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MICHAL MISIUREWICZ

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STRUCTURE OF MAPPINGS OF AN INTERVAL WITH ZERO ENTROPY

by Michał MISIUREWICZ*

0. Complexity of dynamics of a mapping of an interval into itself can be measured by the topological entropy. Systems with positive entropy are much more complicated than those with zero entropy (cf. [3]). But if a system with zero entropy lies in the closure of the set of ones with positive entropy, then its dynamics becomes also complex. Our goal is to study a certain class of such systems. Similar classes were studies in [4], [2] and [1] (see also [7]).

The main tool used in this paper is the Schwarzian derivative. The idea of its using is due to D. Singer [9]. The Schwarzian derivative of a transformation f is given by:

$$Sf=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Mappings with negative Schwarzian derivative have a lot of " good " properties (but not vice versa: the mapping $x\to\sqrt{1+1/2(x^2-1)(9-x^2)}$ of $[1, 3]$ into itself has positive Schwarzian derivative on $[1,1.3492503...)\cup(2.8304104..., 3]$, nonetheless it is smoothly conjugate—by $x\mapsto\frac{1}{8}(x^2-1)$—to the well-known mapping $x\mapsto 4x(1-x)$ of $[0, 1]$ which has negative Schwarzian derivative).

We shall use the following notation: $h(f)$ is the topological entropy of f, $\mathrm{Per}(f)$ is the set of all periodic points of f. The symbol f^n always denotes $f\circ f\circ\ldots\circ f$ (n times).

We assume that the reader is familiar with the kneading theory, and in particular with the paper [3].

The main results of the paper are:

— Theorem (4.3), which says that for any mapping from the class under consideration, preimages of a critical point are dense (and it follows that any two such mappings are topologically conjugate).

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— Theorem (4.7), which describes a structure of such a mapping.

— Theorem (5.1), which claims the existence and uniqueness of a probabilistic non-atomic measure for elements of some larger class of mappings.

I am indebted to J. Guckenheimer who told me about the results of Singer and Feigenbaum, and to W. Szlenk, who conjectured the pointwise convergence of the sequence (f^{2^n}) (Theorem (4.7) (vii)).

1. Denote by $\mathscr{A}$ the set of all mappings $f: [-1, 1] \to [-1, 1]$ which fulfil the following conditions:

(A) f is of class C^1 on $[-1, 1]$ and of class C^3 on $(-1, 0) \cup (0, 1)$,
(B) f is even,
(C) $f(-1) = -1$,
(D) $f'(-1) > 1$,
(E) $f'(x) \neq 0$ for $x \neq 0$,
(F) $Sf < 0$ on $(-1, 0) \cup (0, 1)$,
(G) f has a periodic point of prime period 2^n for any non-negative integer n,
(H) f has no periodic points of prime periods which are not powers of 2.

We shall make some remarks.

Remark **(1.1)**. — From (A) and (B) it follows that $f'(0) = 0$.

Remark **(1.2)**. — Conditions (B), (C) and (E) imply that $\mathscr{A}$ is contained in the class of mappings considered in [5] and [3]. Therefore we can apply the kneading theory to them.

Remark **(1.3)**. — Condition (H) is equivalent to the condition $h(f) = 0$ (see [6]).

Remark **(1.4)**. — f satisfies (G) and (H) if and only if it has the minimal kneading invariant with topological entropy 0 (cf. [3]).

2. We are interested in showing that the class of mappings, introduced in section 1, is not too small. For this we shall use the following proposition:

Proposition **(2.1)**. — *The set of all* C^1*-mappings of a closed interval* I *into itself fulfilling condition* (G) *is closed in* C^1*-topology.*

Proof. — Let $f: I \to I$ be a C^1-mapping which does not fulfil condition (G). For some N, f has no periodic point of prime period 2^N and thus, by Šarkovskiĭ's theorem [10],

[11] every periodic point of f has prime period of the form 2^n, $n<N$. Set $\varphi=f^{2^{N-1}}$. All periodic points of φ are fixed points. We take:

(i) $\varepsilon>0$ such that if $|x-y|<\varepsilon$, then $|(\varphi^2)'(x)-(\varphi^2)'(y)|<\frac{1}{2}$

(ii) $\delta=\frac{\varepsilon}{2}\cdot(\sup_I|(\varphi^2)'|+1)^{-1}$

(iii) $\eta=\inf\{|\varphi^4(x)-x| : \operatorname{dist}(x,\{y : \varphi(y)=y\})\geq\delta\}$.

Then we take a neighbourhood U of φ in the space of all C^1-mappings of I into itself, such that if $\psi\in U$, then:

(iv) $|\psi^4(x)-\varphi^4(x)|<\eta$ for any x

(v) $|\psi^2(x)-\varphi^2(x)|<\frac{\varepsilon}{2}$ for any x

(vi) $|(\psi^2)'(x)-(\varphi^2)'(x)|<\frac{1}{2}$ for any x.

Suppose that some $\psi\in U$ has a periodic point t of prime period 4. By (iv) we have: $|\varphi^4(t)-t|=|\varphi^4(t)-\psi^4(t)|<\eta$. By (iii) there exists a point y such that $\varphi(y)=y$ and $|t-y|<\delta$. By (ii) we have $|\varphi^2(t)-y|=|\varphi^2(t)-\varphi^2(y)|<\frac{\varepsilon}{2}$. Denote $\psi^2(t)=s$. By (v) we have $|s-\varphi^2(t)|<\frac{\varepsilon}{2}$, and hence $|s-y|<\varepsilon$. Since $\frac{\psi^2(t)-\psi^2(s)}{t-s}=-1$, there exists a point z between t and s for which $(\psi^2)'(z)=-1$. We have:

$$|z-y|\leq\max(|t-y|,\ |s-y|)<\max(\delta,\ \varepsilon)=\varepsilon.$$

By (i) we have $|(\varphi^2)'(z)-(\varphi^2)'(y)|<\frac{1}{2}$. By (vi) we have:

$$\begin{aligned}|-1-(\varphi^2)'(y)|&=|(\psi^2)'(z)-(\varphi^2)'(y)|\\&\leq|(\psi^2)'(z)-(\varphi^2)'(z)|+|(\varphi^2)'(z)-(\varphi^2)'(y)|<\frac{1}{2}+\frac{1}{2}=1\end{aligned}$$

and thus $(\varphi^2)'(y)<0$. But $(\varphi^2)'(y)=\varphi'(y)\cdot\varphi'(\varphi(y))=(\varphi'(y))^2\geq 0$, a contradiction. ■

Remark **(2.2)**. — From the above proof it follows that if we have a continuous arc of C^1-mappings of I into itself (i.e. a mapping and its derivative depend continuously on a parameter, but we do not demand any smooth dependence on a parameter), then the first periodic points of periods 2^n and 2^{n+1} cannot occure simultaneously.

Corollary **(2.3)**. — *Let* $f: [-1, 1]\to[-1, 1]$ *be a mapping satisfying conditions* (A), (B), (C), (E), (F) *and also*:

(D') $f'(-1)>2$

(I) $f(0)=1$.

Let g_λ be given by a formula:

$$g_\lambda(x) = \lambda . f(x) + \lambda - 1.$$

Then for some $\lambda \in \left[\frac{1}{2}, 1\right)$, $g_\lambda \in \mathscr{A}$.

Proof. — It is easy to check that for any $\lambda \in \left[\frac{1}{2}, 1\right]$, g_λ maps $[-1, 1]$ into itself and that conditions (A)-(F) are fulfilled for g_λ. We have $g_{1/2}([-1, 1]) = [-1, 0]$, and hence $h(g_{1/2}) = h(g_{1/2}|_{[-1,0]}) = 0$, because $g_{1/2}|_{[-1,0]}$ is a homeomorphism. On the other hand, we have $h(g_1) = \log 2$. Topological entropy of g_λ is a lower semi-continuous function of λ (see [8], [6]). Hence, if we take $\varkappa = \sup\{\lambda : (g_\lambda) = 0\}$, then $h(g_\varkappa) = 0$ and $g_\varkappa$ is a C^1-limit of a sequence of mappings with positive entropy. Every mapping with positive entropy has a periodic point of prime period which is not a power of 2, and hence, by Šarkovskiĭ's theorem, it satisfies (G). Therefore, by Proposition (2.1), $g_\varkappa$ also satisfies (G). By Remark (1.3), $g_\varkappa$ satisfies also (H). ■

Remark **(2.4)**. — Clearly $\varkappa \neq \frac{1}{2}$, because all periodic points of $g_{1/2}$ are fixed points. However, $\varkappa$ can be arbitrarily close to $1/2$. Consider the family of mappings

$$f_\varepsilon(x) = 1 - 2|x|^{1+\varepsilon}.$$

For any $\varepsilon > 0$, f_ε satisfies the hypotheses of Corollary (2.3). Besides, f_ε tends to f_0 in C^0-topology (i.e. uniformly). Denote by $g_{\varepsilon,\lambda}$ a transformation given by:

$$g_{\varepsilon,\lambda}(x) = \lambda . f_\varepsilon(x) + \lambda - 1.$$

Set also $\varkappa(\varepsilon) = \sup\{\lambda : h(g_{\varepsilon,\lambda}) = 0\}$. We claim that $\lim_{\varepsilon \to 0} \varkappa(\varepsilon) = \frac{1}{2}$. It is a consequence of the following lemma:

Lemma **(2.5)**. — *For any* $\lambda_0 \in \left(\frac{1}{2}, 1\right)$ *there exists* $\delta > 0$ *such that for any* $\lambda \in [\lambda_0, 1]$ *and for any continuous mapping* $\varphi : [-1, 1] \to [-1, 1]$, *if* $\sup_{[-1,1]} |g_{0,\lambda} - \varphi| \leq \delta$, *then* $h(\varphi) > 0$.

Proof. — Let $\lambda \in [\lambda_0, 1]$. If $2\lambda^2 \leq 1$, then $g^2_{0,\lambda}$ restricted to $\left[-\frac{2\lambda - 1}{2\lambda + 1}, \frac{2\lambda - 1}{2\lambda + 1}\right]$ is linearly conjugate to $g_{0,2\lambda^2}$. We define:

$$\alpha_0(\lambda) = \lambda, \quad \alpha_{n+1}(\lambda) = 2(\alpha_n(\lambda))^2,$$

$$\beta_0(\lambda) = 1, \quad \beta_{n+1}(\lambda) = \frac{2\alpha_n(\lambda) - 1}{2\alpha_n(\lambda) + 1} \cdot \beta_n(\lambda),$$

$$N(\lambda) = \sup\{n : \alpha_n(\lambda) \leq 1\}.$$

Clearly α_n and β_n are increasing functions of λ. Hence, N is a non-increasing function of λ. It is also easy to see that for any λ, $N(\lambda)$ is finite and the sequence $(\beta_n(\lambda))_{n=0}^\infty$ is decreasing.

We fix λ for a while and therefore omit it as an argument.

The mapping $g_{0,\lambda}^{2^{N+1}}\Big|_{[-\beta_{N+1},\,\beta_{N+1}]}$ is linearly conjugate to the mapping

$$g_{0,\alpha_{N+1}}:\ [-1,1]\to[-1,-1+2\alpha_{N+1}]$$

(remember that $\alpha_{N+1}\in(1,2]$). Let us look at $g_{0,\lambda}^{2^{N+2}}\Big|_{[-\gamma,\gamma]}$, where $\gamma=\beta_{N+1}\cdot\left(1-\dfrac{1}{2\alpha_{N+1}}\right)$. We have:

$$(-1)^N.g_{0,\lambda}^{2^{N+2}}(-\gamma)=(-1)^N.g_{0,\lambda}^{2^{N+2}}(\gamma)\geq\beta_{N+1}$$

and

$$(-1)^N.g_{0,\lambda}^{2^{N+2}}\left(\beta_{N+1}\cdot\left(1-\frac{1}{4\alpha_{N+1}}\right)\right)\leq-\beta_1.$$

Therefore we obtain a horseshoe effect (see [8], [6], [7]) for $g_{0,\lambda}^{2^{N+2}}$ and this effect is persistent under perturbations which are not greater than $\beta_{N+1}-\gamma$ in sup norm. But $\beta_{N+1}-\gamma=\beta_{N+1}\cdot\dfrac{1}{2\alpha_{N+1}}\geq\dfrac{1}{4}\beta_{N+1}=\dfrac{1}{4}\beta_{N(\lambda)+1}(\lambda)\geq\dfrac{1}{4}\beta_{N(\lambda_0)+1}(\lambda_0)$. Set $\dfrac{1}{4}\beta_{N(\lambda_0)+1}(\lambda_0)=\zeta$. If $\sup\limits_{[-1,1]}|\varphi-g_{0,\lambda}^{2^{N+2}}|\leq\zeta$, then $h(\varphi)\geq\log 2$. It can be easily checked by induction that if ψ is Lipschitz continuous with a constant $c\neq 1$ and $\sup\limits_{[-1,1]}|\varphi-\psi|<\eta$, then:

$$\sup_{[-1,1]}|\varphi^n-\psi^n|<\eta.\frac{c^n-1}{c-1}.$$

Hence, if $\sup\limits_{[-1,1]}|\varphi-g_{0,\lambda}|<\zeta.2^{2^{N(\lambda_0)+2}}$, then $h(\varphi)\geq 2^{-N(\lambda_0)-2}.\log 2>0$. ■

Remark **(2.6)**. — Denote by φ an affine mapping from $[-1,1]$ onto $[0,1]$:

$$\varphi(x)=\frac{1}{2}(x+1).$$

Then we have in Corollary (2.3), $\varphi\circ g_\lambda\circ\varphi^{-1}=\lambda.(\varphi\circ f\circ\varphi^{-1})$.

Remark **(2.7)**. — For $\psi(x)=\alpha x+\beta$ we have $S(\psi^{-1}\circ f\circ\psi)(x)=\alpha^2.Sf(\alpha x+\beta)$, and thus the condition $Sf<0$ is preserved under an affine conjugacy.

Examples **(2.8)**. — Take in Corollary (2.3) as f one of the following mappings:

$$\rho(x)=1-2x^2,$$

$$\sigma(x)=2\cos\frac{\pi x}{2}-1,$$

$$\tau(x)=\frac{2}{7}(1-x^2)(7+6x^2+3x^4)-1,$$

$$\xi(x)=(c+2)\left(\frac{c}{c+2}\right)^{x^2}-(c+1)\quad\text{for}\quad c>c_0,$$

where c_0 is a solution of the equation $e^{1/c}=1+\dfrac{2}{c}$ $(c_0=0.795905\ldots)$.

It is easy to check that all of these mappings satisfy the hypotheses of Corollary (2.3).

The transformations ρ, σ and τ are the transformations Q_4, S_1, $C_{64/63}$ from [4] respectively, transformed to $[-1, 1]$ by an affine conjugacy. Note also that if, in Corollary (2.3), we put $f=\rho$, then g_λ is conjugate (by an affine map) to $x \mapsto x^2 - \frac{\lambda(\lambda-2)}{4}$ (cf. [5]).

We can also take any mapping f_ε, $\varepsilon > 0$ from Remark (2.4) (cf. [1]).

The reader can easily obtain further examples by taking various mappings with negative Schwarzian derivative, e.g. from [9]. We may mention also that if the first derivative of a real polynomial of degree at least 2 has no roots in the set

$$\{z \in \mathbf{C} : |\mathrm{Im}\, z| \geq |\mathrm{Re}\, z| - 1\} \setminus \mathbf{R},$$

then this polynomial has negative Schwarzian derivative on $[-1, 1]$ (cf. [9]).

3. Throughout this section f and g will denote C^1-mappings of some interval I into itself, which are C^3 outside the set of critical points and endpoints of I. It is a slightly wider class than the one considered in [9], but the main facts remain valid for its members.

Lemma **(3.1)** (see [9]). — *If* $Sf<0$ *and* $Sg<0$, *then* $S(f \circ g)<0$.

Proof. — We have $S(f \circ g) = (g')^2 . (Sf \circ g) + Sg$ and if $(f \circ g)'(x) \neq 0$ then $f'(g(x)) \neq 0$ and $g'(x) \neq 0$. ■

Corollary **(3.2)** (see [9]). — *If* $Sf<0$, *then also* $S(f^n)<0$ *for all* $n \geq 0$.

Lemma **(3.3)** (see [9]). — *If* $Sf<0$, *then* $|f'|$ *has no positive local minima on* I *(excluding endpoints).*

Proof. — If $|f'|$ has a positive local extremum at x, and x is not an endpoint of I, then $f''(x)=0$ and $f'''(x)$ has the opposite sign than $f'(x)$. ■

We say that the periodic point x of period p is attracting (repelling) if

$$|(f^p)'(x)| < 1 \ (>1).$$

Lemma **(3.4)** (cf. [9]). — *If* $Sf<0$ *and* x *is a non-repelling point, then there exists a point* y *such that* $\lim_{n \to \infty} f^{pn+r}(y) = x$ *for some* $r \in \{0, \ldots, p-1\}$ *and either* $f'(y)=0$ *or* y *is an endpoint of* I.

Proof. — Let x be a periodic point of period p. Since $Sf<0$, we can apply Lemma (3.3) to f^p. But $|(f^p)'(x)| \leq 1$ and hence (provided x is not an endpoint of I), there exists an interval of the form $[x, z)$ or $(z, x]$ such that for all $t \neq x$ from this interval $|(f^p)'(t)| < 1$ and either $(f^p)'(z)=0$, or z is an endpoint of I. But if $(f^p)'(z)=0$, then $f'(y)=0$ for some $y=f^q(z)$. ■

Remark (3.5). — If we take a function f defined as ξ in Examples (2.8), but with $c=\frac{1}{2}$ and then $g_{1/2}$ as in Corollary (2.3), then we obtain a mapping $x\mapsto\frac{5}{4}\left(\left(\frac{1}{5}\right)^{x^2}-1\right)$ which has negative Schwarzian derivative. It has only one critical point (0), and this point is a fixed point. But the derivative in the fixed point -1 is equal to $\ln\sqrt{5}$, which is less than 1, and hence the point -1 is attracting.

The above example is affinely conjugate to the mapping $x\mapsto\frac{1}{8}(5^{4x(1-x)}-1)$ of [0, 1] into itself and we see that Theorem (2.7) of [9] is not stated precisely enough.

4. In this section we shall prove the main results about the elements of the class $\mathscr{A}$.

Proposition (4.1). — *If $f\in\mathscr{A}$, then all periodic points are repelling and there exists at most one orbit of a given prime period (except period 1: there exist two fixed points).*

Proof. — Suppose that some periodic point x is not repelling. In view of Lemma (3.4) there exists $y\in\{0,-1,1\}$ such that for some p, r, $\lim_{n\to\infty} f^{pn+r}(y)=x$. But $f(-1)=f(1)=-1$ and -1 is a repelling point. Hence we have $y=0$. Therefore the kneading invariant of f is eventually periodic and by Remark (1.4) we obtain a contradiction with the results of [3].

Now we shall prove the second statement. Notice first that there is only one admissible kneading invariant of a given period 2^n and greater than the kneading invariant of f, except for $n=0$, when there is one periodic and one anti-periodic admissible kneading invariant (see [3]). Hence, for any two different periodic orbits of the same period $p>1$ we can take one point from each orbit in such a way that they have the same invariant coordinate. But they are both repelling, and hence between them there is either a non-repelling periodic point or a local extremum of f^p. In both cases we have a contradiction. The same arguments can be applied for fixed points. ■

The following lemma is basic for proving further properties of elements of the class $\mathscr{A}$.

Lemma (4.2). — *Let $f\in\mathscr{A}$ and let $f(b)=b$, $b\neq-1$. Then the mapping g, given by $g(x)=-\frac{1}{b}f^2(-bx)$ (which is affinely conjugate to $f^2\big|_{[-b,b]}$), belongs to $\mathscr{A}$.*

Proof. — Notice that if $f^2(0)\notin[-b,b]$, then by the horseshoe effect $h(f^2)\geq\log 2$ (cf. [8], [7]). Hence, $f^2([-b,b])\subset[-b,b]$ and, consequently, $g([-1,1])\subset[-1,1]$.

We must check conditions (A)-(H) for g instead of f. Conditions (A), (B), (C), (E) hold obviously. (D) follows from Proposition (4.1), (F) from Remark (2.7) and Corollary (3.2), (H) from Remark (1.3). To prove (G) it is sufficient to show that there is no periodic orbit of f completely missing $[-b,b]$ (except -1). But clearly $\bigcup_{n\geq 0} f^{-n}([-b,b])=(-1,1)$. ■

Theorem **(4.3).** — *If* $f\in\mathcal{A}$, *then the set* $\bigcup_{n\geq 0} f^{-n}(\{0\})$ *is dense in* $[-1, 1]$.

Proof. — We define by induction a sequence $(b_n)_{n=0}^{\infty}$ such that the transformations g_n, given by $g_n(x)=\frac{(-1)^n}{b_n}.f^{2^n}((-1)^n b_n x)$, belong to $\mathcal{A}$. We put $b_0=1$ (thus $g_0=f$). When g_n is already defined, then we use Lemma (4.2) with g_n instead of f and we take $b_{n+1}=b.b_n$ (b is from Lemma (4.2) for g_n and therefore b also depends on n).

Now we set:

$$I_n=[-b_n, b_n],$$

$$K_n=\begin{cases}[b_{n+1}, b_n] & \text{for } n \text{ even}\\ [-b_n, -b_{n+1}] & \text{for } n \text{ odd,}\end{cases}$$

$$L_n=\begin{cases}[-b_n, -b_{n+1}] & \text{for } n \text{ even}\\ [b_{n+1}, b_n] & \text{for } n \text{ odd.}\end{cases}$$

At the points $\pm b_n$ and $\pm b_{n+1}$ we have $|(f^{2^n})'|>1$, and therefore, by Lemma (3.3):

$$(*)\qquad |(f^{2^n})'|>1 \quad \text{on } K_n\cup L_n.$$

Suppose now that the set $\bigcup_{n\geq 0} f^{-n}(\{0\})$ is not dense in $[-1, 1]$. We take a maximal interval J disjoint from this set.

Suppose first that J contains some periodic point x of period p. We have $\bigcup_{n\geq 0} f^{np}(J)=J$ and since x is repelling, there must be another periodic point in J. But it has the same invariant coordinate as x, a contradiction.

Thus, J is contained either in some $K_k\cup L_k$ or in $\bigcap_{n\geq 0} I_n$. Clearly, the same is true for $f^m(J)$, $m=1, 2, \ldots$ Since the kneading invariant of f is not periodic, at most one of the intervals $J, f(J), f^2(J), \ldots$ can be contained in $\bigcap_{n\geq 0} I_n$. Thus, we may assume that for any $m=0, 1, 2, \ldots$, $f^m(J)$ is contained in $K_{k(m)}\cup L_{k(m)}$ (perhaps then J is not maximal, but we do not need this assumption any more). Then we define by induction:

$$m(0)=0,$$
$$m(r+1)=m(r)+2^{k(m(r))}.$$

From $(*)$ it follows that the length of $f^{m(r+1)}(J)$ is greater than the length of $f^{m(r)}(J)$. The invariant coordinate of points of J is not eventually periodic, and therefore the sets $f^i(J)$, $i=0, 1, 2, \ldots$, are pairwise disjoint, a contradiction. ■

Corollary **(4.4).** — *If* $f\in\mathcal{A}$, *then there are not two distinct points with the same invariant coordinate.*

Corollary **(4.5).** — *Every two elements of* $\mathcal{A}$ *are topologically conjugate.*

Remark **(4.6).** — All points of $f(\bigcap_{n\geq 0} I_n)$ have the same invariant coordinate, and therefore this set consists of one point. Hence, $\bigcap_{n\geq 0} I_n$ consists also of one point (namely 0).

Theorem **(4.7).** — *If* $f\in\mathcal{A}$, *then there exists a set* $S\subset[-1, 1]$ *such that:*

(i) S *is closed and f-invariant,* $0\in S$,

(ii) S *is homeomorphic to the Cantor set,*

(iii) *the system* $(S, f|_S)$ *is minimal,*

(iv) $f|_S : S\to S$ *is a homeomorphism,*

(v) S *is a support of a probabilistic f-invariant non-atomic ergodic measure* μ,

(vi) $\lim_{n\to\infty} \operatorname{dist}(f^n(x), S)=0$ *for every x which is not eventually periodic,*

(vii) $\lim_{n\to\infty} f^{2^n}(x)$ *exists for any x and is equal to x for* $x\in S\cup\operatorname{Per}(f)$,

(viii) *the set of non-wandering points of f is equal to* $S\cup\operatorname{Per}(f)$,

(ix) μ *is the unique probabilistic f-invariant non-atomic measure.*

Proof. — Define $S=\bigcap_{n\geq 0}\bigcup_{k=0}^{2^n-1} f^k(I_n)$. We divide the proof according to the statements of the theorem.

1) Clearly S is closed and $0\in S$. Since $f^{2^n}(I_n)\subset I_n$, S is also f-invariant.

2) We can describe S in another way. Set $M_n=I_n\cup f^{2^{n-1}}(I_n)$.

Then $S=\bigcap_{n\geq 1}\bigcup_{k=0}^{2^{n-1}-1} f^n(M_n)$. The set M_n is a closed interval, the intervals $f^k(M_n)$, $k=0, 1, \ldots, 2^{n-1}-1$, are pairwise disjoint,

$$\bigcup_{k=0}^{2^{n-1}-1} f^k(M_n)\supset\bigcup_{k=0}^{2^n-1} f^k(M_{n+1})$$

and in any set $f^k(M_n)$ there are contained exactly two sets of the form $f^r(M_{n+1})$, $0\leq r\leq 2^n-1$. This gives a natural structure of a Cantor set on S, provided any intersection of the form $\bigcap_{n\geq 1} f^{k_n}(M_n)$ consists of at most one point. But this is true, because either all points of this intersection have the same invariant coordinate, or some image of it is contained in $\bigcap_{n\geq 0} I_n$ and then we use Remark (4.6).

3) If $x\in S$, then for a given n, in any set $f^k(M_n)$, $k=0, 1, \ldots, 2^{n-1}-1$, there is one of the points $f^r(x)$, $r=0, 1, \ldots, 2^{n-1}-1$. Thus, every orbit of an element of S is dense in S, i.e. $f|_S$ is minimal.

4) If $x, y\in S$, $x\neq y$, then for some n, r, s, we have: $x\in f^r(M_n)$, $y\in f^s(M_n)$, $0\leq r$, $s\leq 2^{n-1}-1$, $r\neq s$. Then $f(x)\in f^{r+1}(M_n)$, $f(y)\in f^{s+1}(M_n)$. But $f^{r+1}(M_n)\cap f^{s+1}(M_n)=\emptyset$ and thus $f(x)\neq f(y)$.

5) In order to define a probabilistic measure μ on S it is sufficient to define measures of the sets $f^k(M_n)$ in such a way that if $S \cap \bigcup_{i=1}^{r} f^{k_i}(M_{n_i}) = S \cap f^k(M_n)$, then:

$$\sum_{i=1}^{r} \mu(f^{k_i}(M_{n_i})) = \mu(f^k(M_n))$$

and $\mu(S)=1$. We put $\mu(f^k(M_n))=2^{-n+1}$ and clearly the above conditions are satisfied. It is obvious, that μ is non-atomic; it is f-invariant because $f|_S$ is a homeomorphism and $\mu(f^{k+1}(M_n))=\mu f^k((M_n))$. It follows from the minimality of $f|_S$ that the support of μ is equal to S.

To prove ergodicity of f it is sufficient to show that for any m, n, r, s:

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mu(f^r(M_n)) \cap f^{s+i}(M_m)) = \mu(f^r(M_n)) \cdot \mu(f^S(M_m)).$$

Among every $2^{\min(m,n)-1}$ consecutive terms of the sequence $(\mu(f^r(M_n) \cap f^{s+i}(M_m)))_{i=0}^{\infty}$ there is exactly one which is equal to $2^{-\max(m,n)+1}$ and the rest are zeros. Hence, the average of k first terms of this sequence tends to

$$2^{-\min(n,m)+1} \cdot 2^{-\max(n,m)+1} = 2^{-n+1} \cdot 2^{-m+1} = \mu(f^r(M_n)) \cdot \mu(f^s(M_m)).$$

6) Take x which is not eventually periodic. Set $\ell(n)=\sup\{s : f^n(x) \in \bigcup_{k=0}^{2^s-1} f^k(I_s)\}$ ($\ell(n)=+\infty$ if $f^n(x) \in S$). Since the set $\bigcup_{n=0}^{2^s-1} f^k(I_s)$ is f-invariant, the sequence $(\ell(n))_{n=0}^{\infty}$ is non-decreasing. Suppose that $f^r(x) \in \bigcup_{n=0}^{2^s-1} f^k(I_s)$. Then for some $k \geq r$ we have $f^k(x) \in I_s$. But $I_s = I_{s+1} \cup K_s \cup L_s$. If $f^k(x)$ belongs to K_s, then it is repelled by the point $(-1)^s b_s$ under the action of f^{2^s}, until it reaches I_{s+1}. Hence, $f^{p\cdot 2^s}(f^k(x)) \in I_{s+1}$ for some p. If $f^k(x) \in L_s$, then $f^{2^s}(f^k(x)) \in K_s \cup I_{s+1}$ and we may repeat the above considerations. In any case, we have $f^m(x) \in I_{s+1}$ for some $m \geq k$. This proves that $\lim_{n\to\infty} \ell(n)=+\infty$. But this is equivalent to the statement that $\lim_{n\to\infty} \operatorname{dist}(f^n(x), S)=0$.

7) Any set of the form $f^k(I_s)$ is f^{2^s}-invariant. Hence, if $f^{2^r}(x) \in f^k(I_s)$ for some $r \geq s$, then also $f^{2^n}(x) \in f^k(I_s)$ for any $n \geq r$. But $f^k(I_s)$ is contained either in $f^k(M_s)$ or in $f^{k-2^{s-1}}(M_s)$, and therefore also $\bigcap_{i\geq 0} f^{k_i}(I_{s_i})$ consists of at most one point (provided $s_i \to +\infty$) and this point belongs to S. Thus it follows from the proof of (vi) that if x is not eventually periodic, then $\lim_{n\to\infty} f^{2^n}(x)$ exists and belongs to S. Clearly, if $x \in S$, then this limit is equal to x.

If x is eventually periodic, then by (H), $\lim_{n\to\infty} f^{2^n}(x)$ also exists. Clearly, it is a periodic point and is equal to x if x is periodic.

8) Let $x \notin S \cup \operatorname{Per}(f)$. There exists s such that:

$$x \notin \bigcup_{k=0}^{2^s-1} f^k(I_s).$$

Take N such that $f^{\mathrm{N}}(x)$ belongs to $\bigcup_{j=0}^{2^{s+1}-1} f^j(\mathrm{I}_{s+1})$. This set is contained in the interior of the set $\bigcup_{k=0}^{2^s-1} f^k(\mathrm{I}_s)$ and hence there exists a neighbourhood U of x such that U is disjoint from $\bigcup_{k=0}^{2^s-1} f^k(\mathrm{I}_s)$ and $f^{\mathrm{N}}(\mathrm{U}) \subset \bigcup_{k=0}^{2^s-1} f^k(\mathrm{I}_s)$. Thus, for some smaller neighbourhood V of x we have $f^n(\mathrm{V}) \cap \mathrm{V} = \emptyset$ for $n = 1, 2, \ldots, \mathrm{N}-1$ and also:

$$f^n(\mathrm{V}) \cap \mathrm{V} \subset ([-1, 1] \setminus \bigcup_{k=0}^{2^s-1} f^k(\mathrm{I}_s)) \cap \bigcup_{k=0}^{2^s-1} f^k(\mathrm{I}_s) = \emptyset \quad \text{for} \quad n = \mathrm{N}, \mathrm{N}+1, \ldots$$

9) By (viii) every probabilistic f-invariant non-atomic measure is concentrated on S. But clearly the measure of $f^k(\mathrm{M}_n)$ must be equal to 2^{-n+1} (cf. [7]) and thus $f\big|_{\mathrm{S}}$ is uniquely ergodic. ■

Remark **(4.8)**. — If f and μ are as in Theorem 2, then the transformation π, defined in [7], is a homeomorphism.

Remark **(4.9)**. — D. Sullivan noticed that the system $(\mathrm{S}, f\big|_{\mathrm{S}})$ is topologically conjugate to the rotation on the group of 2-adic integers (so called " adding machine ").

Remark **(4.10)**. — It is not necessary to assume differentiability at 0, but we do not know any example of a mapping satisfying all other conditions but not differentiable at 0.

5. Now we are able to say something about a wider class of transformations than $\mathscr{A}$. Notice that the following theorem makes it possible to use results of [7].

Theorem **(5.1)**. — *If a continuous mapping f of a closed interval* I *into itself satisfies conditions* (G) *and* (H) *and has only one local extremum (except at the endpoints of* I*), then f admits a unique probabilistic f-invariant non-atomic measure.*

Proof. — Since f has a periodic point of prime period 4, f is not a homeomorphism and it has a local extremum at some point c which is not an endpoint of I. We may assume that it is a maximum (otherwise we consider the mapping $x \mapsto -f(-x)$). We take an interval $[a, b]$ which contains I in its interior and a mapping $g : [a, b] \to [a, b]$ such that $g(a) = g(b) = a$, g is increasing on $[a, c]$ and decreasing on $[c, b]$, and $g\big|_{\mathrm{I}} = f$. Clearly g satisfies (G). The only periodic points on $[a, b] \setminus \mathrm{I}$ are fixed points and hence g satisfies also (H).

We take some mapping $\varphi \in \mathscr{A}$. By Remarks (1.3) and (1.4), g and φ have the same kneading invariant. Hence, by Corollary (4.4), there exists a unique mapping $\psi : [a, b] \to [-1, 1]$ which preserves an invariant coordinate. Clearly, we have