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ERGODIC THEORY, SEMISIMPLE LIE GROUPS, AND FOLIATIONS BY MANIFOLDS OF NEGATIVE CURVATURE

by ROBERT J. ZIMMER [1] [2]

1. Introduction.

If M is a manifold, a basic problem that has received considerable attention is to examine the relationship between the topology of M, in particular its fundamental group, and the possible Riemannian structures on M. One particularly striking example of a result relating $\pi_1(M)$ to Riemannian metrics on M is the Mostow-Margulis rigidity theorem which asserts that if M and M′ are suitable locally symmetric spaces of finite volume, then isomorphism of $\pi_1(M)$ and $\pi_1(M')$ implies isometry of M and M′, modulo normalizing scalar multiples. Thus, roughly speaking, for suitable M, $\pi_1(M)$ uniquely determines any locally symmetric Riemannian structure on M.

If $\mathscr{F}$ is a foliation of a manifold (or more generally, of a measure space) one can formulate analogous questions. For example, let T be a transversal to the foliation, so that T has the structure of a measure space with an equivalence relation. We can then enquire as to the relationship between this purely measure theoretic feature of the foliation and the possible Riemannian structures that can be put on the leaves (these structures varying measurably in a suitable sense as we move from leaf to leaf). In [36] we proved an analogue of the Mostow-Margulis theorem in the context of foliations by symmetric spaces. This theorem asserts that if $(M_i, \mathscr{F}_i)$ are two suitable ergodic foliations in which each leaf is a Riemannian symmetric space of noncompact type and rank at least 2, then isomorphism of transversals (as measure spaces with equivalence relations) implies that the measurable foliations are isometric. This means that there is a measure space isomorphism $M_1 \to M_2$ that takes almost every leaf of $\mathscr{F}_1$ isometrically into a leaf of $\mathscr{F}_2$ modulo normalizing scalars. Thus for suitable foliations, the measure theory of the equivalence relation on a transversal uniquely determines possible symmetric Riemannian structures on almost all leaves.

In particular a suitable (see below for precise conditions) foliation in which each

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leaf has constant negative sectional curvature will not be transversally equivalent (*i.e.* have isomorphic transversals in the sense above) to a foliation by symmetric spaces of higher rank. On the other hand, via an amenability argument, it is not hard to show that these foliations by manifolds of constant negative curvature cannot be transversally equivalent to the measurable foliation arising from a measure class preserving free $\mathbf{R}^n$-action. The results of [39] imply that the latter assertion remains true if we replace the condition of constant negative curvature by possibly varying negative curvature bounded away from 0. The central point of this paper is to show that the former assertion is also true in this more general situation. Our main result (Theorem (5.1)) combined with [39] yields the following (see below for all terms involved).

Theorem. — *Let $\mathscr{F}$ be an ergodic Riemannian measurable foliation with transversally invariant measure and finite total volume. Suppose there is $c < 0$ such that for almost every leaf all sectional curvatures K satisfy $K \leq c$. Assume further that almost all leaves are simply connected and complete. Then*

i) *$\mathscr{F}$ is nonamenable, and in particular is not transversally equivalent to an $\mathbf{R}^n$-action.*

ii) *$\mathscr{F}$ is not transversally equivalent to an ergodic irreducible Riemannian measurable foliation with transversally invariant measure and finite total volume, such that (almost) every leaf is isometric to a given symmetric space of noncompact type and rank at least 2.*

In other words, the measure theoretic properties of the equivalence relation on the transversal provide an obstruction to the possible Riemannian structures that can be measurably assigned to the leaves.

This theorem implies a new result concerning orbit equivalence of ergodic actions. We recall that if Γ_i, $i = 1, 2$, are groups acting ergodically on measure spaces S_1, S_2 respectively, the actions are called orbit equivalent if (possibly after discarding null sets) there is a measure space isomorphism $S_1 \to S_2$ taking Γ_1-orbits onto Γ_2-orbits. For the theory of orbit equivalence for actions of amenable groups see [4], and for actions of semisimple Lie groups and their lattices, see [36], [37].

Corollary **(5.3)**. — *Let Γ_1 be a lattice in a simple Lie group G with $\mathbf{R}$-rank(G) ≥ 2, and let Γ_2 be the fundamental group of a compact manifold of negative sectional curvature (or more generally a finite volume complete manifold with negative curvature bounded away from 0). Then Γ_1 and Γ_2 do not have free, finite measure preserving, orbit equivalent ergodic actions.*

While Corollary (5.3) as stated follows directly from the main theorem, the technique of proof of the main theorem can in fact be used to establish a more general assertion about orbit equivalence of group actions. We describe this in the concluding section of this paper, but it implies for example that Corollary (5.3) remains true under the assumption that Γ_2 is any subgroup of the fundamental group of a complete manifold of finite volume and negative sectional curvature bounded away from 0.

Fundamental groups of locally symmetric spaces of non-compact type are of course discrete subgroups of semisimple Lie groups and the Mostow-Margulis theorem can equivalently be considered as a result concerning lattices in semisimple groups. Similarly, the analogous results of [36] for foliations by symmetric spaces can be considered as results about ergodic actions of semisimple Lie groups. In both cases, this formulation allows one to bring the theory of semisimple groups to bear on the problems, and the proof of these theorems are very Lie theoretic in nature. In particular, the study of the relationship of the lattice subgroup or ergodic action to the action of the semisimple Lie group on its maximal Furstenberg boundary is a basic feature of the proofs. For the proof of Theorem (5.1), we also proceed by studying " boundary behavior " but the techniques we employ here are entirely different from those used in [36]. This is necessitated in part by the fact that for manifolds of varying negative curvature there is not necessarily a simple description in Lie theoretic terms, and thus for this foliation we study behavior at infinity in terms of the geometry of asymptotic geodesics in each leaf. Here we employ the formulation of the boundary at infinity for manifolds of nonpositive curvature developed by Eberlein and O'Neill [5]. On the other hand, for the foliations by symmetric spaces we can again use the relationship between an ergodic action of a semisimple Lie group and the action on the Furstenberg boundary. However, the type of information we develop here concerning this relationship is rather different from that we used in [36]. In the latter case, we adapted certain techniques developed by Margulis for proving arithmeticity of lattices [19]. In particular this involved using ergodicity of certain actions to establish rationality of certain *a priori* measurable maps defined on G/P, where G is a semisimple Lie group and P a minimal parabolic subgroup. In the present situation, we again generalize to the framework of ergodic actions a result of Margulis. This result of Margulis [21], which was the basic step in his finiteness theorem, asserting finiteness of either the image or kernel of a homomorphism defined on an irreducible lattice in a higher rank semisimple Lie group, asserts that if $\Gamma \subset G$ is such a lattice, then every factor of the action of Γ on G/P (" factor " in the sense of measurable quotients of ergodic actions) is of the form $G/P \to G/P'$ where $P' \supset P$. In other words, Γ-factors are already G-factors. The generalization of this to ergodic actions we need here (and which we expect will be useful in other contexts) is the following.

Theorem (**4.1**). — *Let* S *be an irreducible ergodic* G*-space with finite invariant measure, where* G *is a connected semisimple Lie group with finite center, no compact factors, and* $\mathbf{R}$*-rank at least* 2. *Let* $P \subset G$ *be a parabolic subgroup. Then every* G*-space* X *which is an intermediate factor between* $S \times G/P$ *and* S (i.e. *we have* G*-maps* $S \times G/P \to X \to S$) *is of the form* $X = S \times G/P'$ *where* $P' \supset P$.

The essential ideas of the proof of this theorem are those used by Margulis in [21]. The connecting link between the boundary behavior of the foliation by symmetric spaces and the foliation by manifolds of negative curvature is provided through the notion of an amenable action [31], and in particular through the fact that G acting on $S \times G/P$ is amenable, if $P \subset G$ is a minimal parabolic subgroup.

The outline of the paper is as follows. Section 2 establishes basic notions we will need throughout the paper. In Section 3 we formulate a boundary theory for foliations by manifolds of negative curvature, and in particular discuss the behavior of measures on the boundary at infinity. Section 4 is devoted to the proof of Theorem (4.1), the generalization of Margulis' theorem. In Section 5 we complete the proof of the main theorem.

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2. Preliminaries.

We shall be dealing with the ergodic theory of group actions, equivalence relations, and foliations, and we briefly recall some definitions. Let G be a locally compact second countable group. By a G-space we mean a standard measure space (S, μ) and a G-action $S \times G \to S$ which is a Borel function such that μ is quasi-invariant under the action, *i.e.* $\mu(Ag) = 0$ if and only if $\mu(A) = 0$, where $A \subset S$ is Borel and $g \in G$. The action is called ergodic if $A \subset S$ is measurable and essentially G-invariant (*i.e.* $\mu(A \bigtriangleup Ag) = 0$ for all $g \in G$) implies A is null or conull. We shall most often in this paper be concerned with the situation in which the measure μ is finite and invariant, *i.e.* $\mu(Ag) = \mu(A)$ for all Borel $A \subset S$ and $g \in G$. If S is an ergodic G-space and S′ an ergodic G′-space, the actions are called orbit equivalent if (possibly after discarding null sets) there is a measure class preserving Borel isomorphism $S \to S'$ that takes G-orbits onto G′-orbits. If G and G′ are amenable discrete groups and S and S′ have finite invariant measure, and are properly ergodic (*i.e.* not transitive on a conull set), then the actions are orbit equivalent. This was established by H. Dye when $G = G' = Z$, and in general by Connes, Feldman, Ornstein, and Weiss [4], [26]. One has similar results for continuous amenable groups. On the other hand, the rigidity theorem for ergodic actions of semisimple Lie groups [36] implies that if $\Gamma \subset G$, $\Gamma' \subset G'$ are lattices in simple Lie groups with finite center, and **R**-rank $(G) \geq 2$, then Γ and Γ' do not have orbit equivalent free finite measure preserving ergodic actions unless G and G′ are locally isomorphic. In fact, if we consider ergodic actions of G and G′ in the centerfree case, then orbit equivalence implies conjugacy modulo an automorphism of G. One of the points of this paper is to establish a new general result on orbit equivalence using methods different from those that have been applied to this question before.

Suppose now that (S, μ) is a finite measure space and that $R \subset S \times S$ is an equivalence relation that is an analytic subset [16]. For example, if S is a G-space, we can take $R \subset S \times S$ to be $R_G = \{(s, sg) \mid g \in G\}$. The equivalence relation is called countable if each equivalence class is countable. For R_G this is obviously the case if G is discrete. If R is a countable equivalence relation, by a partial automorphism of R we mean a Borel isomorphism $\varphi : A_1 \to A_2$, where $A_i \subset S$ are Borel, such that for $y \in A_1$, $\varphi(y) \sim y$. The measure μ on S is called invariant under R if for every partial automorphism of R, $\varphi_*(\mu \mid A_1) = \mu \mid A_2$. Quasi-invariance is similarly defined. If $R = R_G$, then μ is (quasi-) invariant under G if and only if it is (quasi-) invariant under R_G. This is immediate once one observes that by countability of G, for every partial automorphism $\varphi : A_1 \to A_2$ of R_G, A_1 can be written as a countable disjoint union of sets on each of which φ acts by some element of G. A measurable set $A \subset S$ is called invariant under R if $x \in A$, $x \sim y$ implies $y \in A$, and a countable equivalence relation with quasi-invariant measure is called ergodic if invariant sets are either null or conull.

Countable equivalence relations with quasi-invariant measure arise naturally not only from actions of countable groups but as transversals for continuous group actions and foliations. Let (S, μ) be a measure space and $R \subset S \times S$ a (not necessarily countable) equivalence relation on S. By a countable section for R we mean a Borel set $T \subset S$ such that T intersects μ-almost every equivalence class and the intersection with each equivalence class is at most countable. Then $R \mid T = R \cap (T \times T)$ is a countable equivalence relation on T. A measure ν on T is called μ-compatible if for $A \subset T$ a Borel set, $\nu(A) = 0$ if and only if $\mu([A]) = 0$ where $[A]$ is the saturation of A under R, *i.e.* $[A] = \{x \in S \mid x \sim y$ for some $y \in A\}$. Such a measure ν is quasi-invariant under the countable equivalence relation $R \mid T$, and its measure class is uniquely determined by the condition of μ-compatibility. If there is a μ-compatible measure on one countable section, then one exists on every countable section and we then call μ transversally quasi-invariant. We call μ transversally invariant if the measure induced on countable sections can be chosen to be invariant for the equivalence relations on the countable sections. Again, for this to hold it suffices to see that it holds on one countable section. Furthermore, as countable equivalence relations with quasi-invariant measure, any two countable sections are stably isomorphic [9], *i.e.* have subsets of positive measure which are isomorphic as countable equivalence relations with quasi-invariant measure. We then have the following theorem, due to A. Ramsay [27] and J. Feldman, P. Hahn, and C. C. Moore [10].

Theorem (**2.1**). — *Let* G *be a locally compact group and* (S, μ) *an ergodic* G*-space. Then countable sections for* R_G *exist, and* μ *is transversally quasi-invariant.*

We remark that if μ is G-invariant and the action is locally free (*i.e.* stabilizers are discrete) then μ will be transversally invariant if and only if G is unimodular. This follows from a more precise quantitative result of C. Series [30].

Suppose now that $R \subset S \times S$ is an ergodic equivalence relation such that each equivalence class has the structure of a C^∞-manifold of a fixed dimension n. This is the case for example if $R = R_G$ where G acts locally freely on S and G is a Lie group, since each equivalence class inherits a C^∞-structure from that of G. For such an equivalence relation a countable section $T \subset S$ is called a transversal if for each equivalence class L, $T \cap L$ has no accumulation points in L and if there is a Borel isomorphism $\varphi : T \times D \to A$ where D is the open ball in $\mathbf{R}^n$, $A \subset S$ is a Borel set with $T \subset A$, such that

(i) $\varphi(t, 0) = t$;
(ii) for each $t \in T$, the map $\varphi_t : D \to S$ is a C^∞ diffeomorphism onto an open neighborhood of t in the C^∞-structure of the equivalence class of t.

Such a set A is called a flow box.

Definition **(2.2)**. — *By a measurable foliation of a measure space* (S, μ) *we mean an equivalence relation* $\mathscr{F} \subset S \times S$ *such that*

(i) μ *is transversally quasi-invariant for* $\mathscr{F}$.
(ii) *Each equivalence class (hereafter "leaf") of* $\mathscr{F}$ *has a* C^∞*-structure.*
(iii) *There exists a countable collection of flow boxes whose union contains* μ*-almost all leaves.*

Every locally free ergodic action of a Lie group defines a measurable foliation [10], [35], and every ergodic C^∞-foliation of a manifold M does as well, where μ is Lebesgue measure on M. Here the transversals can be taken to be transversal manifolds and Lebesgue measure on these manifolds will be μ-compatible. This latter remark fails if the foliation is only assumed to be continuous, but there are many natural examples of continuous foliations which will still be measurable foliations in the sense of definition (2.2) [40].

Definition **(2.3)**. — *We call two ergodic measurable foliations* $\mathscr{F}$, $\mathscr{F}'$ *transversally equivalent if there are transversals* T, T' *for* $\mathscr{F}$, $\mathscr{F}'$ *respectively which are isomorphic as countable equivalence relations with quasi-invariant measure. By the remarks preceding Theorem* (2.1), *this is equivalent to the assertion that for any transversal* T *of* $\mathscr{F}$ *and* T' *of* $\mathscr{F}'$, T *and* T' *are stably isomorphic.*

If $\mathscr{F}$ is a measurable foliation of (S, μ) then for each $s \in S$ we can assign to s the tangent space at s to the leaf of $\mathscr{F}$ through s. We can clearly speak of smooth sections of this bundle, Riemannian structures, etc., by demanding that they be smooth on the leaves and measurable over all S. The latter condition can be simply defined by, for example, pulling back from a flow box to $T \times D$ where the tangent bundle is just a product and so measurability over $T \times D$ has an obvious meaning. By a Riemannian measurable foliation we mean a measurable foliation with a Riemannian structure (smooth on leaves, measurable over S).

Example **(2.4)**. — *(a)* Consider a free ergodic action of $\mathbf{R}^n$ on (S, μ). Then each leaf of $R_{\mathbf{R}^n}$ inherits the Euclidean Riemannian structure from $\mathbf{R}^n$, and thus is a Riemannian measurable foliation in a natural way.

(b) Let G be a connected semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup. Let (S, μ) be a free ergodic G-space. Since K is compact, S/K is a standard Borel space and we let $\nu = p_*(\mu)$ where $p : S \to S/K$ is the natural map. The equivalence relation R_G on S projects to an equivalence relation $\mathscr{F}$ on S/K in which each leaf can be identified with $X = G/K$. Assigning the G-invariant metric to X, each leaf of $\mathscr{F}$ inherits this metric [38] and hence $\mathscr{F}$ is a Riemannian measurable foliation in which each leaf is isometric to the symmetric space X. (The local triviality of $\mathscr{F}$ can be verified as in [38].)

(c) Let M be a complete Riemannian manifold of finite volume, $\Gamma = \pi_1(M)$, and for simplicity suppose the universal covering $\widetilde{M}$ is diffeomorphic to Euclidean space. Then $\widetilde{M}$ is a Riemannian manifold in a natural way, and Γ acts properly discontinuously and by isometries on $\widetilde{M}$. Let (X, μ) be a free ergodic Γ-space. Form the product action of Γ on $X \times \widetilde{M}$ and let S be the quotient space $S = (X \times \widetilde{M})/\Gamma$. Since Γ acts properly discontinuously on $\widetilde{M}$, S will be a standard Borel space, and since the action on S is free, each $\{x\} \times \widetilde{M}$ projects injectively into S, and these images define an equivalence relation on S which it is easy to verify is a Riemannian measurable foliation in a natural way, each leaf being isometric to $\widetilde{M}$. There is a natural map $\varphi : S \to M$ which is a covering map on each leaf, and the fiber over the base point in M is just the Γ-space X. In fact, $\varphi^{-1}(m)$ will be a transversal for each $m \in M$ and the equivalence relation on it is clearly isomorphic to $R_\Gamma \subset X \times X$. Thus S is a "foliated bundle" over M [14].

We now recall the definition of 1-cohomology for ergodic actions and countable equivalence relations. If S is an ergodic G-space and H is a standard Borel group, a Borel function $\alpha : S \times G \to H$ is called a cocycle if for all $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$ *a.e.* Two cocycles $\alpha, \beta : S \times G \to H$ are called equivalent if there is a Borel function $\varphi : S \to H$ such that for each g, $\alpha(s, g) = \varphi(s)\beta(s, g)\varphi(sg)^{-1}$ *a.e.* The cocycle α is called an orbital cocycle if $\alpha(s, G_s) = e$ *a.e.*, where G_s is the stabilizer of s in G. Any homomorphism $\pi : G \to H$ defines a cocycle $\alpha_\pi : S \times G \to H$ by $\alpha_\pi(s, g) = \pi(g)$. Any orbit equivalence of free ergodic actions also defines a cocycle, namely, if $\theta : S \to S'$ is an orbit equivalence of a free ergodic G-space S with a free ergodic G'-space S', then $\alpha : S \times G \to G'$ given by $\theta(s).\alpha(s, g) = \theta(sg)$ is a cocycle. If $G = G'$ and α is a cocycle defined by an orbit equivalence, then $\alpha \sim \alpha_\pi$ for some inner automorphism π of G implies that the orbit equivalent actions are actually conjugate ([36], Proposition (2.4)). If $\alpha : S \times G \to H$ and Y is a (left) H-space, by an α invariant function $\varphi : S \to Y$ we mean a Borel map φ such that for all $g \in G$, $\alpha(s, g)\varphi(sg) = \varphi(s)$ *a.e.* This just means that φ is a fixed point in the space F(S, Y) of Borel maps $S \to Y$ under the α-twisted action $(g.\varphi)(s) = \alpha(s, g)\varphi(sg)$. For example, if $G = H$ and $\alpha = \alpha_{id}$ where $id : G \to G$ is the identity, then φ is α-invariant if and only if φ is a G-map (after we switch to the right action on Y).

Suppose now that R is a countable ergodic equivalence relation on S with quasi-invariant measure. If H is a group, a Borel function $\alpha : R \to H$ is a cocycle if there is a conull set $S_0 \subset S$ such that for all $x, y, z \in S_0$ with $(x,y), (y,z) \in R$, $\alpha(x,y)\alpha(y,z) = \alpha(x,z)$. Cocycles $\alpha, \beta : R \to H$ are called equivalent if there is a Borel function $\varphi : S \to H$ such that $\alpha(x,y) = \varphi(x)\beta(x,y)\varphi(y)^{-1}$ *a.e.* Clearly the notions of cocycle and equivalence on R_G, where G is a discrete group acting on S, and the notions of orbital cocycle and equivalence for the action as defined in the previous paragraph correspond. We can also speak of an α-invariant function $\varphi : S \to Y$, namely a Borel function φ such that for almost all $(x,y) \in R$, $\alpha(x,y)\varphi(y) = \varphi(x)$.

We recall that if G is a group, G is called amenable if whenever G acts by affine transformations on a compact convex set there are G-fixed points. This notion can be extended to a notion of amenable ergodic action or amenable equivalence relation using α-invariant functions in place of fixed points [31], [32]. Namely, suppose S is an ergodic G-space, E a separable Banach space, and $\alpha : S \times G \to \mathrm{Iso}(E)$ a cocycle, where Iso(E) is the group of isometric isomorphisms. Suppose for each $s \in S$ we have a compact convex set $A_s \subset E_1^*$, the latter being the unit ball in the dual of E, such that $\{(s,x) \mid x \in A_s\}$ is a Borel subset of $S \times E_1^*$, and $s \to A_s$ is α-invariant in the sense that $\alpha(s,g)A_{sg} = A_s$ *a.e.* The action of G on S is called amenable [31] if for all (E, α, A_s) as above, there is an α-invariant $\varphi : S \to E_1^*$ such that $\varphi(s) \in A_s$ *a.e.* Equivalently, we can form $A \subset L^\infty(S, E^*)$, $A = \{\varphi \mid \varphi(s) \in A_s \ a.e.\}$. Then A is a compact convex G-invariant set in $L^\infty(S, E^*) = L^1(S, E)^*$. We say that such a compact convex G-invariant set is a compact, convex set over S. Thus, while amenability of G demands a fixed point in all compact convex sets, amenability of the action demands a fixed point in all compact convex sets over S. While the definition may appear somewhat technical, this is a very natural and useful class of actions from a variety of viewpoints. (See [4], [31], [32], [33].) We summarize some basic properties. Proofs can be found in [31], [33].

Proposition **(2.5)**. — (a) *Every ergodic action of an amenable group is amenable.*

(b) *An action with finite invariant measure is amenable if and only if the acting group is amenable.*

(c) *If* $X \to Y$ *is a* G*-map of ergodic* G*-spaces and* Y *is an amenable* G*-space, so is* X.

(d) *The transitive action of* G *on* G/H *is amenable if and only if* H *is an amenable group.*

(e) *If* $\Gamma \subset G$ *is closed and* G *acts amenably on* X *then the restricted action of* Γ *on* X *is also amenable.*

Proposition **(2.6)**. — *For free ergodic actions, amenability is an invariant of orbit equivalence.*

If R is a countable equivalence relation with quasi-invariant measure, one can define amenability of R analogously [32]. Here, amenability will be an invariant of stable isomorphism. If R is a general ergodic equivalence relation with transversally quasi-invariant measure, we say that R is amenable if and only if any one (and hence by the remarks preceding Theorem (2.1), all) transversals are amenable. In particular, we may speak of amenable measurable foliations. For a free action of a group G, R_G will be amenable in this sense if and only if the action of G is amenable.

3. Boundary Theory for Measurable Foliations.

If X is a symmetric space of noncompact type, say $X = G/K$ where G is a semisimple Lie group without compact factors, and K is a maximal compact subgroup, there are a variety of ways to compactify X or obtain a "boundary" of X [3], [11], [22], [29], [5]. The action of a discrete subgroup of G on the boundary of X plays an important role in understanding some of the deep properties of the discrete subgroup. For many (but not all) purposes, the proper choice of a boundary of G/K will be the homogeneous space G/P where $P \subset G$ is a minimal parabolic subgroup. For example, if X is the hyperbolic n-ball, so that $G = SO(1, n)$, then G/P will be the boundary $n-1$ sphere [24]. If $\mathscr{F}$ is a measurable foliation of (S, μ) by symmetric spaces, one would like to consider the space obtained by adjoining a boundary to each leaf. However, in general, one cannot do this and still have a standard Borel structure on the resulting space. Instead, for each point $s \in S$, one can consider the boundary through the leaf of s, and consider the resulting "bundle of boundaries" on S. This, roughly speaking, is the approach we will follow here, not just for foliations by symmetric spaces but for foliations by complete simply connected manifolds of nonpositive curvature. If a foliation by symmetric spaces comes from a G-action on S as in Example (2.4) *(b)*, the bundle of boundaries (except for passing from S to S/K) is the product space $S \times G/P$ which of course has a natural G-action on it, and this was our "boundary object" in the proof of rigidity of ergodic actions in [36].

We first recall some facts concerning a simply connected complete manifold H of nonpositive sectional curvature, following Eberlein and O'Neill [5], [6]. As is well known, H is diffeomorphic to Euclidean space. If α and β are geodesics in H, α and β are called asymptotic if $d(\alpha(t), \beta(t))$ is bounded for $t \geq 0$. This is clearly an equivalence relation on the set of geodesics, and we let $H(\infty)$ denote the set of equivalence classes. If α is a geodesic, then $\alpha(\infty)$ denotes its equivalence class in $H(\infty)$. $H(\infty)$ is a natural boundary for H in the following sense. There is a natural topology (the "cone topology" [5]) on $\overline{H} = H \cup H(\infty)$ which makes $\overline{H}$ homeomorphic to the closed unit ball in $\mathbf{R}^n$ and $H(\infty)$ homeomorphic to the boundary sphere. The extended geodesic $\alpha : [-\infty, \infty] \to \overline{H}$ is then continuous. See [5], [6] for details of the construction and further results concerning it.

Now let $\mathscr{F}$ be a Riemannian measurable foliation of a measure space (S, μ). For $s \in S$, let $L_s \subset S$ be the leaf through s. We assume henceforth that L_s is a simply connected complete manifold of nonpositive curvature. We then have the tangent bundle $T(\mathscr{F}) = \bigcup_{s \in S} T(L_s)_s$ where $T(L_s)_s$ is the tangent space to L_s at s, and as indicated above (see also [35]), $T(\mathscr{F})$ is also a standard Borel space, and the natural map $p : T(\mathscr{F}) \to S$ is of course Borel. Let $T \subset S$ be a transversal for the foliation, and ν a μ-compatible measure on T. For each $s \in T$, let $B_s \subset T(L_s)_s$ be the open unit ball (recall $T(L_s)_s$ has an inner product assigned to it), and $\overline{B}_s$ its closure. Let

$T_0(\mathscr{F}) \subset T(\mathscr{F})$ be given by $T_0(\mathscr{F}) = p^{-1}(T)$ and let $B(\mathscr{F}) = \{v \in T_0(\mathscr{F}) \mid v \in B_{p(v)}\}$. Define $\overline{B}(\mathscr{F})$ similarly. Let $f: [0, 1] \to [0, \infty]$ be a homeomorphism with $f(0) = 0$. Let $\Psi : B(\mathscr{F}) \to S$ be given by $\Psi(v) = \exp_s(f(\|v\|_s).v)$ where $p(v) = s$ and $\exp_s : T(L)_s \to L_s$ is the exponential map with respect to the given Riemannian metric on L_s. We recall that for each s, $\exp_s$ is a diffeomorphism. Let $\Psi_s = \Psi \mid B_s$, so that $\Psi_s : B_s \to L_s$ is a diffeomorphism. For $(s, t) \in \mathscr{F} \mid T = \mathscr{F} \cap (T \times T)$, define $\beta(s, t) : B_t \to B_s$ by $\beta(s, t) = \Psi_s^{-1} \circ \Psi_t$. By [5] (Theorem (2.10)), the diffeomorphism $\beta(s, t)$ extends to a homeomorphism $\overline{B}_t \to \overline{B}_s$. $T_0(\mathscr{F})$ is a measurable bundle of finite dimensional real Hilbert spaces over the measure space T, and a standard argument allows us to measurably choose an orthonormal basis in each fiber. Equivalently, if the leaves are of dimension n, for each $s \in T$ we can choose a linear isometry $f_s : T(L_s)_s \to \mathbf{R}^n$ and this can be done measurably in s. Let $B \subset \mathbf{R}^n$ be the unit ball, $\overline{B}$ its closure. Let $\Phi : T \times B \to S$ be given by $\Phi(s, v) = \Psi(f_s^{-1}(v))$, and define $\alpha : \mathscr{F} \mid T \to \mathrm{Diff}_b(B)$ by $\alpha(s, t) = f_s \beta(s, t) f_t^{-1}$, where $\mathrm{Diff}_b(B)$ is the set of diffeomorphisms of B that extend to homeomorphisms of $\overline{B}$. We remark that $\mathrm{Diff}_b(B)$ is a standard Borel group (as is Diff(B), the group of all diffeomorphisms) and that α is a cocycle on $\mathscr{F} \mid T$. The space $T \times B$ is a measurable foliation in a natural way, where the leaves are $\{s\} \times B$, $s \in T$, and on each $\{s\} \times B$ we have a Riemannian metric given by $\omega_s = (\Psi_s \circ f_s^{-1})^*((\cdot, \cdot))_{L_s}$ where $(\cdot, \cdot)_{L_s}$ is the given Riemannian metric on L_s. We summarize this discussion in the following proposition.

Proposition **(3.1)**. — *Let $\mathscr{F}$ be a Riemannian measurable foliation of a measure space* (S, μ) *so that (almost) every leaf is a simply connected complete manifold of nonpositive curvature and dimension n. Let* $T \subset S$ *be a transversal and* ν *a* μ*-compatible measure on* T. *Let* B *be the unit ball in* $\mathbf{R}^n$ *and* $\overline{B}$ *its closure. Then viewing* $T \times B$ *as a measurable foliation by leaves* $\{s\} \times B$, *there exist:*

(i) *a Riemannian structure on* $T \times B$ *(smooth on leaves, measurable over* $T \times B$*), for which we denote* ω_s *the Riemannian structure on* B *induced by the leaf* $\{s\} \times B$;

(ii) *a smooth map* $\Phi : T \times B \to S$; *and*

(iii) *a cocycle* $\alpha : \mathscr{F} \mid T \to \mathrm{Diff}_b(B)$, *such that*

   (a) *for each* $s \in T$, $\Phi_s : B \to L_s$, $\Phi_s(v) = \Phi(s, v)$ *is an isometry of Riemannian manifolds where* B *has the metric* ω_s, *and* $\Phi_s(0) = s$;

   b) *for* $s, t \in T$, $s \sim t$, *we have* $\Phi_t = \Phi_s \circ \alpha(s, t)$, *and hence in light of* a), $\alpha(s, t)^* \omega_s = \omega_t$.

We remark that in particular we can view α as a cocycle $\mathscr{F} \mid T \to \mathrm{Homeo}(\partial B)$, and it is properties of this " boundary cocycle " on which we will be focusing. The formalism of Proposition (3.1) allows us to describe the situation in which (almost) all leaves are isometric.

Theorem **(3.2)**. — *Let $\mathscr{F}$ be as in Theorem* (3.1), *with $\mathscr{F}$ ergodic, and assume all leaves of $\mathscr{F}$ are isometric to a Riemannian manifold* X *on which the isometry group, say* G, *acts transitively.*

Then there is an ergodic G*-space* (M, μ) *such that* $\mathscr{F}$ *is isometric to the foliation on* M/K *as described in Example* (2.4) *(b), where* $K \subset G$ *is a maximal compact subgroup.*

Proof. — Let (T, ν), α be as in Proposition (3.1). Let $\mathscr{M}$ be the set of Riemannian metrics on B. Then Diff(B) acts naturally on $\mathscr{M}$. Let $\mathscr{M}_X$ be the subset of $\mathscr{M}$ for which (B, ω), $\omega \in \mathscr{M}_X$, is isometric to X. Then $\mathscr{M}_X$ is an orbit in $\mathscr{M}$ under Diff(B), and we fix an element $\omega_0 \in \mathscr{M}_X$. Our hypotheses imply that for almost all $s \in T$, $\omega_s \in \mathscr{M}_X$, where $\omega : T \to \mathscr{M}$ is as in Proposition (3.1). The Borel structure on $\mathscr{M}_X$ is standard and we have a map $\mathrm{Diff(B)} \to \mathscr{M}_X$, $\varphi \to \varphi^*(\omega_0)$. By Kallman's sharpening of the classical von Neumann selection theorem ([15], Proposition (7.1)), there is a Borel section q of this map, and we can assume $q(\omega_0) = \mathrm{id}$. Let $h : T \to \mathrm{Diff(B)}$ be $h = q \circ \omega$. Thus $h(t)^*(\omega_0) = \omega_t$. Define a cocycle $\gamma : \mathscr{F} | T \to \mathrm{Diff(B)}$ by $\gamma(s, t) = h(s)\alpha(s, t)h(t)^{-1}$. Then $\gamma(s, t)^*(\omega_0) = \omega_0$, and so $\gamma(s, t)$ all lie in the isometry group of (B, ω_0), which we identify with G. In other words $\gamma : \mathscr{F} | T \to G$ is a G-valued cocycle.

Now consider the function $f: T \to S$ given by $f(s) = \Phi_s([h(s)](s))$. Observe that $f(s) \in L_s$ for all $s \in T$. Since S can be covered by countably many flow boxes, by passing to a subset $T_1 \subset T$ of positive ν-measure, we can assume $f(T_1)$ lies in a single flow box. Let us call $\widetilde{T}$ the transversal of this flow box and p the projection of the flow box onto $\widetilde{T}$. The map $p \circ f : T_1 \to \widetilde{T}$ is countable-to-one since it preserves the equivalence relation on transversals, and hence there is a Borel set $T_0 \subset T_1$ of positive measure on which $p \circ f$ is injective. It follows that $T' = f(T_0)$ is a transversal (recall the ergodicity assumption on $\mathscr{F}$) and $f | T_0 : T_0 \to T'$ is a Borel isomorphism preserving the equivalence relation defined by the leaves. Finally, define the cocycle

$$\gamma' : \mathscr{F} | T' \to G \quad \text{by} \quad \gamma'(x,y) = \gamma(f^{-1}(x), f^{-1}(y)),$$

and a function

$$\Phi' : T' \times B \to S \quad \text{by} \quad \Phi'(x, w) = \Phi(f^{-1}(x), h(f^{-1}(x))^{-1}w),$$

so that $\Phi'_x = \Phi_{f^{-1}(x)} \circ h(f^{-1}(x))^{-1}$.

It is a straightforward unraveling of the definitions to see that:

a) $\Phi'_x : (B, \omega_0) \to L_s$ is an isometry;

b) $\Phi'_x(0) = x$;

c) for $x, y \in T'$, $x \sim y$, $\Phi'_y = \Phi'_x \circ \gamma'(x,y)$.

In other words, we have the same conclusion as in Proposition (3.1) except that now the cocycle in question takes values not in $\mathrm{Diff}_b(B)$, but in G, the isometry group of (B, ω_0).

As is well known, G is a Lie group and the stabilizer of a point is a maximal compact subgroup K. We claim that for a fixed $x \in T'$, $\{\gamma(x,y) \mid y \in T', y \sim x\}$ has no accumulation points in G. This follows by observing that from *b)* and *c)* above, $y = \Phi'_y(0) = \Phi'_x(\gamma'(x,y)(0))$. Since T' is a transversal, $L_x \cap T'$ has no accumulation points in L_x, and hence $(\Phi'_x)^{-1}(L_x \cap T')$ has no accumulation points in B. It follows that $\{\gamma'(x,y)(0) \mid y \sim x\}$ has no accumulation points in B, verifying our assertion. This enables us to construct the required ergodic action of G using the Mackey range construction [18], [27]. Namely, consider the equivalence relation on $T' \times G$ given by $(x, g) \sim (y, h)$ if and only if $x \sim y$ and $h = (\gamma'(x,y))^{-1}g$. Let M be the space of equivalence classes in $T' \times G$ under this equivalence relation. By the observation on nonexistence of accumulation points for γ in G, M is a standard Borel space. Let $p : T' \times G \to M$ be the natural map, and $m = p_*(\nu' \times \lambda)$ where ν' is a μ-compatible probability measure on T' and λ is a probability measure on G in the class of Haar measure. G acts on $T' \times G$ by $(x, g).h = (x, gh)$ and this action permutes the equivalence classes of the above relation. Thus there is an induced action of G on M, and it is not hard to see that (M, m) is an ergodic G-space on which G acts freely.

We now claim that the foliation on M/K as described in Example (2.4) *(b)* is isometric to $\mathcal{F}$. Here we take K to be the stabilizer of $0 \in B$ in G. Let $\theta : T' \times G \to S$ be given by $\theta(x, g) = \Phi'(x, g.0)$. Then

$$\begin{aligned}\theta(y, \gamma'(x,y)^{-1}g) &= \Phi'_y \circ \gamma'(x,y)^{-1}(g.0)\\ &= \Phi'_x(g.0) = \theta(x, g).\end{aligned}$$

Thus θ induces a map $M \to S$ which clearly factors to a map $M/K \to S$. It is straightforward to check that this is an isometry of the Riemannian measurable foliations of these spaces, and this completes the proof.

The geometric version of the rigidity theorem for ergodic actions [36] concerns foliations by symmetric spaces $X = G/K$ of the form S/K where S is an ergodic G-space and G is a centerfree semisimple Lie group. By Theorem (3.2), any Riemannian measurable foliation for which the leaves are isometric to X is of the form S'/K' where S' is an ergodic G'-space, $S' = \mathrm{Iso}(X)$. It is well known that G is a subgroup of finite index G'. It follows that any foliation with leaves isometric to X has a finite extension that comes from a G-action, where finite extension is taken in the following sense.

Definition **(3.3)**. — *Suppose* ($\mathcal{F}$, M) *and* ($\mathcal{F}'$, M′) *are ergodic Riemannian measurable foliations. We say that* ($\mathcal{F}'$, M′) *is a finite Riemannian extension of* ($\mathcal{F}$, M) *if there is a finite-to-one smooth map of measurable foliations* $M' \to M$ *that maps each leaf of* $\mathcal{F}'$ *isometrically onto a leaf of* $\mathcal{F}$.

Example **(3.4)**. — If S, S′ are free ergodic G-spaces and S′ is a finite extension of S (in the sense that there is a finite-to-one measure class preserving G-map $S' \to S$), then the foliation $\mathcal{F}'$ on S′/K is a finite Riemannian extension of the foliation $\mathcal{F}$ on S/K.

Corollary **(3.5)**. — *Suppose* G *is a connected semisimple Lie group with trivial center,* $K \subset G$ *a maximal compact subgroup,* $X = G/K$, *and* $\mathcal{F}$ *a Riemannian measurable foliation in which (almost) every leaf is isometric to* X. *Then* $\mathcal{F}$ *has a finite Riemannian extension of the form* S/K *where* S *is an essentially free ergodic* G-*space.*

Proof. — Let G' be the group of isometries of X. By Theorem (3.2), we can assume $\mathcal{F}$ is of the form $(\mathcal{F}, S'/K')$ where S' is an ergodic G'-space. Restrict the G'-action to G, and consider S'/K. We clearly have a finite-to-one map $S'/K \to S'/K'$, and S'/K has a foliation on it which is a finite extension of $\mathcal{F}$. The G-action on S' need not be ergodic; however there are only finitely many ergodic components. To see this, we simply observe that G'/G must act ergodically, and hence transitively, on the space of G-ergodic components. Let $E \subset S'$ be one of these ergodic components. It clearly suffices to see that the image of E in S'/K' contains almost all leaves. However $GK' = G'$ and hence K' acts transitively on the space of G-ergodic components in S, and it is easy to see that this suffices.

Remark. — With $\mathcal{F}$ as in Theorem (3.5), we call $\mathcal{F}$ irreducible if the G-action on S can be taken to be irreducible.

We now return to the general situation in Proposition (3.1) of a foliation by simply connected manifolds of nonpositive curvature. We shall also assume that we are in the situation of a transversally invariant measure, rather than just a transversally quasi-invariant measure. In this case, fixing compatible invariant measures on the transversals (in the ergodic case this is unique up to a scalar multiple), we can define a measure on S as follows. If $B \cong T \times D^n$ is a flow box, then on each $\{s\} \times D^n$ there is a volume form defined by the Riemannian metric on L_s, and this defines a measure μ_s on $\{s\} \times D^n$. If ν is the compatible invariant measure on T, we have a measure μ_B on B given by $\mu_B = \int \mu_s \, d\nu$. By transversal invariance, if we are given two flow boxes B, B', then $\mu_B = \mu_{B'}$ on $B \cap B'$. Thus $\{\mu_B\}$ piece together to define a σ-finite measure μ on S (having the same saturated null sets as the original measure on S). This construction is in the same spirit as the Ruelle-Sullivan current [28].

Definition **(3.6)**. — *If $\mathcal{F}$ is a Riemannian measurable foliation of* S *with transversally invariant measure, we say that* $(\mathcal{F}, S)$ *has finite volume if the measure μ constructed above satisfies* $\mu(S) < \infty$. *We can then assume* $\mu(S) = 1$, *and we then call μ the canonical measure on* S.

If G is a semisimple Lie group, (S, μ) a free ergodic G-space with $\mu(S) = 1$, and $\mathcal{F}$ the foliation of S/K by symmetric spaces, then the canonical measure on S/K is just $p_*(\mu)$ where $p: S \to S/K$ is projection. This follows, for example, from Series' local description of an invariant measure [30]. We also remark that a finite Riemannian extension of a Riemannian measurable foliation with finite volume also has finite volume.

We shall now examine the behavior of the boundary cocycle $\alpha : \mathcal{F} \mid T \to \mathrm{Homeo}(\partial B)$ given by Proposition (3.1). Let $M(\partial B)$ be the space of probability measures on ∂B, so that $\mathrm{Homeo}(\partial B)$ acts on $M(\partial B)$. We recall that a function $\mu : T \to M(\partial B)$ is called α-invariant if for almost all $s, t \in T$ with $s \sim t$, $\alpha(s,t)_*(\mu_t) = \mu_s$. The main result we need concerning behavior of α acting on $M(\partial B)$ is the following. Most of the remainder of this section will be devoted to its proof.

Theorem (**3.7**). — *Let $\mathcal{F}$ be an ergodic Riemannian measurable foliation of* S *by simply connected complete manifolds such that the sectional curvature* K *of the leaves satisfies* $K \leq c < 0$ *for some c. Assume $\mathcal{F}$ has a transversally invariant measure and finite total volume. Let* T, α *as in Proposition* (3.1). *Then for any ergodic subrelation* $R \subset \mathcal{F} \mid T$ *and* $(\alpha \mid R)$*-invariant function* $\mu : T \to M(\partial B)$, *we have* μ_s *is supported on at most two points for almost all s.*

For the proof, we shall find it convenient to introduce an analogue of the classical notion of limit set [5].

Definition (**3.8**). — *We use the notation of Proposition* (3.1). *Fix* $p \in B$. *If* $b \in \partial B$, *we say that* $b \in L_p(\alpha)$, *the limit set of* α *with respect to p, if for any* $A \subset T$, $\nu(A) > 0$, *for almost all* $s \in A$ *there is a sequence* $t_n \in A$, $s \sim t_n$, *such that* $\alpha(s, t_n)p \to b$.

Proposition (**3.9**). — *If* $p, q \in B$, *then* $L_p(\alpha) = L_q(\alpha)$. *Thus we can speak of* $L(\alpha) = L_p(\alpha)$, *the limit set of* α.

Proof. — For $s \in T$, let $d_s(\cdot, \cdot)$ be the metric on B defined by the Riemannian metric on $\{s\} \times B$. Given A, $\nu(A) > 0$, we can write $A = \bigcup A_n$, $\mu(A_n) > 0$, such that $d_s(p, q)$ is bounded for $s \in A_n$. Suppose $b \in L_p(\alpha)$. Then for almost all $s \in A_n$, we can find $t_j \in A_n$ such that $s \sim t_j$ and $\alpha(s, t_j)p \to b$. Now $d_s(\alpha(s, t_j)p, \alpha(s, t_j)q) = d_{t_j}(p, q)$ (by conclusion (b) in Proposition (3.1)) which is bounded. Therefore by the law of cosines in (B, ω_s) (see [6], top of p. 496), $\alpha(s, t_j)q \to b$. Thus $L_p(\alpha) \subset L_q(\alpha)$, and the reverse inclusion follows similarly.

The argument of the above proposition also shows the following technical fact that we shall need.

Lemma (**3.10**). — *Suppose* $p : T \to B$ *is Borel such that* $d_s(p(s), 0)$ *is bounded. Then* $\alpha(s, t_j)(0) \to b$ *if and only if* $\alpha(s, t_j)(p(t_j)) \to b$.

The following is basic.

Theorem (**3.11**). — *Suppose $\mathcal{F}$ has transversally invariant measure and finite volume. Then* $L(\alpha) = \partial B$.

To prove Theorem (3.11) we need to use the geodesic flow on $\mathcal{F}$. Namely, let $T_1(\mathcal{F})$ be the unit tangent bundle of $T(\mathcal{F})$. Then the geodesic flow of $\mathcal{F}$ is a well defined **R**-action on $T_1(\mathcal{F})$.

Lemma (**3.12**). — *The geodesic flow on* $T_1(\mathcal{F})$ *is finite measure preserving, and hence recurrent* [12]. *There are no periodic points under this flow. Moreover as* $t \to \infty$, *a geodesic leaves all compact subsets of a leaf.*

The fact that the geodesic flow is measure preserving follows as in the case of the geodesic flow on a manifold [40]. The last two statements are clear.

Proof of Theorem (3.11). — We first remark that by passing to a subset of T, we may assume that there is an open ball $D \subset B$ centered at o such that $\Phi \mid T \times D : T \times D \to S$ is a flow box. We may further assume (again by passing to a subset) that $d_t(o, \partial D)$ is bounded for $t \in T$. Take $p = o \in B$ and let $b \in \partial B$. For each $s \in T \times D$, $s = (t, y)$, there is a unique geodesic $\varphi_s(a)$, $a \in \mathbf{R}$, in the Riemannian manifold (B, ω_t) such that $\varphi_s(0) = y$ and $\varphi_s(\infty) = b$ [5], [6]. Let $F_a(s, v)$, $s \in S$, $v \in T(L_s)_s$, $a \in \mathbf{R}$, represent the geodesic flow. For $s \in T \times D$, let $v_s = \varphi_s'(0)$ and

$$(\mathcal{O}_n)_s = \{v \in T(B)_y \mid \|v\| = 1 \text{ and } \|v_s - v\| < 1/n\}.$$

We can consider $\Phi : T \times D \to S$ as a map on the tangent bundles of these measurable foliations as well. Let D_n be a decreasing sequence of open balls in D such that $\bigcap D_n = \{0\}$, and let $A \subset T$ have positive measure. Now for each n apply recurrence of the geodesic flow to the set

$$\Phi\Big(\bigcup_{s \in A \times D_n} \{(s, v) \mid v \in (\mathcal{O}_n)_s\}\Big).$$

(We recall that recurrence of an $\mathbf{R}$-action means that for any set of positive measure W, for almost all $z \in W$ there exist $a_n \in \mathbf{R}$, $a_n \to \infty$ such that $z.a_n \in W$. Here we can in fact suppose a_n are integers.) Thus, for almost all $t \in A$, we can find $p_n(t), q_n(t) \in D_n$, $t_n \sim t$, $t_n \in A$, $w_n(t) \in (\mathcal{O}_n)_{(t, p_n(t))}$, $z_n(t) \in (\mathcal{O}_n)_{(t_n, q_n(t))}$, and a sequence $a_n(t)$ of positive integers with $a_n(t) \to \infty$ as $n \to \infty$, such that $F_{a_n(t)}(\Phi(t, p_n(t), w_n(t))) = \Phi(t_n, q_n(t), z_n(t))$. Thus we have $F_{a_n(t)} \circ \Phi_t(p_n(t), w_n(t)) = \Phi_{t_n}(q_n(t), z_n(t))$. Since Φ_t is an isometry, we have, letting F_a^t represent the geodesic flow in (B, ω_t)

$$\Phi_t(F^t_{a_n(t)}(p_n(t), w_n(t))) = [\Phi_t \circ \alpha(t, t_n)](q_n(t), z_n(t))$$

(using b) of Proposition (3.1)). Hence

$$F^t_{a_n(t)}(p_n(t), w_n(t)) = \alpha(t, t_n)(q_n(t), z_n(t)).$$

As $n \to \infty$, we have $p_n(t) \to 0$, $w_n(t) \to v_{(t,0)}$, $a_n(t) \to \infty$, and hence by [5] (Proposition (2.13)), the projection of $F^t_{a_n(t)}(p_n(t), w_n(t))$ to B converges to b. Thus $\lim_{n\to\infty} \alpha(t, t_n)(q_n(t)) = b$. Since $d_t(q_n(t), p)$ is uniformly bounded over t, n it follows from Lemma (3.10) that $\alpha(t, t_n)p \to b$. Therefore $b \in L(\alpha)$, completing the proof of Theorem (3.11).

We now turn to the proof of Theorem (3.7).

Proof of Theorem (3.7). — For each $s \in T$ and points $b_1, b_2 \in \overline{B} - \{0\}$ there are unique geodesics in (B, ω_s), φ_1, φ_2, with $\varphi_i(0) = 0$ and $\varphi_i(a_i) = b_i$, $a_i \in (0, \infty]$ [5]. Let $A^s(b_1, b_2)$ be the angle formed by $\varphi_i'(0)$. Let $\widetilde{\partial B}$ be a compact neighborhood of ∂B in $\overline{B}$ with $0 \notin \widetilde{\partial B}$. Then $A^s : \widetilde{\partial B} \times \widetilde{\partial B} \to [0, \pi]$ is a continuous function [5] (p. 55), and since $s \to \omega_s$ is measurable, the map $T \to C(\widetilde{\partial B} \times \widetilde{\partial B}; \mathbf{R})$, $s \to A^s$ is measurable. Similarly, for any $y \in B$ we can form the functions A^s_y where geodesics and angles are based at y instead of o. We also have a measurable map $T \to M(\partial B)$, $s \to \mu_s$. By the ergodicity of the subrelation $R \subset \mathscr{F} \mid T$, measurability of A^s and μ_s, and compactness of $\overline{B}$, it follows that for almost all $s \in T$, there is a sequence $t_n \in T$ of distinct points, $(s, t_n) \in R$, such that

(i) $\mu_{t_n} \to \mu_s$;
(ii) $A^{t_n} \to A^s$ uniformly on $\widetilde{\partial B} \times \widetilde{\partial B}$; and
(iii) $\alpha(s, t_n)(0) \to p$, $\alpha(s, t_n)^{-1}(0) \to q$ for some $p, q \in \partial B$.

Following the notation of [5], p. 63, if $W \subset \overline{B}$, we denote by

$$A^s(W) = \sup\{A^s(b_1, b_2) \mid b_i \in W,\ b_i \neq 0\}$$

and
$$A^s(b, W) = \sup\{A^s(b, b_1) \mid b_1 \in W,\ b_1 \neq 0\}$$

(assuming of course $b \neq 0$, $W \neq \{0\}$). Suppose now that $V \subset \partial B$ is a compact subset with $p \notin V$. We have, for $s \sim t_n$, $A^{t_n}(\alpha(s, t_n)^{-1}V) = A^s_{\alpha(s,t_n)(0)}(V)$ by the isometric nature of $\alpha(s, t_n)$, and since $\alpha(s, t_n)(0) \to p$, we have by [5], Proposition (4.7) that $A^{t_n}(\alpha(s, t_n)^{-1}V) \to 0$, as $n \to \infty$. We can write

$$A^s(\alpha(s, t_n)^{-1}V) = A^{t_n}(\alpha(s, t_n)^{-1}V) + [A^s(\alpha(s, t_n)^{-1}V) - A^{t_n}(\alpha(s, t_n)^{-1}V)],$$

and since $A^{t_n} \to A^s$ uniformly on $\partial B \times \partial B$, we have $A^s(\alpha(s, t_n)^{-1}V) \to 0$. Passing to a subsequence, we can assume the sets $\alpha(s, t_n)^{-1}V$ converge, and since $A^s(\alpha(s, t_n)^{-1}V) \to 0$, $\alpha(s, t_n)^{-1}V$ converges to a point, say $y \in \partial B$. We also have

$$A^{t_n}(\alpha(s, t_n)^{-1}(0), \alpha(s, t_n)^{-1}V) \to A^s(q, y)$$

since $A^{t_n} \to A^s$ uniformly on $\widetilde{\partial B} \times \widetilde{\partial B}$, and $\alpha(s, t_n)^{-1}(0) \in \widetilde{\partial B}$ for n sufficiently large. However, $A^{t_n}(\alpha(s, t_n)^{-1}(0), \alpha(s, t_n)^{-1}V) = A^s_{\alpha(s,t_n)(0)}(0, V)$, and again by [5] (Proposition (4.7)), this converges to 0. Thus $A^s(q, y) = 0$, and this implies $y = q$.

Let $\varepsilon > 0$ and B_ε the closed ε-ball in ∂B around q (with any suitable metric on ∂B). Let $f_\varepsilon : \partial B \to [0, 1]$ with $f_\varepsilon = 1$ on $B_{\varepsilon/2}$ and $f_\varepsilon = 0$ outside B_ε. Then for n sufficiently large $\chi_{\alpha(s,t_n)^{-1}V} \leq f_\varepsilon$, so

$$\int f_\varepsilon\, d\mu_{t_n} \geq \mu_{t_n}(\alpha(s, t_n)^{-1}V) = \mu_s(V),$$

by $\alpha \mid R$-invariance of μ. But $\mu_{t_n} \to \mu_s$, so $\int f_\varepsilon\, d\mu_s \geq \mu_s(V)$. Hence $\mu_s(B_\varepsilon) \geq \mu_s(V)$. Summarizing, we have the following stiuation. There are points $p, q \in \partial B$ (possibly equal) so that for every compact set $V \subset \partial B$ with $p \notin V$, and every $\varepsilon > 0$, we have $\mu_s(B_\varepsilon) \geq \mu_s(V)$. It is then a straightforward exercise to show that μ_s is supported on $\{p, q\}$, and this completes the proof of Theorem (3.7).

4. Intermediate Subalgebras for Actions of Semisimple Lie Groups: An Extension of a Theorem of Margulis.

In the previous section we exhibited a type of boundary behavior for foliations by manifolds of negative curvature bounded away from 0. In this section we prove some results concerning the boundary behavior of a foliation by symmetric spaces of higher rank. In Section 5 we will exhibit the incompatibility of these types of behavior with a transversal equivalence. The result we prove in this section is an extension of the following result of G. A. Margulis. Let G be a semisimple Lie group of **R**-rank at least 2, and $\Gamma \subset G$ an irreducible lattice. Let $P \subset G$ be a minimal parabolic subgroup. If P′ is another parabolic subgroup, then there is a measure class preserving Γ-map $G/P \to G/P'$. Margulis proves in [21] that every measurable Γ-space factor of G/P is of this form. The generalization we need is the following. Let S be an irreducible ergodic G-space. (We recall that irreducibility asserts that H is ergodic on S for every noncentral normal subgroup $H \subset G$.) Then we have a measure class preserving G-map $S \times G/P \to S$. If $P' \subset G$ is another parabolic subgroup, we clearly have an intermediate G-space, *i.e.* we have measure class preserving G-maps $S \times G/P \to S \times G/P' \to S$. Our extension of Margulis' result is that these are the only intermediate factors.

Theorem (**4.1**). — *Let* G *be a connected semisimple Lie group with no compact factors, trivial center, and* **R**-*rank* $(G) \geq 2$. *Let* X *be an irreducible ergodic* G-*space with finite invariant measure. If* Y *is an ergodic* G-*space for which there exist measure class preserving* G-*maps* $X \times G/P \to Y \to X$ *whose composition is the projection, then there is a parabolic subgroup* $P' \subset G$ *such that* Y *is isomorphic as a* G-*space to* $X \times G/P'$ *in such a way that the maps* $X \times G/P \to Y$ *and* $Y \to X$ *are identified with the natural ones (modulo null sets, of course).*

The proof of this theorem relies heavily on Margulis' arguments. In fact, when $X = G/\Gamma$, Theorem (4.1) is easily seen to be equivalent to Margulis' theorem.

Before beginning the proof, we recall some of the structure of parabolic subgroups, for ease of reference following in large measure Margulis' notation in [21]. We shall also write our actions on the left for the course of this proof. Let S be a maximal **R**-split Abelian subgroup in G. Then one has the associated standard parabolic subgroups with respect to a choice of ordering on the set of roots of G relative to S, and every parabolic subgroup is conjugate to a standard one. If P_0 is any standard parabolic, let $\overline{P}_0$ be the opposite parabolic and V_0, $\overline{V}_0$ be the unipotent radicals of P_0 and $\overline{P}_0$ respectively. If R_0 is the reductive Levi component of P_0, it is also the Levi component of $\overline{P}_0$ and we have $P_0 = R_0 \ltimes V_0$, $\overline{P}_0 = R_0 \ltimes \overline{V}_0$. The natural map $G \to G/P_0$ takes $\overline{V}_0$ diffeomorphically onto an open subset of G/P_0 of full measure, and thus for many purposes involving measure theory on G/P_0, we can view $G/P_0 \approx \overline{V}_0$ as measure spaces. If P is the minimal parabolic, we have a natural map $G/P \to G/P_0$, and we want to interpret this map in terms of $\overline{V}$ and $\overline{V}_0$. We have $\overline{P} \subset \overline{P}_0 = R_0 \ltimes \overline{V}_0$, and $\overline{V}_0 \subset \overline{V}$. Let $\overline{L}_0 = R_0 \cap \overline{V} = P_0 \cap \overline{V}$. We then have $\overline{V} = \overline{L}_0 \ltimes \overline{V}_0$, and the natural map $G/P \to G/P_0$ is identified modulo null sets with the projection map $\overline{V} \to \overline{V}_0$. Margulis' argument depends upon choosing an element $s \in S$ such that $\overline{V}$ behaves nicely under conjugation by s, where " nicely " refers to the decomposition $\overline{V} = \overline{L}_0 \ltimes V_0$.

More precisely, Margulis needs $s \in S$ such that conjugation by s leaves $\overline{L}_0$ pointwise fixed and contracts $\overline{V}_0$. So given P_0, we let S_0 be the subgroup $S_0 = Z(R_0)$, the center of R_0, and let S_0' (denoted by R_0 in [21]) be $S_0' = \{s \in S_0 \mid \mathrm{Int}(s) \text{ contracts } V_0 \text{ and } \mathrm{Int}(s)^{-1}$ contracts $\overline{V}_0\}$ where $\mathrm{Int}(s)$ is conjugation by s. (By contraction we mean that for any open neighborhood of the identity and any compact set, sufficiently high powers of the automorphism all bring the compact set within the given neighborhood.) If $P_0 = G$, then of course S_0 is trivial. On the other hand, if $P_0 \neq G$, then it is not hard to see that $S_0' \neq \varnothing$. Margulis (and we as well) need $S_0' \neq \varnothing$ for parabolics P_0 which are minimal among those containing P but not equal to P. This accounts for the assumption that $\mathbf{R}$-rank $(G) \geq 2$, for of course we have $\mathbf{R}$-rank $(G) = 1$ if and only if such a P_0 is in fact G.

Still following Margulis' notation, if $C \subset \overline{V}$, we let $\psi_0(C) = \overline{V}_0 . (C \cap \overline{L}_0)$. We then have the following important lemma of Margulis.

Lemma **(4.2)** (Margulis [21] (Lemma (1.4.2))). — *If* $s \in S_0'$ *and* $C \subset \overline{V}$ *is measurable, then for almost all* $u \in \overline{V}$, *the sequence* $\{s^n u C s^{-n}\}_{n \geq 0}$ *converges in measure to* $\psi_0(uC)$.

In [21], Margulis then uses ergodicity of the integer action defined by powers of s, via the consequence that for almost all $u \in \overline{V}$, $\{\Gamma u s^{-n}\}_{n \geq 0}$ is dense in G, where $\Gamma \subset G$ is an irreducible lattice ([21], Lemma (1.9)), to deduce the following ([21], Lemma (1.14.1)). If $B(G/P)$ is the measure algebra of G/P and $B \subset B(G/P)$ is a Γ-invariant σ-subalgebra, and $C \in B$, then for almost all $u \in \overline{V}$, $g.\psi_0(uC) \in B$ for all $g \in G$.

Here we have identified $B(G/P)$ with $B(\overline{V})$, the measure algebra of $\overline{V}$. We are again assuming $S_0' \neq \varnothing$. We will need an analogous result in our situation which we now formulate.

We have maps $X \times G/P \to Y \to X$, and passing to measure algebras we have $B(X) \subset B(Y) \subset B(X \times G/P)$. Let $B = B(Y)$, so that B is a G-invariant sub-Boolean-σ-algebra of $B(X \times G/P)$ which contains $B(X)$. Thus a standard argument concerning direct integrals of Abelian von Neumann algebras shows that we have $B = \int^{\oplus} B_x$, where $B_x \subset B(G/P)$ is a sub-Boolean-σ-algebra, and the $\{B_x\}$ are uniquely determined up to null sets in X. G-invariance of B means that for each $g \in G$, $g.B_x = B_{gx}$ for almost all x. By a standard technical result [17] we can choose B_x such that on a fixed conull set of x, $g.B_x = B_{gx}$ for x in the conull set.

Lemma **(4.3)**. — *Assume* X *is an irreducible ergodic* G*-space,* $P_0 \supset P$ *a parabolic with* $S_0' \neq \varnothing$. *Identify* $\overline{V}$ *with* G/P *as measure spaces. Let* $C \in B$, $C = \int^{\oplus} C_x$. *Then for almost all* $(x, u) \in X \times \overline{V}$, *we have* $g.\psi_0(uC_x) \in B_x$ *for all* $g \in G$.

Proof. — We begin by recalling that we can identify $B(G/P)$ as a closed subset of $L^{\infty}(G/P) = L^1(G/P)^*$ with the weak-*-topology. Thus $B(G/P)$ is a compact metrizable space, say with metric d. Let ρ be the Hausdorf metric on $\mathscr{C}$, the set of closed subsets of $B(G/P)$, so that $(\mathscr{C}, \rho)$ is a compact metric space. Since $X \to \mathscr{C}$, $x \to B_x$ is measurable, for each positive integer N we can write $X = \bigcup_i X_i^N$ (a finite union) such that for $x, y \in X_i^N$, $\rho(B_x, B_y) < 1/N$. Let $s \in S_0'$. For each N, i, let $Z_i^N = \{(x, g) \in X_i^N \times G \mid \{\alpha g s^{-n} \mid n \geq 0,\ \alpha \in G \text{ with } \alpha . x \in X_i^N\} \text{ is dense in } G\}$. The action of $G \times Z$ on $X \times G$ given by $(\alpha, n).(x, g) = (\alpha.x, \alpha g s^{-n})$ is easily seen to be ergodic owing to the ergodicity of $\{s^n\}$ on X, which in turn follows from Moore's ergodicity theorem [23] (cf. [34], Theorem (5.4)). From this we deduce that Z_i^N is conull in $X_i^N \times G$, and hence that $Z^N = \bigcup_i Z_i^N$ is conull in $X \times G$. (We remark that we can assume $n \geq 0$ in the definition of Z_i^N since for an ergodic integer action with finite invariant measure, one in fact has ergodicity of Z^+.)

Now let $U^N = Z^N \cap X \times \overline{V}$. We claim that U^N is conull in $X \times \overline{V}$ where the latter has the measure $\mu \times \mu_{\overline{V}}$, $\mu_{\overline{V}}$ being Haar measure on $\overline{V}$. If $x \in X$ with $(x, g) \in Z^N$ for almost all g, then for almost all $u \in \overline{V}$, we have $(x, up) \in Z^N$ for almost all $p \in P$. But the argument of [21], Lemma (1.9) shows that for all x, $gP \cap \{h \in G \mid (x, h) \in Z^N\}$ is a closed set. It follows that U^N is conull in $X \times \overline{V}$. Let $U = \bigcap_N U^N$, so that U is also conull in $X \times \overline{V}$.

We have identified G/P with $\overline{V}$, and thus, as in [21], (1.14), we have a corresponding action of G on $\overline{V}$, $(g, u) \to g \circ u$. Under this action we have $s \circ u = sus^{-1}$ for $s \in S$, and $u' \circ u = u'u$ for $u' \in \overline{V}$. Identifying $B(G/P)$ with $B(\overline{V})$ and B_x as a subalgebra of $B(\overline{V})$, G invariance implies that for almost all $x \in X$, $B_{\alpha.x} = \alpha \circ B_x$ for almost all $\alpha \in G$. Now let $C = \int^{\oplus} C_x$, $C_x \in B_x$, and fix $g \in G$. Let

$$W = \{(x, u) \in X \times \overline{V} \mid s^n u C_x s^{-n} \text{ converges in measure } \mu_{\overline{V}} \text{ to } \psi_0(uC_x)\},$$

which by Lemma (4.2) is conull. If $(x, u) \in U \cap W$, we can choose a sequence $\alpha_n \in G$ and a sequence of positive integers $n_j \to \infty$ such that

(i) $\alpha_n u s^{-n} \to g$, and
(ii) $\rho(B_x, B_{\alpha_n . x}) \to 0$.

Let $g_n = \alpha_n u s^{-n}$, so that $\alpha_n = g_n s^n u$, and $g_n \to g$. For almost all x we have $\alpha_n \circ C_n \in B_{\alpha_n . x}$ by invariance of $\int^{\oplus} B_x$. But

$$\alpha_n \circ C_x = (g_n s^n u) \circ C_x = g_n \circ (s^n u C_x s^{-n}) \to g \circ \psi_0(uC_x) \quad \text{since } (x, u) \in W.$$

But we also have by (ii) that $d(\alpha_n \circ C_x, B_x) \to 0$. Since $\alpha_n \circ C_x$ converges, the limit must be in B_x. This proves the lemma.

We are now ready to prove Theorem (4.1).

Proof of Theorem (4.1). — The theorem is equivalent to the following assertion: If B is a G-invariant sub-Boolean-σ-algebra, $B(X) \subset B \subset B(X \times G/P)$, then $B = B(X \times G/P_0)$

for some parabolic P_0. Let B_0 be a maximal Boolean σ-algebra in $B(X \times G/P_0)$ which satisfies

(i) $B_0 = B(X \times G/P_1)$ for some parabolic P_1, and
(ii) $B(X) \subset B_0 \subset B$.

We want to show that $B_0 = B$, and suppose to the contrary. Let $C \subset B - B_0$ and write $C = \int^{\oplus} C_x$. Then for x in a set of positive measure, we have $C_x \notin B(G/P_1)$. Arguing as in [21], Theorem (1.14.2), we note that P_1 is generated by P and $\overline{L}_i$ where as above, $\overline{L}_i = P_i \cap \overline{V}$, and P_i runs over the standard parabolics with $P \subsetneqq P_i \subset P_1$, and there are no parabolics between P and P_i. It follows that for some parabolic P_0, $P \subsetneqq P_0 \subset P_1$, with no parabolics between P and P_0, we have for x in a set of positive measure that $C_x \notin B(G/P_0)$. Once again, identifying $B(G/P) \cong B(\overline{V})$, the sub-$\sigma$-algebra $B(G/P_0) \subset B(G/P)$ is identified with $p^*(B(\overline{V}_0))$ where $p : \overline{V} = \overline{L}_0 \ltimes \overline{V}_0 \to \overline{V}_0$ is projection. Thus for any x with $C_x \notin p^*(B(\overline{V}_0))$, we have for a set of $u \in \overline{V}$ of positive measure that $\overline{L}_0 \cap uC_x$ is neither null nor conull in $\overline{L}_0$. From this remark, Lemma (4.3), and Fubini's theorem, we deduce that there exists $u \in \overline{V}$ such that

(i) for x in a set of positive measure, $\overline{L}_0 \cap uC_x$ is neither null nor conull in $\overline{L}_0$; and
(ii) for almost all x, $g \circ \psi_0(uC_x) \in B_x$ for all $g \in G$.

We observe that (i) implies $\psi_0(uC_x) \notin B(G/P_0)$, and in particular, $\psi_0(uC_x) \notin B(G/P_1)$. Fix $u \in \overline{V}$ satisfying (i) and (ii).

Let $B_1(x)$ be the Boolean σ-algebra in $B(\overline{V}) = B(G/P)$ generated by $B(G/P_1)$ and $\{g \circ \psi_0(uC_x) \mid g \in G\}$. Then by Lemma (4.3), $B_1(x) \subset B_x$ *a.e.*, and by condition (i), $B_1(x) \supsetneqq B(G/P_1)$. However, $B_1(x)$ is clearly a G-invariant subalgebra of $B(G/P)$, and (using the fact that all G-invariant subalgebras of $B(G/P)$ are of the form $B(G/H)$ for some H, and finiteness of the number of conjugacy classes of parabolics), this implies that for a set of x of positive measure, $B_1(x) = B(G/P_2)$ where P_2 is a parabolic with $P \subset P_2 \subset P_1$ and $P_2 \neq P_1$. Thus we have $B(G/P_2) \subset B_x$ for x in a set of positive measure, and since $g.B_x = B_{gx}$, ergodicity of G on X implies that $B(G/P_2) \subset B_x$ *a.e.* But this contradicts the maximality assumption on $B(X \times G/P_1)$, completing the proof of the theorem.

We conclude this section with an observation concerning measures on G/P_0 invariant under certain subgroups of G where $P_0 \subset G$ is a (proper) parabolic subgroup. Let $A \subset G$ a noncompact Abelian group. By amenability of A, A fixes a point $\lambda \in M(G/P_0)$, the latter being, as above, the space of probability measures on G/P_0. We now wish to observe that if $\mathbf{R}\text{-rank}(G) \geq 2$, we can choose A and λ so that λ is nonatomic, and so that a further technical condition is satisfied.

Proposition **(4.4)**. — *Let* $\mathbf{R}$-rank(G) ≥ 2, *and* X *an irreducible ergodic* G-*space with finite invariant measure. Let* $P_0 \subset G$ *a proper parabolic subgroup, and* $Y \subset X \times G/P_0$ *a*

$(\mu \times \nu)$*-conull set, where* μ *is the given measure on* X *and* ν *is quasi-invariant on* G/P_0. *Then there is a noncompact Abelian subgroup* $A \subset G$ *and a nonatomic probability measure* λ *on* G/P_0 *such that* (i) λ *is* A*-invariant; and* (ii) *for almost all* $x \in X$, $\lambda(Y_x) = 1$ *where* $Y_x = \{y \in G/P_0 \mid (x, y) \in Y\}$.

Proof. — Since $\mathbf{R}$-rank$(G) \geq 2$, it is easy to see that there is a nontrivial subgroup $A \subset S$ (where S is as above, a maximal $\mathbf{R}$-split Abelian subgroup of P) such that $C(A)/C(A) \cap P_0$ is of positive dimension, where C(A) is the centralizer of A in G. Since A leaves $[C(A)/C(A) \cap P_0] \subset G/P_0$ pointwise fixed, were it not for condition (ii) we could simply choose λ to be a suitable measure supported on $C(A)/C(A) \cap P_0$. To deal with this technical point, suppose $Z \subset G/P_0$ is conull. Then for each $c \in C(A)$, we have $P_0cg \in Z$ for almost all $g \in G$. Thus for each $c \in C(A)$, we have $P_0cg \in Y_x$ for almost all $(x, g) \in X \times G$. By Fubini's theorem, there exists $g \in G$ such that for almost all $x \in X$, $P_0cg \in Y_x$ for almost all $c \in C(A)$. In other words, $[P_0g]g^{-1}cg \in Y_x$ for almost all $c \in C(A)$. Thus, using the group $g^{-1}Ag$ instead of A, we can ensure that condition (ii) holds as well.

5. Completion of the Proof.

We now prove the main result.

Theorem (**5.1**). — *For* $i = 1, 2$, *let* $\mathscr{F}_i$ *be an ergodic Riemannian measurable foliation of* (X_i, μ_i) *with transversally invariant measure and finite total volume. Assume* $\mathscr{F}_1$ *is an irreducible foliation by symmetric spaces of noncompact type and rank at least* 2, *and that the sectional curvature* k *of* $\mathscr{F}_2$ *satisfies* $k \leq c < 0$ *for some* c. *Then* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *are not transversally equivalent. (We also assume almost all leaves of* $\mathscr{F}_2$ *are complete and simply connected.)*

Proof. — Let T_i be isomorphic transversals of $\mathscr{F}_i$. By irreducibility (see remark following Corollary (3.5)), T_1 has a finite extension $\widetilde{T}_1$ that is a transversal to a foliation of the form S/K where S is an irreducible ergodic G-space, G is a connected semisimple Lie group with trivial center and no compact factors, and $K \subset G$ is a maximal compact subgroup. Using Proposition (3.1), it is easy to see that $\mathscr{F}_2$ has a finite extension (Definition (3.3)) with a transversal isomorphic to $\widetilde{T}_1$. Namely, owing to the isomorphism to T_1 and T_2, we have a finite extension $p : \widetilde{T}_1 \to T_2$. Let $\alpha : T_2 \to \text{Diff}(B)$ be the cocycle given by Proposition (3.1), and let $\beta : \widetilde{T}_1 \to \text{Diff}(B)$ be the cocycle $\beta(t_1, t_2) = \alpha(p(t_1), p(t_2))$. Let $\widetilde{X}_2$ be the quotient of $\widetilde{T}_1 \times B$ by the equivalence relation $(t_1, b_1) \sim (t_2, b_2)$ if and only if $t_1 \sim t_2$ and $\beta(t_1, t_2)b_2 = b_1$. Then it is straightforward to check that $\widetilde{X}_2$ is a finite extension of X_2 and has transversal $\widetilde{T}_1$. We may thus assume without loss of generality that $\mathscr{F}_1$ itself is of the form $X_1 = S/K$. We can also assume that T_1 is the bijective image of a transversal (which we also denote by T_1) to the G-action on S. (See [24], Section 2, *e.g.*)

Let $F \subset S$ be a flow box for the G-action around the transversal T_1. We can find a Borel map $\varphi : S \to T_1$ (defined *a.e.*) such that φ preserves the equivalence relations and such that $\varphi|$ F is simply projection onto T_1. Let $\theta : T_1 \to T_2$ be the isomorphism of equivalence relations and let $\alpha : T_2 \to \text{Diff}(\bar{B})$ be chosen as in Proposition (3.1). Let $\beta : S \times G \to \text{Homeo}(\partial B)$ be defined by $\beta(s, g) = \alpha(\theta(\varphi(s)), \theta(\varphi(sg)))$, so that β is also a cocycle. Let $\tilde{\beta} : S \times G/P \times G \to \text{Homeo}(\partial B)$ be $\tilde{\beta}(s, x, g) = \beta(s, g)$, so that $\tilde{\beta}$ is a cocycle on the ergodic G-space $S \times G/P$. However, by Proposition (2.5), $S \times G/P$ is an amenable G-space, and hence there is a measurable map $\mu : S \times G/P \to M(\partial B)$ such that $\tilde{\beta}((s, x), g)\mu(sg, xg) = \mu(s, x)$. More precisely, for each $g \in G$,

(*) $$\beta(s, g)\mu(sg, xg) = \mu(s, x) \quad \text{for almost all } (s, x) \in S \times G/P.$$

Lemma **(5.2)**. — *For almost all* $(s, x) \in S \times G/P$, $\mu(s, x)$ *is supported on at most two points of* ∂B.

Proof. — The argument of the proof of [31], Theorem (1.9) shows that (*) implies that for almost all $x \in G/P$, we have for all $h \in P$

$$\beta(s, g^{-1}hg)\mu(sg^{-1}hg, x) = \mu(s, x)$$

for almost all $s \in S$, where $[g] = x \in G/P$. In other words, for almost all x, $s \to \mu(s, x)$ is a $\beta \mid (S \times g^{-1}Pg)$-invariant function. Since $g^{-1}Pg$ is ergodic on S, to prove the lemma it suffices to show that for any subgroup $H \subset G$ acting ergodically on S, that a $\beta \mid S \times H$ invariant function $\gamma : S \to M(\partial B)$ takes values in the set of measures supported on at most two points. Recall the flow box $F = T_1 \times D$. Define an equivalence relation R_H on T_1 by $t_1 \sim t_2$ if and only if there exist $d_1, d_2 \in D$ and $h \in H$ such that $(t_1, d_1).h = (t_2, d_2)$. This is clearly a subrelation of the relation on T_1 defined by the G-action, and R_H is easily seen to be ergodic since F has positive measure in S and H is ergodic on S. It follows that for almost all $d \in D$, $\gamma \mid T_1 \times \{d\}$ corresponds under the isomorphism $\theta : T_1 \to T_2$ to an α-invariant function $T_2 \to M(\partial B)$ and hence has its image in the set of measures supported on at most two points by Theorem (3.7). Hence for almost all $s \in F$, $\gamma(s)$ is supported on at most two points, and by ergodicity of H and $\beta \mid S \times H$ invariance, $\gamma(s)$ is so supported for almost all $s \in S$. This proves the lemma.

Returning to the proof of the theorem, let $n(s, x)$ be the number of atoms of $\mu(s, x)$. By ergodicity and β-invariance, $n(s, x)$ will be essentially constant, say $n(s, x) = n$ *a.e.*, where $n = 1$ or 2. We can therefore suppose that the map $\mu : S \times G/P \to M(\partial B)$ is a map $\mu : S \times G/P \to (\partial B)^n/S_n$, where S_n is the permutation group on n letters, and equation (*) still holds.

Define an action of G on $S \times [(\partial B)^n/S_n]$ by $(s, \mu).g = (sg, \beta(s, g)^{-1}\mu)$, and we denote the space by $S \times_\beta [(\partial B)^n/S_n]$ when it is endowed with this action. Let $f : S \times G/P \to S \times [(\partial B)^n/S_n]$ be defined by $f(s, x) = (s, \mu(s, x))$. Then

$$f(sg, xg) = (sg, \mu(sg, xg)) = (sg, \beta(s, g)^{-1}\mu(s, x)) = (s, \mu(s, x)).g = f(s, x).g.$$

In other words, f is essentially a G-map. Thus if we let $\nu = f_*(\mu_S \times \mu_{G/P})$, we have a sequence of G-spaces $S \times G/P \to S \times_\beta [(\partial B)^n/S_n] \to S$. Thus by Theorem (4.1),

$S \times_\beta [(\partial B)^n/S_n]$ is essentially isomorphic to $S \times G/P_0$ for some parabolic P_0. Thus there is a conull G-invariant set $Y \subset S \times G/P_0$ and an injective G-map $h : Y \to S \times_\beta [(\partial B)^n/S_n]$ such that $p_1 \circ h = p_2$ where p_1 is projection of $S \times [(\partial B)^n/S_n]$ onto S and p_2 is projection of $S \times G/P_0$ onto S.

Suppose $P_0 = G$. Then h is a G-map $h : S \to S \times_\beta [(\partial B)^n/S_n]$, *i.e.* h_2 (the second coordinate of h) is a β-invariant function $S \to (\partial B)^n/S_n$. Identifying $(\partial B)^n/S_n$ as a subset of $M(\partial B)$, and arguing as above, this implies there is an α-invariant function $T_2 \to M(\partial B)$ which is impossible by the proof of Theorem 2 of [39]. Thus, we can assume $P_0 \neq G$. Then choose $A \subset G$ and $\lambda \in M(G/P_0)$ as in Proposition (4.4), where Y in Proposition (4.4) is chosen as in the previous paragraph. Once again, we choose a flow box $F = T_1 \times D$ for the G-action on S and define an equivalence relation R_A on T_1 by $t_1 \sim t_2$ if and only if there exist $d_1, d_2 \in D$ and $a \in A$ such that $(t_1, d_1) . a = (t_2, d_2)$. As with R_H above, this will be (by [23]) an ergodic subrelation on T_1. For almost all s, $h_s : G/P_0 \to (\partial B)^n/S_n$ defined *a.e.* by $h_s(x) = h_2(s, x)$ is injective, and using Fubini's theorem, we see that for some $d \in D$, the map $T_1 \to M((\partial B)^n/S_n)$, $s \to (h_{(s,d)})_*\lambda$ is a $\beta \mid R_A$-invariant function with $(h_{s,d})_*\lambda$ nonatomic by injectivity of h_s. Hence, via the isomorphism θ, there exists an $\alpha \mid R'$-invariant function $H : T_2 \to M((\partial B)^n/S_n)$ with $R' \subset R$ an ergodic subrelation and $H(t)$ nonatomic for almost all $t \in T_2$. From this it is easy to see that there is an $\alpha \mid R'$-invariant function $H' : T_2 \to M((\partial B)^n)$ with $H'(t)$ nonatomic for almost all t. For each i, let p_i be projection on the i-th factor of $(\partial B)^n$. If a measure on $(\partial B)^n$ projects to an atomic measure under all p_i, it must be atomic. Therefore, for some i, the measure $(p_i)_*(H'(t))$ will be nonatomic on ∂B for a set of $t \in T_2$ of positive measure, and $t \to (p_i)_*(H'(t))$ is $\alpha \mid R'$-invariant. This contradicts Theorem (3.7), completing the proof.

Corollary (5.3). — *Let* Γ_1 *be a lattice in* G, *a connected, noncompact simple Lie group with trivial center and* **R**-*rank at least* 2. *Let* $\Gamma_2 = \pi_1(M)$ *where* M *is a finite volume Riemannian manifold, complete, and with negative sectional curvature bounded away from* 0. *Then* Γ_1 *and* Γ_2 *do not have orbit equivalent free ergodic actions with finite invariant measure.*

Proof. — If Γ_1 is torsion free, we can apply Theorem (5.1) to the foliated bundle of Example (2.4) *(c)* derived from the Γ_i-actions. If not, then Γ_1 has a torsion free subgroup of finite index, say Γ_0. It is easy to see that R_{Γ_0}, the relation defined by the restriction of the Γ_1 action to Γ_0, is isomorphic to a transversal of a finite extension of the foliated bundle defined by the Γ_2-action. Thus the result in general follows from the torsion free case.

6. Concluding Remarks.

We now indicate how the techniques of the proof of Theorem (5.1) can be used to generalize Corollary (5.3).

Definition **(6.1)**. — *Suppose* Γ *is a discrete group. We say that* Γ *is of type* (H) *(for hyperbolic) if there is a compact metric space* Y *on which* Γ *acts continuously such that*

(i) *The stabilizer in* Γ *of every point in* Y *is an amenable subgroup of* Γ.

(ii) *For any* $\varepsilon > 0$, *there is a finite subset* $F \subset \Gamma$ *such that for all* $\gamma \in \Gamma - F$, *there are open subsets* $V_1, V_2 \subset Y$ *of diameter at most* ε *such that* $\gamma(Y - V_1) \subset V_2$.

We remark that if $\Gamma = \pi_1(M)$ where M is a complete manifold of finite volume and negative sectional curvature bounded away from 0, then Γ is of type (H) where Y is the boundary of the universal covering of M. (For condition (i) in Definition (6.1), see [41], Corollary (3.3).) Furthermore, any infinite subgroup of a type (H) group is type (H).

We then have the following generalization of Corollary (5.3).

Theorem **(6.2)**. — *Let* Γ_1 *be as in* (5.3), *and suppose* Γ_2 *is of type* (H). *If* S_i *is a free ergodic* Γ_i*-space with finite invariant measure* $(i = 1, 2)$, *then the* Γ_1*-action on* S_1 *and the* Γ_2*-action on* S_2 *are not orbit equivalent.*

Proof. — Let $\alpha : S_2 \times \Gamma_2 \to \Gamma_2$ be the projection on the second coordinate, and R the equivalence relation on S_2 defined by the Γ_2-action. We can identify α with a cocycle $R \to \Gamma_2$.

Suppose the actions are orbit equivalent. Arguing as in the final two paragraphs of the proof of Theorem (5.1), we deduce that there exists either

a) a Γ_2-map $h : S_2 \to M(Y)$; or

b) an ergodic subrelation $R^1 \subset R$ and an $\alpha | R^1$-invariant function $h : S_2 \to M(Y)$ such that $h(s)$ is non-atomic for *a.e.* $s \in S_2$.

In case *a)*, the existence of a Γ_2-invariant measure for S_2 implies the existence of a Γ_2-invariant measure on M(Y), and since M(Y) is a compact convex set, this implies the existence of a Γ_2-fixed point in M(Y). In other words, there is a Γ_2-invariant measure on Y. But from condition (ii) of Definition (6.1), it follows that this measure must be supported on at most two points, and from (i) of Definition (6.1), we then conclude that Γ_2 is amenable. Hence, the Γ_2-action on S_2 is amenable, and by orbit equivalence, so is the Γ_1-action on S_1. However, this is impossible since Γ_1 is not amenable and has a finite invariant measure on S_1. This shows that case *a)* is impossible.

We next observe that using condition (ii) of (6.1), we can show by an argument similar to the proof of Theorem (3.7) that case *b)* is impossible as well. Namely, as in (3.7), we have that for any such function *h*, $h(s)$ must be supported on at most two points.

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