedded in Int $W_{j,n}$ given by (2.5), and denote by Y the vector field given by (2.5). The foliations of T′ and T″ induced by $\mathscr{F}$ are conjugate by the orbits of Y and this foliation is equivalent to an irrational flow on $\mathbf{T}^2$ or the product foliation $\mathbf{S}^1 \times \{\Theta\}$ of $\mathbf{T}^2$. Now T′ and T″ bound a submanifold W of $W_{j,n}$ such that the foliation of W induced by $\mathscr{F}$ is equivalent to the product of the induced foliation on T′ by I; the orbits of Y define the conjugation. Thus in W we can construct a vector field X_0, transverse to $\mathscr{F}$ such that X_0 points into W on T′ and each orbit of X_0 starting at a point of T′ goes to a point of T″. Since each orbit of N starting at $\partial W_{j,n}$ intersects T′ or T″, we can extend X_0 to $W_{j,n}$ to coincide with N in a neighborhood of $\partial W_{j,n}$ and to be transverse to $\mathscr{F}$. Denote this extension by $X_{j,n}$. Now we define X_j on C_j to equal $X_{j,n}$ on $W_{j,n}$ and N on the compact orbits of $\mathscr{F}$. Each orbit of X_j starting at a point of $p_K^1(\mathbf{T}^2 \times \{0\})$ goes to a point of $p_K^1(\mathbf{T}^2 \times \{1\})$; after reparametrizing the orbits of X_j we can assume the orbits take a time 1 to go from one boundary component of C_j to the other. This completes the proof of (2.7).

(2.8) *The foliation $\mathscr{F}$ on C_j is equivalent to a foliation $\mathscr{F}(\chi)$ of* $\mathbf{T}^2 \times I$, *for some representation* χ.

Proof. — By identifying the orbits of X_j to a point we define a fibration $C_j \to \mathbf{T}^2$ with fibre I and $\mathscr{F}$ is transverse to the fibres. Such foliations are determined by a representation $\chi : \pi_1(\mathbf{T}^2) \to \mathscr{G}$. The conjugation $H_j : C_j \to (\mathbf{T}^2 \times I, \mathscr{F}(\chi))$ can be constructed so that $H_j \circ p_K^1 = \text{identity}$ on $\partial(\mathbf{T}^2 \times I)$ (see [1]).

(2.9) *The foliation $\mathscr{F}$ on W_K^0, for* K *between* 1 *and r, is equivalent to a foliation $\mathscr{F}(\alpha, 0)$ or $\mathscr{F}(C, 0)$.*

Proof. — If all the leaves of $\mathscr{F}$ in the interior of W_K^0 are planes, then we have proved in [1] that $\mathscr{F}$ is equivalent to a foliation $\mathscr{F}(\alpha, 0)$ for some irrational α. We construct in [1] a conjugation $H_K^0 : (W_K^0, \mathscr{F}) \to (\mathbf{T}^2 \times I, \mathscr{F}(\alpha, 0))$ such that $H_K^0 p_K^0 =$ identity on $\partial(\mathbf{T}^2 \times I)$.

Now suppose the leaves of $\mathscr{F}$ in Int W_K^0 are cylinders. This case is much easier to deal with than the planar case because of the existence of the vector field Y given by (2.5). Let T′ and T″ be the transverse tori given by (2.5). Between T′ and T″ in W_K^0 we have a manifold W and the foliation $\mathscr{F}$ on W is equivalent to the foliation $\mathbf{S}^1 \times \{\Theta\} \times I$ of $\mathbf{T}^2 \times I$; the equivalence is defined using the orbits of Y. Let A and B be the closure of the connected components of $W_K^0 - W$. The conjugation H_K^0 is defined in $A \cup B$ by the holonomy of the compact leaves, i.e., the boundary components of W_K^0. We do this precisely in [1]; H_K^0 is defined so that $H_K^0 p_K^0 =$ identity on $\partial(\mathbf{T}^2 \times I)$. Now this gives H_K^0 on $A \cup W$ and B. The construction above might give two different values for H_K^0 on T″ (for, on $A \cup W_K^0$ its value is determined as soon as it is determined on $p_K^0(\mathbf{T}^2 \times (0))$ and, on B, it is determined by its value on $p_K^0(\mathbf{T}^2 \times (1))$.

Let H′ and H″ be the restrictions of H_K^0 on T″ resulting from the two different definitions. Then $H = H'^{-1} H''$ is homotopic and hence isotopie to the identity and sends the leaves of the induced foliation $\mathscr{F} \cap T''$ onto themselves. Let then F be the diffeomorphism from T′ onto T″ associated with the orbits of Y. It is clear that Y may be modified into a field Y′ (tangent to the leaves) in such a way that $F' = HF$ (F′ obviously means the diffeomorphism associated with the orbits of Y′). Extension of H_K^0 using the orbits of Y′ gives them the same value for the definitions of H_K^0 on $A \cup W$ and B.

Now piecing together the conjugations H_j of (2.8) and H_K^0 of (2.9), theorem 3 is proved.

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