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## A REMARK ON DENJOY'S INEQUALITY AND HERMAN'S THEOREM

by LENNART CARLESON

**1.** In the preceding proof [1] by M. Herman of the Arnold conjecture, the Hurewicz (or the Chacon-Ornstein) ergodic theorem plays an important role and the proof is in this way non-constructive. The purpose of this note is to give a constructive argument which gives a remainder estimate in the basic Denjoy inequality. This argument also makes it possible to avoid the reduction to the case when

$$\int_0^1 |Df(x)| dx = V$$

is small, which was used by Herman.

Let us first recall the situation and some basic results from Herman's paper:  $f(x)$  is an increasing continuous function on  $-\infty < x < \infty$  such that  $f(x+1) - f(x) = 1$ ,  $0 < f(0) < 1$  and  $f^i(x)$  are the iterates. Sometimes  $f(x)$  will be considered on the torus  $\mathbf{T}$  (modulo 1) and this will be clear from the context;  $\alpha$  is the rotation number, *i.e.*:

$$|f^n(x) - x - n\alpha| < 1;$$

$\alpha$  is assumed irrational with continued fraction expansion  $[a_1, a_2, \dots]$  and  $p_i/q_i$  are the convergents. There is a homomorphism  $h$  of  $(0, 1) \pmod{1}$ ,  $t = h(x)$ , so that

$$h^{-1} \circ f \circ h(t) = t + \alpha.$$

Herman's theorem asserts that if  $f(x)$  is smooth, then, for almost all  $\alpha$ , it follows that  $h(t)$  is also smooth.

There is a unique probability measure  $\mu$  on  $(0, 1)$  which is invariant under  $f(x) \pmod{1}$  and:

$$\left| \sum_{i=0}^{q-1} \varphi(f^i(x_0)) - q \int_0^1 \varphi(f(x)) d\mu(x) \right| \leq \text{Var}(\varphi)$$

for all denominators  $q = q_j$  in the convergents of  $\alpha$ . This is Denjoy's inequality.

We shall prove the following theorem—without use of Herman's result but using his ideas:

*For almost all  $\alpha$  there are constants  $C$  and  $\beta$  so that*

$$\left| \sum_{i=0}^{q_j-1} \varphi(f^i(x_0)) - q_j \int_0^1 \varphi(x) d\mu(x) \right| < Cj^{-\beta}$$

for all  $\varphi$  on  $\mathbf{T}$  with  $|\varphi''(x)| \leq 1$ . The same result holds if  $\varphi'(x)$  only satisfies some Hölder condition.

Once this is proved it follows that

$$(1.1) \quad |\log Df^{q_j}| < Cj^{-\beta}$$

and for almost all  $\alpha$ :

$$|\log Df^n| \leq C \sum_{j=1}^{c \log n} \frac{a_j}{j^\beta} = O((\log n)^{1-\beta'})$$

with  $\beta' < \beta$ . This is the crucial estimate needed for Herman's argument. (1.1) also implies the estimate  $|f^q(x) - x - p| < q^{\delta-1}$ ,  $\delta > 0$ . See [1], Chapter VIII.

We shall use the letters  $C, c$  to denote different constants whose values are immaterial in the context.

2. Let  $q$  be one of the  $q_j$  and  $x_0$  a fixed point. We define the measure on  $(0, 1)$ :

$$\nu_q = k \sum_{i=0}^{q-1} Df^i(x_0) \delta_{f^i(x_0)}$$

where  $k$  is chosen so that  $\nu_q(0, 1) = 1$ . Let  $I$  be an interval so that  $f(I)$  does not contain  $x_0$  or  $f^q(x_0)$ . Then it is easy to see that

$$\nu_q(f(I)) = f'(\xi) \nu_q(I) \quad \text{for some } \xi \in I.$$

If  $x_0$  or  $f^q(x_0) \in f(I)$ , the situation is a little more complicated.

We first observe that:

$$(2.1) \quad |f^q(x_0) - x_0| < q^{-\lambda}, \quad \lambda > 0 \pmod{1}$$

(see [1], VIII, (2.1)) for almost all  $\alpha$  since  $\sqrt[q]{q_j} \rightarrow \text{const}$ , almost everywhere. Assume that the length  $|I|$  of  $I$  is greater than  $q^{-\lambda/2}$ , and that e.g.  $x_0 \in f(I)$ . Suppose also that e.g.  $I$  extends by  $\frac{1}{2}|I|$  to the right of  $f^{-1}(x_0)$ . For every second  $q_i$ ,  $f^{q_i}(x_0) > x_0$  and by the inequality (2.1):

$$f^{q_i}(x_0) \in f(I) \quad \text{if } q_i > \sqrt{q}.$$

This is true for at least  $c \log q$  different  $i$ 's, so that

$$\sum_{f^i(x_0) \in f(I)} Df^i(x_0) \geq c \log q$$

since  $Df^{q_i}(x_0) \geq c > 0$ , as was observed by Denjoy.

Now:

$$\begin{aligned} \nu_q(f(I)) &= k \sum_{\substack{f^i(x_0) \in f(I) \\ i=0, \dots, q-1}} Df^i(x_0) = kf'(\xi) \sum_{\substack{f^i(x_0) \in I \\ i=-1, 0, \dots, q-2}} Df^i(x_0) \\ &= f'(\xi) \nu_q(I) + O(kf'(\xi)) \\ &= f'(\xi) \left( 1 + O\left(\frac{1}{\log q}\right) \right) \nu_q(I) \end{aligned}$$

since  $v_q(I) > ck \log q$ . We obtain the following lemma:

*Lemma 1.* — Let  $I$  be an interval of length  $> q^{-\lambda/2}$ . Then:

$$v_q(f(I)) = f'(\xi)v_q(I), \quad \xi \in I, \quad \text{if } x_0, f^q(x_0) \notin f(I),$$

and 
$$v_q(f(I)) = f'(\xi) \left( 1 + O\left(\frac{1}{\log q}\right) \right) v_q(I), \quad \xi \in I,$$

in all cases, provided  $\alpha$  is not in an exceptional set of measure zero.

**3.** Next we need some information on the mapping  $x = h(t)$ . Let  $\omega$  be an interval on the  $t$ -axis and assume that

$$(3.1) \quad \frac{1}{q_{i-1}} > |\omega| \geq \frac{1}{q_i}, \quad a = \frac{q_{i+2}}{q_{i-1}}.$$

We bisect  $\omega$  into two equal intervals  $\omega_1$  and  $\omega_2$  and we want to estimate  $|h(\omega_1)|$  compared to  $|h(\omega)|$ . From (3.1) follows that

$$\bigcup_{v=1}^{2aq_i} f^v(h(\omega)) \supset (0, 1)$$

and every point is covered at most  $4a^2$  times. A similar statement is true for  $\omega_1$ .

Namely, if  $\alpha = \frac{p_{i+1}}{q_{i+1}} + \frac{\delta}{q_{i+1}^2}$ , then  $\frac{q_{i+1}}{q_{i+2}} < |\delta| < 1$ , so that  $2aq_i$  iterations of an interval of length  $q_{i+1}^{-1}$  gives a complete covering. Furthermore:

$$|f^v(h(\omega))| = \left( \prod_{\mu=0}^{v-1} f'(\xi_\mu) \right) |h(\omega)|, \quad \xi_\mu \in f^\mu(h(\omega)),$$

and similarly for  $\omega_1$ . Hence:

$$\begin{aligned} \frac{|f^v(h(\omega))|}{|f^v(h(\omega_1))|} &= \frac{|h(\omega)|}{|h(\omega_1)|} \cdot \prod_{\mu=0}^{v-1} \frac{f'(\xi_\mu)}{f'(\xi'_\mu)} \\ &\geq \frac{|h(\omega)|}{|h(\omega_1)|} e^{-ca^2} \end{aligned}$$

and 
$$\begin{aligned} Ca^2 \geq \sum_{v=1}^{4aq_i} |f^v(h(\omega))| &\geq \frac{|h(\omega)|}{|h(\omega_1)|} \sum_{v=1}^{4aq_i} |f^v(h(\omega_1))| e^{-ca^2} \\ &\geq e^{-ca^2} \frac{|h(\omega)|}{|h(\omega_1)|}. \end{aligned}$$

This gives the following lemma:

*Lemma 2.* — Let  $\frac{1}{q_i} \leq |\omega| < \frac{1}{q_{i-1}}$  and bisect  $\omega$  into  $\omega_1$  and  $\omega_2$ . Then, for almost all  $\alpha$ :

$$|h(\omega_1)| \geq \exp\left(\left(-c \frac{q_{i+2}}{q_{i-1}}\right)^2\right) |h(\omega)|$$

and (see (2.1))

$$|h(\omega)| \leq |\omega|^\lambda C \frac{q_i + 2}{q_i - 1}.$$

4. We shall now describe the exceptional set of  $\alpha$ .

Let  $\delta$  be a small positive number and  $n$  a large integer. Denote by  $B_\ell$  the interval:

$$B_\ell(n) : \delta \cdot \ell n \leq k < \delta(\ell + 1)n$$

$$\delta^{-1} \frac{1}{2} \frac{2^n}{n} \leq \ell < \delta^{-1} \frac{3}{4} \frac{2^n}{n}.$$

The intervals  $B_\ell(n)$ ,  $n = 1, 2, \dots$  and  $\ell$  as above, are disjoint.

For every  $B_\ell$ , define the number  $b_{\ell,n}(\alpha)$ :

$$b_{\ell,n}(\alpha) = \text{Max}_{k \in B_\ell} \frac{q_{k+1}}{q_k}.$$

For fixed  $(\ell, n)$ ,  $b_{\ell,n}(\alpha) \leq C$  on a set of measure  $\geq 2^{-n\delta}$ . For fixed  $n$ ,  $b_{\ell,n}(\alpha) \leq C$  for  $\geq 2^{3/4n}$  values of  $\ell$ , if we exclude a set  $E_n$  of measure  $\leq 2^{-n}$  and if  $\delta < 1/4$ . We now do this for all  $n$  and consider those  $\alpha$  which do not belong to infinitely many  $E_n$ . We also exclude those sets of measure zero mentioned earlier.

5. We shall now prove that  $\nu_q$  converges weakly to Lebesgue measure and shall also obtain an estimate of the error. We first prove that for some suitable  $\gamma > 0$  and  $C < \infty$ :

$$(5.1) \quad C^{-1} \leq \frac{\nu_q(h(\omega))}{|h(\omega)|} \leq C, \quad \text{if } |\omega| > q^{-\gamma}.$$

Take some  $q_i$  so that  $\sqrt{q} < q_i < q$  and so that  $\frac{q_{i+1}}{q_i} < C$ . This is possible for almost all  $\alpha$ . Then:

$$\alpha = \frac{p_i}{q_i} + \frac{\delta_i}{q_i^2}, \quad 1 > \delta_i > c > 0 \quad (\text{or } < -c).$$

It follows that if  $\frac{1}{q_i} < |\omega| < \frac{2}{q_i}$ , then  $\bigcup_{\nu=1}^{eq_i} f^\nu(h(\omega)) \supset (0, 1)$  and every point is covered a bounded number of times. Since both  $\nu_q(I)$  and  $|I|$  are transformed by the rules in lemma 1 it follows that:

$$\frac{\nu_q(h(\omega))}{|h(\omega)|} \frac{1}{C} \leq \frac{\nu_q(f^i(h(\omega)))}{|f^i(h(\omega))|} \leq C \frac{\nu_q(h(\omega))}{|h(\omega)|}$$

and since both measures are additive, (5.1) follows.

We now wish to prove (5.1) with a constant  $C$  very close to 1. Let us define  $M_k$  by

$$\sup_{|\omega| \geq q_k^{-1}} \frac{\nu_q(h(\omega))}{|h(\omega)|} = M_k.$$

Suppose that  $q = q_s$  and choose  $n$  so that

$$2^{n-1} < s \leq 2^n.$$

The number of blocks  $B_{\ell,n}$  so that  $M_k$  increases in  $B_{\ell,n}$  by more than a factor  $(1 + 2^{-n/2})$  is less than  $C \cdot 2^{n/2}$ . Hence there exists  $\ell$  so that (with  $k = \delta \ell n + 2$ )

- (i)  $b_{\ell,n}(\alpha) \leq C,$
- (ii)  $M_{k+\delta n} \leq (1 + 2^{-n/2}) M_k.$

Now pick an interval  $\omega$  of length between  $q_k^{-1}$  and  $2q_k^{-1}$ , for which

$$(5.2) \quad v_q(h(\omega)) = M_k |h(\omega)|.$$

Divide  $\omega$  into  $e^{c\delta n}$  equal intervals  $\omega'$  by successive bisections. We assert that for every  $\omega'$ :

$$(5.3) \quad v_q(h(\omega')) \geq M_k |h(\omega')| (1 - e^{-c\delta n}).$$

To see this, recall that, by lemma 2,

$$|h(\omega')| \geq \exp(-c\delta n) |h(\omega)|.$$

Hence if (5.3) is false for one interval  $\omega'$ , it follows by (ii) that

$$v_q(h(\omega)) \leq M_k ((1 + 2^{-n/2}) - e^{-c\delta n} \cdot e^{-c\delta n}) |h(\omega)| < M_k |h(\omega)|$$

if  $\delta$  is small enough. This contradicts the choice (5.2).

Let  $\omega^*$  be an arbitrary interval of length  $|\omega^*|$  so that

$$\frac{1}{20} q_k^{-1} < |\omega^*| < \frac{1}{10} q_k^{-1}.$$

Then for some  $\omega''$  of the same length and  $\omega'' \subset \omega$ :

$$h(\omega^*) = f^m(h(\omega'')), \quad m < Cq_k.$$

Divide  $\omega^*$  and  $\omega''$  into intervals of length  $e^{-c\delta n} |\omega^*|$  and let  $\omega_0^*$  and  $\omega_0''$  be two corresponding intervals. Then:

$$v_q(h(\omega_0^*)) = \prod_{v=1}^m f'(\xi_v) v_q(h(\omega_0'')) (1 + O(2^{-n}))$$

$$|h(\omega_0^*)| = \prod_{v=1}^m f'(\xi_v) |h(\omega_0'')|$$

so that: 
$$\frac{v_q(h(\omega_0^*))}{|h(\omega_0^*)|} = \frac{v_q(h(\omega_0''))}{|h(\omega_0'')|} \cdot \exp\left(\sum_{v=0}^m |f^v(h(\omega_0''))|\right)$$

$$= M_k (1 + O(e^{-cn\delta})) (1 + O(e^{-cn\delta}))$$

because, by lemma 2,  $|f^v(h(\omega_0^*))| \leq e^{-cn\delta} |f^v(h(\omega^*))|$ , and  $\sum_{v=0}^m |f^v(h(\omega^*))| \leq C.$

We cover  $(0, 1)$  by disjoint intervals  $\omega^*$  and obtain:

$$1 = \sum_{\omega^*} v_q(h(\omega^*)) = M_k \sum_{\omega^*} |\omega^*| (1 + O(e^{-cn\delta}))$$

so that:  $M_k = 1 + O(e^{-cn\delta}).$

Hence, if  $|\omega| > q_k^{-1}$ , it follows that

$$v_q(h(\omega)) \leq |h(\omega)| (1 + O((\log q)^{-\beta}))$$

and the reverse inequality is proved similarly.

If we observe that  $|h(\omega)| \leq (\log q)^{-K}$  for all  $K$  if  $|\omega| < q^{-\epsilon}$ , we can conclude that

$$(5.4) \quad \int_0^1 \varphi(x) dv_q(x) = \int_0^1 \varphi dx + O((\log q)^{-\beta}) \quad \text{if } \varphi \in C^1.$$

It remains to prove the same remainder estimate in Denjoy's inequality.

We denote by  $\omega_j$  the interval  $\left(\frac{r}{q}, \frac{r+1}{q}\right)$  containing  $h^{-1}(x_0) + j\alpha$  and denote by  $\omega_{j_0}$  the subinterval  $\left(\frac{r+\eta}{q}, \frac{r+1-\eta}{q}\right)$  of  $\omega_j$  where  $\eta = q_k/q$  and  $q_k$  is the integer defined above. We first observe that

$$\left| \varphi(f^j(x_0)) - q \int_{h(\omega_j)} \varphi(x) d\mu(x) \right| < C |h(\omega_j)|.$$

Divide  $(0, q-1)$  into blocks  $C_1, \dots, C_m$  of length  $q_k$ . Since  $q_k$  does not divide  $q$  we have to skip a set  $\Gamma$  of less than  $q_k$  numbers. This set  $\Gamma$  is chosen so that

$$\sum_{\Gamma} |h(\omega_j)| < \frac{q_k^2}{q} < q^{-\epsilon}.$$

To estimate  $\sum_{C_\nu} (\varphi(f^j(x_0)) - q \int_{h(\omega_j)} \varphi(x) d\mu(x))$  we write  $h(\omega_j) = h(\omega_{j_0}) \cup h(\omega_j \setminus \omega_{j_0})$ . Then:

$$(5.5) \quad \sum_{C_\nu} \left( \sum_{C_\nu} \left| \varphi(f^j(x_0)) - 2\eta - q \int_{h(\omega_j \setminus \omega_{j_0})} \varphi(x) d\mu(x) \right| \right) < C\eta.$$

If  $C_\nu = (\lambda, \lambda + q_k)$  we set  $y_0 = f^\lambda(x_0)$ . For  $y \in h(\omega_{\lambda_0})$  we have  $j = \lambda + s$ ,

$$\varphi(f^s(y_0)) - \varphi(f^s(y)) = \varphi'(z_s) \int_y^{y_0} Df^s(\xi) d\xi$$

and  $f^s(y) \in h(\omega_j)$  and  $z_s$  is some number in  $h(\omega_j)$ . Hence:

$$\begin{aligned} & \sum_{C_\nu} \left( \varphi(f^j(x_0)) (1 - 2\eta) - q \int_{h(\omega_{j_0})} \varphi(x) d\mu(x) \right) \\ &= (1 - 2\eta) \sum_{s=0}^{q_k-1} \int_{h(\omega_{\lambda_0})} \varphi'(z_s) d\mu(y) \int_y^{y_0} Df^s(\xi) d\xi + \text{error} \\ &= (1 - 2\eta) \sum_{s=0}^{q_k-1} \int_{h(\omega_{\lambda_0})} d\mu(y) \int_y^{y_0} Df^s(\xi) \varphi'(f^s(\xi)) d\xi \\ &+ O(\text{Max}_\lambda |h(\omega_{\lambda_0})|) + \text{error}. \end{aligned}$$

We have used  $|\varphi''(x)| \leq 1$ . The error occurs because the intervals  $h(\omega_{j_0})$  are not exactly maps of  $h(\omega_{\lambda_0})$ . The error is estimated as in (5.5). For the final sum we use (5.4) for  $q = q_k$  and find the bound:

$$\sum_{s=0}^{q_k-1} \int_{h(\omega_{\lambda_0})} d\mu(y) \int_y^{y_0} Df^s(\xi) d\xi. O((\log q)^{-\beta}) = O\left(\frac{q_k}{q} (\log q)^{-\beta}\right).$$

This proves the remainder estimate.

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