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MARKOV PARTITIONS AND K_2

by J. B. WAGONER*

TABLE OF CONTENTS

1. Introduction

Let $\mathscr{S}$ be a finite or a countably infinite collection of "states", and let $A : \mathscr{S} \times \mathscr{S} \to \{0, 1\}$ be a zero-one matrix with the corresponding subshift $\sigma_A : X_A \to X_A$. See [Wi], [PW], [F], or [W2] for example. Let $\mathrm{Aut}(\sigma_A)$ denote the discrete group of uniformly continuous homomorphisms $\alpha : X_A \to X_A$ which commute with σ_A and which have α^{-1} uniformly continuous also. This is the group of *uniform equivalences* or *symmetries* of σ_A. A central problem or theme in symbolic dynamics is to examine the algebraic structure and homological properties of $\mathrm{Aut}(\sigma_A)$. The aim of this paper is to introduce a method for studying $\mathrm{Aut}(\sigma_A)$ by making it act on a contractible simplicial complex $\mathscr{P}_A$ constructed in a very natural way from the set of uniform Markov partitions for σ_A on X_A.

The first paper giving a systematic account of $\mathrm{Aut}(\sigma_A)$ for the Bernoulli 2-shift $\sigma_2 : X_2 \to X_2$ was written by Hedlund [H]. It was proved there that $\mathrm{Aut}(\sigma_2)$ contains every finite group as well as elements of infinite order not of the form σ_2^k for some $k \in \mathbf{Z}$. These extra infinite order elements were obtained by taking products of non-commuting elements of finite order. More recent results can be found in [BK] and [BLR]. Despite notable progress in the last several years, there is an old and central problem which still remains wide open. This is the finite order generation conjecture (dubbed FOG by D. Lind) which states that $\mathrm{Aut}(\sigma_p)$ for the full Bernoulli p-shift with p prime is

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generated by σ_p together with elements of finite order. This problem arose from the early work by Hedlund and coworkers such as Arnold, Curtis, Lyndon, Rhodes, and Welch. In more recent papers, all exotic elements of infinite order have been of this form. For example, Boyle-Lind [BLR] have shown $\mathrm{Aut}(\sigma_2)$ contains the free non-abelian group on two generators, each of which is constructed as a product of elements of finite order. Concrete computations of gyration numbers by Boyle-Krieger [BK] lend credence to the possibility raised by Rhodes that $\mathrm{Aut}(\sigma_2)$ is generated by σ_2 and by involutions. This would imply that the first Eilenberg-MacLane homology group $H_1(\mathrm{Aut}(\sigma_2))$ is the direct sum of $\mathbf{Z}$ and a vector space over $\mathbf{Z}/2$ which, by [BK], would be infinite dimensional. Conversely, information about H_1 could bear on Rhodes' question or on FOG.

A general open problem is to obtain information about the higher Eilenberg-MacLane homology groups $H_i(\mathrm{Aut}(\sigma_A))$, and the action of $\mathrm{Aut}(\sigma_A)$ on $\mathscr{P}_A$ is a natural setting for this type of question. Gromov asked whether $\mathscr{P}_A$ had certain combinatorial properties similar to a space with non-positive curvature. For example, he asked whether every loop with L edges in $\mathscr{P}_A$ could be spanned by a disc with at most cL^2 triangles for some universal constant c. This turns out to be the case for $c = 40$. We hope that more detailed analysis of $\mathscr{P}_A$ will prove useful for FOG.

The present paper concentrates on H_1 and uses the action of $\mathrm{Aut}(\sigma_A)$ on $\mathscr{P}_A$ to construct homomorphisms of $\mathrm{Aut}(\sigma_A)$ into various other simpler groups such as the group of automorphisms of the dimension group or the algebraic K-theory group K_2. These are obtained via a canonical homomorphism

$$\psi_A : \mathrm{Aut}(\sigma_A) \to \pi_1(S(\bar{\mathscr{E}}), A)$$

into the fundamental group of the space of all shift equivalences in the category $\bar{\mathscr{E}}$ of non-negative integral matrices. When A is finite, we show

$$\pi_1(S(\bar{\mathscr{E}}), A) \cong \mathrm{Aut}(G(A), G(A)_+, s_A)$$

where the right-hand side is the group of automorphisms of the dimension group $G(A)$ considered as an ordered group as in [E] and a $\mathbf{Z}[t, t^{-1}]$-module via the action of the automorphism s_A of $G(A)$ coming from A.

The first example of the general theory developed here was a commutative diagram

$$\begin{tikzcd} & K_2(F(t)) \arrow[d, "\partial"] \\ \mathrm{Aut}(\sigma_A) \arrow[ur, "K_A"] \arrow[r, "\kappa_A"'] & \bigoplus_{\mathscr{P}} F^{\bullet}_{\mathscr{P}} \end{tikzcd}$$

where A is assumed to be finite, $F(t)$ is the field of rational functions over a field F, and ∂ is the tame symbol [M1]. Another typical example comes from random walk on a discrete group G. See § 4.

Here is why one might expect a connection between $\mathrm{Aut}(\sigma_A)$ and K_2 in the first place. Let $f: M \to M$ be a Smale diffeomorphism [F] of a compact manifold M. In [F] it is shown that $\zeta_f^h(t) \equiv \zeta_f(t) \bmod 2$ where $\zeta_f^h(t)$ is the " homology " zeta function obtained by counting the periodic points of f algebraically using the Lefschetz numbers of f^n and $\zeta_f(t)$ is the " symbolic " zeta function obtained by counting periodic points in the non-wandering set of f. Knowing the contribution to $\zeta_f(t)$ from each basic set enables one in principle to compute the entropy of f. On the other hand, let M_f denote the mapping torus of f. This is just $M \times I$ modulo the identification of $(x, 0)$ with $(f(x), 1)$. From algebraic K-theory and simple homotopy theory there is a rational function $\tau(M_f)$ called the Reidemeister torsion of M_f which is an invariant of the manifold M_f. This is a special case of a Whitehead torsion invariant which arises in connection with the algebraic K-theory group K_1. See [M2], [M3]. It is well known [F], [M2] that $\zeta_f^h(t) = \tau(M_f)$. So in a loose sense we say " entropy is a K_1-type invariant ". Now in differential topology it is an old story that just as K_1 provides useful invariants for manifolds, the algebraic K-theory group K_2 gives invariants for diffeomorphisms of a manifold. See [HW], [W1]. Consider the group of symmetries $\mathrm{Aut}(f)$ of our original Smale diffeomorphism $f: M \to M$. By definition, this consists of all homeomorphisms $g: M \to M$ satisfying $gf = fg$ on M. Each such g induces a homeomorphism of M_f by the formula $G(x, t) = (g(x), t)$. In this setting, it is then possible to define a homomorphism κ_f^h from $\mathrm{Aut}(f)$ into $K_2(F(t))$. It was consequently natural to ask whether there was a homomorphism such as κ_A, and this was the starting place for our paper. Incidentally, in his thesis [Z] Zizza shows for a fitted Smale diffeomorphism f and the field $F = \mathbf{Z}/2$ that κ_f^h is an alternating product of the various " symbolic " κ_A's coming from the basic sets in the non-wandering set of f. The argument makes use of $\mathscr{P}_A$.

In summary, the idea is that if K_1 is related to the dynamical system (X_A, σ_A), then K_2 should be (and is) related to its symmetry group $\mathrm{Aut}(\sigma_A)$. To carry over the analogy from differential topology to symbolic dynamics, the space of C^∞ functions on a manifold is replaced by the space $\mathscr{P}_A$. Put differently, a given Markov partition for σ_A on X_A is like a particular triangulation of a finite complex.

In view of the embedding of $\mathrm{Aut}(\sigma_A)$ into the automorphisms $\mathrm{Aut}(\mathscr{M}_A^u)$ of the Cuntz-Krieger type C^*-algebra $\mathscr{M}_A^u$ discussed in [W2], it is reasonable to expect a " non-commutative " version of the theory for automorphisms of operator algebras. In [W2] a K_2 invariant for $\mathrm{Aut}(\sigma_A)$ was constructed using the definition of the dimension group $G(A)$ in terms of unstable manifolds of σ_A on X_A. In case A is aperiodic, this leads to a homomorphism $\psi: \mathrm{Aut}(\sigma_A) \to \mathbf{R}_+^*$ which turns out to be the inverse of Connes' module homomorphism $m_\lambda: \mathrm{Aut}(\mathscr{M}_A^u) \to \mathbf{R}_+^*$. The map m_λ comes from seeing how an automorphism multiplies the line of semi-finite traces on $\mathscr{M}_A^u$. See [C]. In [CS] Connes-Störmer define the entropy of an automorphism θ of a type II_1 von Neumann algebra preserving a faithful normal trace τ. Can one define a homomorphism $\mathrm{Aut}(\theta) \to \mathbf{R}^*$ using " non-commutative " partitions? Or other methods? The module homomorphism is of course trivial on $\mathrm{Aut}(\theta)$ because the trace is normalized to satisfy $\tau(1) = 1$.

In § 2 we define the simplicial structure on $\mathscr{P}_A$ and develop some of its basic properties. The main theorem (2.12) is that $\mathscr{P}_A$ is contractible. $\mathscr{P}_A$ is locally finite when

A is a finite matrix. In § 3 we prove the new algebraic Triangle Identities (3.3) coming from the analysis of geometric triangles in $\mathscr{P}_A$. This is basic to the remainder of the paper.

The first theme of § 4 is invariants of $\mathrm{Aut}(\sigma_A)$ that come from " inverting functors ", of which the dimension group is the prime example when A is finite. Let $\bar{\mathscr{E}}$ be the category of non-negative integral matrices and, for simplicity, let $\mathscr{D}$ be an abelian category such as modules and module homomorphisms over a commutative ring Λ. Consider a new category in which the objects are endomorphisms (square matrices) A of $\bar{\mathscr{E}}$ and where a morphism from A to B is a non-negative integral matrix X such that $AX = XB$. A functor F on this new category assigns an object $F(A)$ in $\mathscr{D}$ to each endomorphism A and a morphism $f(X) : F(B) \to F(A)$ in $\mathscr{D}$ to each morphism X. In particular, we have $f(A) : F(A) \to F(A)$. We say F is *inverting* provided each $f(A)$ is an isomorphism. In addition to the dimension group, there are a number of other inverting functors arising in dynamical systems and C^*-algebras. See [BF], [CK], [Cu1], [Cu2], [E], [F], [K] and [W2]. The principal result of this section constructs a homomorphism

$$\psi_{F,A} : \mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(f(A))$$

into the automorphisms of $F(A)$ commuting with $f(A)$ and a commutative diagram

$$\begin{tikzcd}
 & \pi_1(S(\bar{\mathscr{E}}), A) \arrow[d, "\eta_{F,A}"] \\
\mathrm{Aut}(\sigma_A) \arrow[ur, "\psi_A"] \arrow[r, "\psi_{F,A}"'] & \mathrm{Aut}(f(A))
\end{tikzcd}$$

When A is finite, this implies (4.19) that $\psi_{F,A}$ factors through $\mathrm{Aut}(G(A), G(A)_+, s_A)$. Some examples are given, including the relation of $\mathrm{Aut}(\sigma_A)$ to K_2.

To finish § 4, we briefly discuss one way to further explore the implications of the Triangle Identities by constructing the space $SS(\mathscr{E})$ of strong shift equivalences for the set $\mathscr{E}$ of zero-one matrices. Like $S(\mathscr{E})$, the space $SS(\mathscr{E})$ is built independently of any dynamical system. Its simplices are formed using the algebraic identities (3.3). The homomorphism ψ_A factors through a canonical homomorphism from $\pi_1(SS(\mathscr{E}), A)$ to $\pi_1(S(\mathscr{E}), A)$. An open problem is to obtain more information about $\pi_1(SS(\mathscr{E}), A)$. It is a new but hard to compute invariant of strong shift equivalence. Whether it is also an invariant of shift equivalence is far from clear at the present time.

The material in this paper can be extended in several ways. For example, stochastic shift equivalence and strong shift equivalence for finite irreducible matrices was developed by Parry-Williams in [PW]; see also [PT], [T]. Their theory goes through in the infinite case. Also, if μ is a Markov measure invariant under σ_A, basically all the results on $\mathscr{P}_A$ and $\mathrm{Aut}(\sigma_A)$ can be carried over to the contractible simplicial complex $\mathscr{P}_\mu$ of μ-Markov partitions and the subgroup $\mathrm{Aut}_\mu(\sigma_A)$ of μ-preserving symmetries of σ_A. In another direction, the classical " marker method " produces many ways of embedding any finite group G into $\mathrm{Aut}(\sigma_A)$ when A is aperiodic [BLR]. The G-fixed point sets of $\mathscr{P}_A$ are contractible and can be used to give a criterion for conjugacy classes of G in $\mathrm{Aut}(\sigma_A)$ in terms of G-equivariant strong shift equivalence.

To finish this introduction, we would like to thank M. Boyle, D. Lind, W. Krieger, and F. Zizza for useful discussions and comments over the last few years while this paper underwent several revisions. We would also like to thank IMPA (Rio de Janeiro), the University of Geneva (Switzerland), and IHES (Paris) for their hospitality during Spring 1986.

2. Markov partitions

In this section we develop the basic properties of the simplicial complex $\mathscr{P}_A$ of Markov partitions. In particular we show in (2.12) that $\mathscr{P}_A$ is contractible and is locally compact when A is finite.

Let $\mathscr{S}$ be a countable set of " states " and let $A : \mathscr{S} \times \mathscr{S} \to \{0, 1\}$ be a zero-one matrix such that each row and each column has a non-zero entry. We let X_A denote the space of sequences $x = (x_i)$, $-\infty < i < \infty$, such that $A(x_i, x_{i+1}) = 1$. The metric on X_A is defined to be $d(x, y) = 0$ if $x = y$ and for $x \neq y$

$$d(x,y) = \frac{1}{k+1}$$

where k is the least non-negative integer such that $x_k \neq y_k$ or $x_{-k} \neq y_{-k}$. The space X_A is complete. It is locally compact if and only if A is locally finite and compact if and only if A is finite. See [W2]. The shift $\sigma_A : X_A \to X_A$ is to the *left*; i.e. $\sigma_A(x)_i = x_{i+1}$. Both σ_A^{-1} and σ_A are uniformly continuous. A homeomorphism $\alpha : X_A \to X_B$ between two shift spaces satisfying $\alpha\sigma_A = \sigma_B\,\alpha$ will be called a *uniform equivalence* or an *isomorphism* provided both α and α^{-1} are uniformly continuous. We let $\mathrm{Isom}(\sigma_A, \sigma_B)$ denote the set of all isomorphisms from σ_A to σ_B. When $A = B$, we often write $\mathrm{Aut}(\sigma_A)$ for $\mathrm{Isom}(\sigma_A, \sigma_A)$ and will call $\alpha \in \mathrm{Aut}(\sigma_A)$ a *symmetry* of σ_A.

There are generally two definitions of a Markov partition in the literature. We start with the approach of [PT] and later in this section will discuss the definition used in [F, p. 100].

If $U = \{U_i\}$ and $V = \{V_j\}$ are coverings of X_A, we let

$$U \cap V = \{ U_i \cap V_j \text{ where } U_i \cap V_j \neq \emptyset \}.$$

We say V *refines* U, written $U < V$, provided each V_j is a subset of some U_i. If $m, n \in \mathbf{Z}$ and $m \leqslant n$, let $U(m, n) = \sigma_A^{-m}(U) \cap \ldots \cap \sigma_A^{-n}(U)$ where $\sigma_A^k(U) = \{\sigma_A^k(U_i)\}$ for $k \in \mathbf{Z}$. Let $U^A = \{U_s^A\}$ where for each $s \in \mathscr{S}$ we define $U_s^A = \{x \in X_A \mid x_0 = s\}$. This will be a standard example of a Markov partition for σ_A on X_A. Observe that the sets in $U^A(-n, n)$ are all disjoint, closed, open, and form a basis for the topology of X_A as n runs over the non-negative integers.

Definition **2.1.** — A *topological Markov partition* for σ_A on X_A is a covering $U = \{U_i\}$ of X_A such that

(a) the U_i are disjoint and open (and therefore closed);

(b) any intersection $\bigcap_{-\infty}^{\infty} \sigma_A^{-n}(U_{i(n)})$ consists of at most one point;

(c) if $U_{i(n)} \cap \sigma_A^{-1}(U_{i(n+1)}) \neq \emptyset$ for $n \in \mathbf{Z}$, then $\bigcap_{-\infty}^{\infty} \sigma_A^{-n}(U_{i(n)}) \neq \emptyset$.

Moreover, we say U is *uniform* provided

(d) there are $m, n \geqslant 0$ such that $U^A < U(-m, m)$ and $U < U^A(-n, n)$.

We let $\mathscr{P}_A$ denote the set of all uniform topological Markov partitions for σ_A.

Remark **2.2.** — Any two coverings U and V satisfying *(d)* with respect to U^A have mutual refinements in the sense that there are $m, n \geqslant 0$ so that $U < V(-m, m)$ and $V < U(-n, n)$. If A is finite, then any two coverings satisfying *(a)*, *(b)*, *(c)* also satisfy this mutual refinement condition. So as in [PT] the condition *(d)* is not needed in the definition when A is finite.

If $\alpha : X_A \to X_B$ is a homeomorphism and $U = \{U_i\}$ is a covering of X_A, let $\alpha(U) = \{\alpha(U_i)\}$. Observe also that uniform continuity can be expressed as follows: Given a refinement $U^B(-n, n)$ there is a refinement $U^A(-m, m)$ such that

$$U^B(-n, n) < \alpha(U^A(-m, m)).$$

It is not hard to verify that if $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ and $U \in \mathscr{P}_A$, then $\alpha(U) \in \mathscr{P}_B$. Thus we have a bijection

$$\mathscr{P}_A \cong \mathscr{P}_B \tag{2.3}$$

given by the correspondence $U \to \alpha(U)$.

For future reference we observe

Lemma **2.4.** — *If* $U \in \mathscr{P}_A$, *then* $U \cap \sigma_A^{-1}(U)$ *and* $\sigma_A(U) \cap U$ *are in* $\mathscr{P}_A$. *If* $U \in \mathscr{P}_A$ *and* $V = \{V_j\}$ *is a cover of* X_A *by disjoint, open sets such that* $U < V < U \cap \sigma_A^{\varepsilon}(U)$ *where* $\varepsilon = +1$ *or* $\varepsilon = -1$, *then* $V \in \mathscr{P}_A$ *also.*

The proof of this is straightforward.

Remark. — It is *not* true in general that $U, V \in \mathscr{P}_A$ implies $U \cap V \in \mathscr{P}_A$. However, this property does hold for Markov partitions as defined in [F, p. 100]; see (2.17) below.

If $U = \{U_i\}$ is in $\mathscr{P}_A$, let the matrix $M = M(U)$ associated to U be the function $M : U \times U \to \{0, 1\}$ defined as usual by

$$M(U_i, U_j) = \begin{cases} 1, & U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

The definition of M as a function from $U \times U$ to $\{0, 1\}$ is independent of the choice of a bijection $U \cong I$ between the sets of U and some countable indexing set I which is tacitly assumed when we write $U = \{ U_i \}$ for $i \in I$. However, such an identification does give a matrix $M : I \times I \to \{ 0, 1 \}$ where $M(i,j) = M(U_i, U_j)$. In particular, there is a canonical bijection $\mathscr{S} \cong U^A$ under which $M(U^A) = A$. Just as in the case when A is finite we have

Lemma **2.6.** — *Let* $U \in \mathscr{P}_A$ *and* $B = M(U)$. *Then there is a uniform equivalence* $(X_A, \sigma_A) \cong (X_B, \sigma_B)$.

Proof. — Let $U = \{ U_t \}$ where t runs through the countable indexing set T. Define $\alpha : X_A \to X_B$ by the condition that $\alpha(x)_i = t$ if and only if $\sigma_A^i(x) \in U_t$. Thus α is continuous by *(a)* of (2.1), injective by *(b)* of (2.1), and surjective by *(c)* of (2.1). Uniform continuity of α follows from $U < U^A(-n, n)$ and uniform continuity of α^{-1} follows from $U^A < U(-m, m)$.

We now define the simplicial structure on $\mathscr{P}_A$. If $U, V \in \mathscr{P}_A$, we let

$$(2.7) \qquad U \xrightarrow[(-m,\, n)]{} V$$

mean that $U < V < U(-m, n)$. The special cases $U \xrightarrow[(0,\, 1)]{} V$ and $U \xrightarrow[(-1,\, 0)]{} V$ will be denoted respectively by

$$(2.8) \qquad U \xrightarrow[+]{} V \quad \text{and} \quad U \xrightarrow[-]{} V.$$

If $U < V$, then from (2.2) we have $V < U(-m, n)$ for some $m, n \geqslant 0$. Let

$$(2.9) \qquad \ell(U, V) = \min\{ m + n \mid V < U(-m, n) \}.$$

Then $\ell(U, V)$ is like a length function, but it is only defined when $U < V$ and it is not symmetric. If $U < V < W$, then $\ell(U, W) \leqslant \ell(U, V) + \ell(V, W)$.

Definition **2.10.** — If $U, V \in \mathscr{P}_A$, then we write $U \to V$ if and only if $U \xrightarrow[+]{} U \cap V \xleftarrow[-]{} V$.

Observe by (2.4) that this condition implies $U \cap V \in \mathscr{P}_A$.

Definition **2.11.** — The simplicial complex $\mathscr{P}_A$ has as n-simplices those $(n + 1)$-tuples $\langle V_0, \ldots, V_n \rangle$ where each $V_i \in \mathscr{P}_A$ and $V_i \to V_j$ for $i \leqslant j$.

It is clear that the bijection $\mathscr{P}_A \cong \mathscr{P}_B$ of (2.3) produces an isomorphism of simplicial complexes.

Proposition **2.12.** — $\mathscr{P}_A$ *is contractible and is locally compact if* A *is finite.*

Proof. — First we verify the easy part that $\mathscr{P}_A$ is locally compact when A is finite. Under this hypothesis there are, for a given $U \in \mathscr{P}_A$, only finitely many $V \in \mathscr{P}_A$ such that $U < V < U \cap \sigma_A^{\pm 1}(U)$. Consequently there are only finitely many V such that either $V \to U$ or $U \to V$. In particular, a given vertex U can belong to only finitely many simplices. This implies that $\mathscr{P}_A$ is locally compact.

Now we prove that $\mathscr{P}_A$ is contractible in three stages:

Step I: $\mathscr{P}_A$ is connected;
Step II: $\pi_1(\mathscr{P}_A) = 0$;
Step III: $H_n(\mathscr{P}_A) = 0$ for $n \geqslant 2$.

Contractibility then follows from the Whitehead theorem [Sp].

Step I. — Connectivity of $\mathscr{P}_A$ was essentially proved by Williams in [Wi] where he introduced the notion of strong shift equivalence. Another exposition of this is found in [PW]. For completeness, we give the argument here.

Let U and V be vertices in $\mathscr{P}_A$. We show that there is a path from U to V in $\mathscr{P}_A$ having edges of the form $\langle U_0, U_1 \rangle$ where $U_0 \xrightarrow[\varepsilon]{} U_1$ for $\varepsilon = \pm 1$. By *(d)* of (2.1) we know that $U < U^A(-n, n) > V$ for some $n \geqslant 0$; so we may as well assume $U < V$. The proof then proceeds by induction on $\ell(U, V)$. By (2.2) we know $U \xrightarrow[(-m,\,n)]{} V$ for some $m, n \geqslant 0$. If $n \geqslant 1$, then

$$U \xrightarrow[(-m,\,n)]{} U(-m, n) \xleftarrow[(-m,\,n-1)]{} V \cap \sigma_A^{-1}(U) \xleftarrow[(0,\,1)]{} V.$$

Similarly, if $m \geqslant 1$, then

$$U \xrightarrow[(-m,\,n)]{} U(-m, n) \xleftarrow[(-m+1,\,n)]{} \sigma_A(U) \cap V \xleftarrow[(-1,\,0)]{} V.$$

Finally, we have

$$U \xrightarrow[+]{} U(0, 1) \xrightarrow[+]{} \dots \xrightarrow[-]{} U(0, n) \xrightarrow[-]{} U(-1, n) \xrightarrow[-]{} \dots \xrightarrow[-]{} U(-m, n).$$

Observe that by (2.4) each of the partitions $V \cap \sigma_A^{-1}(U)$, $\sigma_A(U) \cap V$, and $U(-p, q)$ are still in $\mathscr{P}_A$.

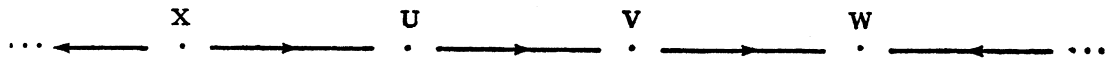
Step II. — Simple connectivity of $\mathscr{P}_A$ can be proved in two forms, (2.13) and (2.14). As mentioned in the introduction, it was Gromov who asked whether such properties held for $\mathscr{P}_A$, and he remarked that they might indicate $\mathscr{P}_A$ is something like a space with non-positive curvature.

By a path in $\mathscr{P}_A$ we always mean a simplicial path, and by a homotopy between paths with the same endpoints we mean a sequence of intermediate paths having the same endpoints such that one path differs from the next one by replacing a single edge by the two opposite edges of a triangle or vice versa.

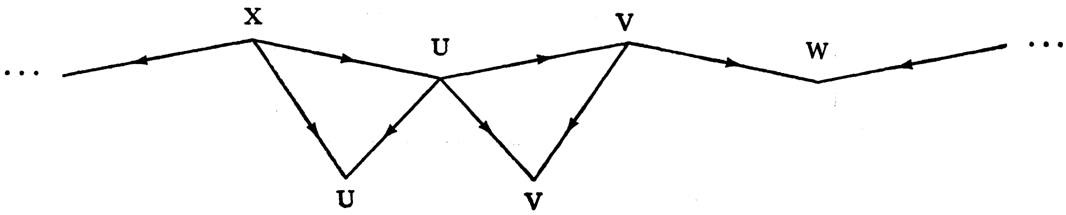
Proposition **2.13.** — *A path in $\mathscr{P}_A$ having L edges can be spanned by a (possibly singular) triangulated 2-disc in $\mathscr{P}_A$ with at most* $40L^2 - 39L$ *triangles.*

Remark **2.14.** — An argument similar to the one for (2.13) shows that any two paths of length at most L are homotopic through intermediate paths of length at most $6L + 2$. The specific constants in these propositions do not seem all that important, and they can very likely be improved.

Proof of 2.13. — Step 1: Consider a segment of a loop α such as

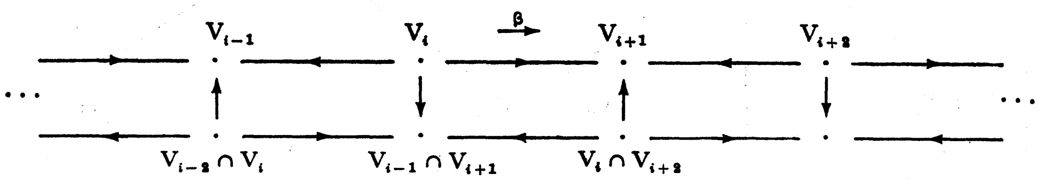


The diagram



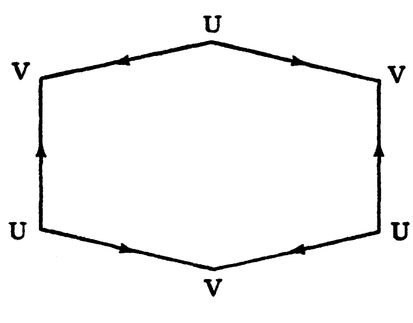
shows how to deform α to a loop β which is *alternating* in the sense that the natural direction (2.10) of each edge switches as one moves around the loop. Equivalently, the two edges containing any vertex either point both toward that vertex or both point away from it. If α has at most L edges, then β has at most 2L edges and the number of simplices used to deform α to β is at most L.

Step 2: Assume β is an alternating loop with 2L edges and successive vertices V_0, $V_1, \ldots, V_{2L-1}, V_0$, and consider the diagram



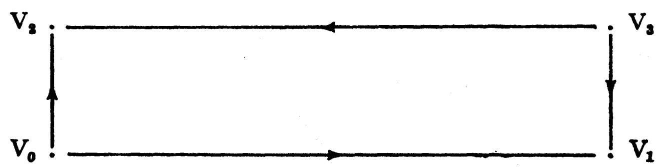
In Step 3 we will show how to triangulate each square, but for the moment observe that the bottom horizontal loop γ is alternating with 2L edges having successive vertices $V_{2L-1} \cap V_1$, $V_0 \cap V_2$, $V_1 \cap V_3, \ldots, V_{2L-2} \cap V_0$. Repeating this construction a total of $L - 1$ times produces an alternating loop of length 2L with only two distinct vertices $V_1 \cap \ldots \cap V_{2L-1}$ and $V_0 \cap \ldots \cap V_{2L-2}$. Observe that the number of squares involved is $2L(L-1) = 2L^2 - 2L$.

Step 3: Consider a loop with 2L edges having just two alternating vertices U and V as in the diagram

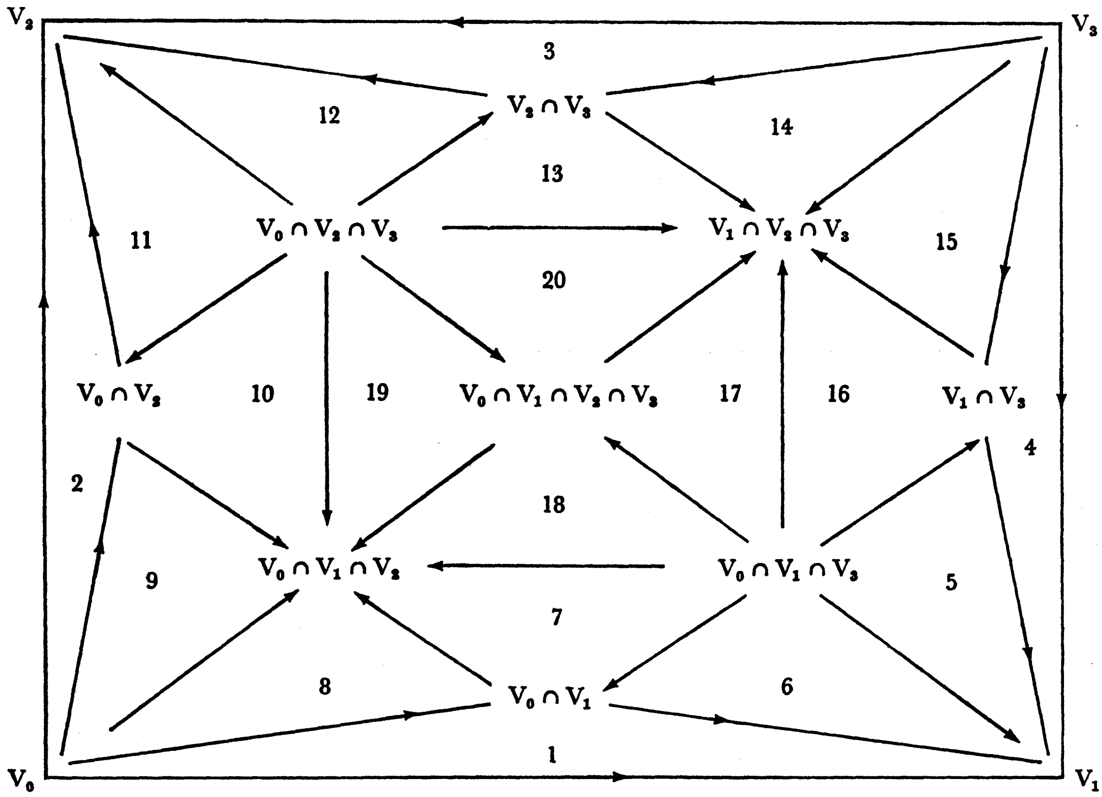


for L = 3. The loop has two identical sides of length L emanating from U, say, and ending in a common vertex which is either U or V. These sides may be identified to fill in the 2-disc. No extra triangles are added at this stage.

Step 4: A square of the type



can be filled in by a triangulation with 20 triangles as follows:



We finally see that the total number of triangles required to fill in the loop using Steps 1 through 4 is at most

$$L + 2L(L-1)\,20 = 40L^2 - 39L.$$

This completes the proof of (2.13).

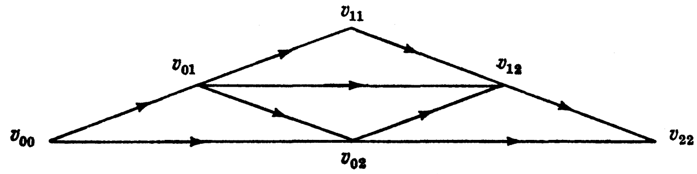
Step III. — To prove $H_n(\mathcal{P}_A) = 0$ for $n \geqslant 2$ it suffices to show for any finite complex $K \subset \mathcal{P}_A$ that the homomorphism $H_n(K) \to H_n(\mathcal{P}_A)$ is zero.

Step 1: Subdivision. — Let K be any finite simplicial complex such that each simplex is equipped with an ordering of the vertices which is compatible with the face maps. For example, K could be any finite complex of $\mathscr{P}_A$. We construct a subdivision K′ of K as follows:

The vertices of K′ are pairs $v_{ij} = (v_i, v_j)$ where v_i and v_j are vertices occurring in a simplex $\langle v_0, \ldots, v_i, \ldots, v_j, \ldots, v_n \rangle$ of K and v_i comes before v_j in the ordering; that is, $i \leqslant j$. More generally, an n-simplex of K′ is an $(n+1)$-tuple $\langle v_{i_0 j_0}, v_{i_1 j_1}, \ldots, v_{i_n j_n} \rangle$ satisfying

(i) all $v_{ij} = (v_i, v_j)$ come from vertices lying in a simplex $\langle v_0, v_1, \ldots, v_p \rangle$ of K,
(ii) $i_\alpha \leqslant i_{\alpha+1}$, $j_\alpha \leqslant j_{\alpha+1}$,
(iii) $i_{\alpha+1} - i_\alpha + j_{\alpha+1} - j_\alpha = 1$.

For example, if $K = \langle v_0, v_1, v_2 \rangle$, then K′ is



Note that neither $\langle v_{00}, v_{12} \rangle$ nor $\langle v_{01}, v_{22} \rangle$ is a simplex of K′ in this example.

A homeomorphism $\theta : K' \to K$ is defined by taking a point in a simplex $\langle v_{i_0 j_0}, \ldots, v_{i_r j_r} \rangle$ represented in terms of barycentric coordinates as the linear combination

$$\Sigma_{\alpha=0}^{r} \lambda_\alpha v_{i_\alpha j_\alpha}$$

with $\Sigma_\alpha \lambda_\alpha = 1$ and $\lambda_\alpha \geqslant 0$ to the point

$$\Sigma_{\alpha=0}^{r} \left(\frac{\lambda_\alpha}{2} v_{i_\alpha} + \frac{\lambda_\alpha}{2} v_{j_\alpha} \right)$$

inside $\langle v_0, \ldots, v_p \rangle$.

Now let $K \subset \mathscr{P}_A$ be a subcomplex. Each simplex of K is of the form $\langle V_0, \ldots, V_p \rangle$ where each $V_i \in \mathscr{P}_A$. We let L denote the isomorphic simplicial complex with a simplex $\langle v_0, \ldots, v_p \rangle$ corresponding to each simplex $\langle V_0, \ldots, V_p \rangle$ of K and we let $\eta : L \to K \subset \mathscr{P}_A$ be the simplicial map where $\eta(v_i) = V_i$. Let $\tau : L' \to \mathscr{P}_A$ be defined on $\langle v_{i_0 j_0}, \ldots, v_{i_p j_p} \rangle$ by the formula

$$\tau(v_{ij}) = V_i \cap V_j.$$

That this is a simplicial map follows from the observation that if $U \to V$, then $U \to U \cap V \to V$; and if $U \to X$ and $V \to Y$, then $U \cap V \to X \cap Y$.

Let $\overline{K} \subset \mathscr{P}_A$ be the subcomplex of $\mathscr{P}_A$ consisting of all possible simplices having vertices of the form $V_0 \cap V_1 \cap \ldots \cap V_r \in \mathscr{P}_A$ where the V_i are vertices of K. The vertices V_i need not all belong to the same simplex and not every such intersection is in $\mathscr{P}_A$. But certainly some do, and $\overline{K}$ consists of the simplices which can be formed in this way. Let $\theta : L' \to L$ be the homeomorphism as above.

Claim: $\eta\theta$ and τ are homotopic as maps from L' into $\overline{K} \subset \mathscr{P}_A$.

To see this define $\rho : L' \to \overline{K} \subset \mathscr{P}_A$ by sending a simplex $\langle v_{i_0 j_0}, \ldots, v_{i_r j_r} \rangle$ of L' to the simplex $\langle V_{i_0}, \ldots, V_{i_r} \rangle$ of $K \subset \mathscr{P}_A$. Then τ and ρ are homotopic as follows: $L' \times I$ is triangulated by simplices of the form

$$\langle (v_{i_0 j_0}, 0), \ldots, (v_{i_r j_r}, 0), (v_{p_0 q_0}, 1), \ldots, (v_{p_s q_s}, 1) \rangle$$

where $\langle v_{i_0 j_0}, \ldots, v_{i_r j_r} \rangle$ and $\langle v_{p_0 q_0}, \ldots, v_{p_s q_s} \rangle$ are sub-simplices of the same simplex in L' and $i_r \leqslant p_0$, $j_r \leqslant q_0$. The homotopy from ρ to τ takes this simplex to

$$\langle V_{i_0}, \ldots, V_{i_r}, V_{p_0} \cap V_{q_0}, \ldots, V_{p_s} \cap V_{p_s} \rangle.$$

Now observe that the two *continuous* maps $\rho : L' \to K \subset \mathscr{P}_A$ and $\eta\theta : L' \to K \subset \mathscr{P}_A$ have the property that if S is a simplex of L', then $\rho(S)$ and $\eta\theta(S)$ lie in the same simplex of K. Hence, the one parameter family $(1 - t)\rho + t\eta\theta$, $0 \leqslant t \leqslant 1$, is a homotopy from ρ to $\eta\theta$.

This step can now be summarized in the following way: Suppose we have a chain

$$\alpha = \Sigma f_p S_p, \quad f_p \in Z$$

representing an n-dimensional homology class in $H_n(K)$ where each $S = S_p$ is a non-degenerate n-simplex of the form $S = \langle V_0, \ldots, V_n \rangle$. Let α also denote the corresponding chain in the isomorphic simplicial complex L. Let

$$\alpha' = \Sigma f_p S'_p$$

be the chain on L' where S'_p is the subdivision of S_p as above. Let $\theta_n : H_n(L') \underset{\simeq}{\to} H_n(K)$ be the induced map on homology. Then $\theta_n(\alpha') = \alpha$ in $H_n(K)$ and, moreover,

$$\tau_n(\alpha') = \alpha$$

as homology classes in $H_n(\overline{K})$ where $\tau_n : H_n(L') \to H_n(\overline{K})$ is the map induced on homology by $\tau : L' \to \overline{K}$.

Step 2: Shrinking. — This is the main inductive step. Let $\alpha = \Sigma_p f_p S_p$ be an n-cycle representing an n-dimensional homology class in $H_n(\mathscr{P}_A)$. We may assume that each of the n-simplices S_p is non-degenerate. Let $K \subset \mathscr{P}_A$ be the finite subcomplex obtained by taking all faces of the simplices S_p. From Step 1 we know that α in $H_n(\overline{K})$ is represented by the cycle $\tau_n(\alpha')$. We will show that $\tau_n(\alpha') \in H_n(\overline{K})$ is represented by an n-cycle $\xi(\bar{\alpha})$ where $\bar{\alpha}$ is an n-cycle having only non-degenerate n-simplices S on a complex M' which is simplically isomorphic to L' and $\xi : M' \to \overline{K}$ is a simplicial map with the property that whenever $\xi(S)$ is a non-degenerate simplex of $\overline{K}$ its vertices are of the form

$$V_0 \cap \ldots \cap V_r$$

where at least two of the vertices $V_i \in K$ are distinct. Since $\overline{K}$ is finite and $\overline{\overline{K}} = \overline{K}$, we can continue this process to eventually represent $\alpha \in H_n(\overline{K})$ in the form $\xi(\overline{\alpha})$ where $\xi : M \to \overline{K}$ is a simplicial map from some high order subdivision M of K which takes each n-simplex of M into a simplex of lower dimension in $\overline{K}$. This says that $\alpha = \xi(\overline{\alpha})$ must be zero in $H_n(\overline{K})$.

Let E denote the set of vertices v of L′ which correspond to the vertices V of K in $\mathscr{P}_A$ under the homeomorphism $\eta\theta$. Let F denote the remaining vertices of L′. These are all of the form $v_{ij} = (v_i, v_j)$ for $v_i, v_j \in E$. Under τ_n each $v \in E$ goes to $V \in \mathscr{P}_A$ and each $v_{ij} \in F$ goes to $V_i \cap V_j \in \mathscr{P}_A$.

For each vertex $v \in E$, let C_v denote the set of all simplices in L′ containing v. Let

$$C = \bigcup_{v \in E} C_v.$$

Let D denote the subcomplex of L′ consisting of those simplices with vertices in F. Then

$$L' = C \cup D$$

and $C \cap D$ lies in the $(n-1)$-skeleton of L′. Observe that any $(n-1)$-simplex of C_v having v as a vertex does not belong to C_u for $u \neq v$. Let β_v be the part of α' supported in C_v, let $\beta = \Sigma_{v \in E} \beta_v$, and let γ be the part of α' supported in D. We have $\alpha' = \beta + \gamma$.

Now fix a vertex v on E. We will construct a complex

$$M' = D \cup \widetilde{C}_z \cup [\bigcup_{u \neq v} C_u]$$

simplicially isomorphic to L′ where $\widetilde{C}_z$ is isomorphic to C_v but with a different ordering of the vertices. We will also construct a complex P′ containing M′ and L′ as deformation retracts and will produce an n-cycle $\overline{\alpha}$ on M′ which agrees with α' on $M' \cap L'$ and is homologous to it in P′. In fact, P′ will have exactly one more vertex z than L′. Finally, we will produce a simplicial map $\xi : P' \to \overline{K}$ which agrees with τ_n on L′ and takes the vertex z to a vertex in $\overline{K}$ of the form $V_0 \cap \ldots \cap V_r$ where at least two of the V_i are distinct. Continuing this for each vertex $v \in E$ gives a simplicial map $\xi : M' \to \overline{K}$ taking each vertex to a higher order intersection $V_0 \cap \ldots \cap V_r$ and a cycle $\overline{\alpha}$ on M′ such that $\xi(\overline{\alpha}) = \tau_n(\alpha')$ in $H_n(\overline{K})$.

So once again fix a vertex $v \in E$. A typical simplex R in C_v looks like

$$\langle x_1, \ldots, x_a, v, y_1, \ldots, y_b \rangle$$

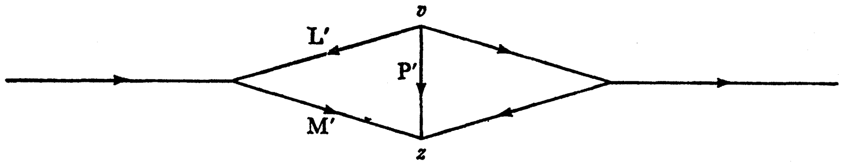
with $a + b = n$, $x_i \in F$, and $y_j \in F$. Under τ_n this goes vertex by vertex to a simplex

$$\langle X_1, \ldots, X_a, V, Y_1, \ldots, Y_b \rangle$$

where each X_i and Y_j is of the form $U \cap V$ for vertices U and V in K, where $V \underset{-}{\to} X_i$, and $V \underset{+}{\to} Y_j$. We allow for the possibility that there are no x_i's or y_j's and therefore no X_i's or Y_j's.

Case 1: There are no x_i's appearing in any simplex of C_v.

Define P' to be L' together with all faces of $\langle v, y_1, \ldots, y_n, z\rangle$ where $\langle v, y_1, \ldots, y_n\rangle$ is a simplex in C_v. Define M' to be $D \cup \widetilde{C}_z \cup [\bigcup_{u \neq v} C_u]$ where $\widetilde{C}_z$ consists of all faces of the $\langle y_1, \ldots, y_n, z\rangle$. The construction in dimension one is illustrated by the diagram



First we define the cycle $\bar{\alpha}$ on M'. The chain β_v is a sum of terms hR where $h \in \mathbf{Z}$ and $R = \langle v, y_1, \ldots, y_b\rangle$. Let

$$S = (-1)^n \langle y_1, \ldots, y_n, z\rangle$$

and define $\bar{\beta}_v$ by replacing each term hR of β_v by hS. Let

$$\bar{\alpha} = \bar{\beta}_v + \Sigma_{u \neq v}\, \beta_u + \gamma.$$

Next we show $\bar{\alpha}$ is homologous to α' in P'. This implies in particular that $\bar{\alpha}$ is a cycle. Corresponding to an n-simplex R of C_v, let

$$T = (-1)^{n+1} \langle v, y_1, \ldots, y_n, z\rangle.$$

Let $\bar{\delta}$ denote the sum of all the terms hT as hR runs over the terms of β_v. We claim that

$$\partial(\bar{\delta}) = \alpha' - \bar{\alpha}.$$

To see this, first recall that the formula for the boundary of a term hR of β_v is

$$\partial(hR) = h\langle y_1, \ldots, y_n\rangle + \sum_{j=1}^{n} (-1)^j h\langle v, y_1, \ldots, \hat{y}_j, \ldots, y_n\rangle.$$

Since $\partial\alpha' = 0$, the disjointness property of the $(n-1)$-simplices in the various β_v implies that summing up all the factors like $(-1)^j h\langle v, y_1, \ldots, \hat{y}_j, \ldots, y_n\rangle$ for $1 \leqslant j \leqslant n$ over all the terms hR in all the simplices of β_v gives zero. Now compute the boundary of hT:

$$\partial(hT) = (-1)^{n+1} h\langle y_1, \ldots, y_n, z\rangle + h\langle v, y_1, \ldots, y_n\rangle$$
$$+ \sum_{j=1}^{n} (-1)^{n+1}(-1)^j h\langle v, y_1, \ldots, \hat{y}_j, \ldots, y_n, z\rangle.$$

Summing up the terms

$$(-1)^{n+1}(-1)^j h\langle v, y_1, \ldots, \hat{y}_j, \ldots, y_n, z\rangle$$

for $1 \leqslant j \leqslant n$ over all terms hT will produce a coefficient equal to $(-1)^{n+1}$ times the coefficient obtained by summing the terms $(-1)^j h\langle v, y_1, \ldots, \hat{y}_j, \ldots, y_n\rangle$ in $\partial(hR)$.

Hence this coefficient is zero, and we therefore conclude that summing the terms $\partial(hT)$ is the same as summing terms

$$(-1)^{n+1} h\langle y_1, \ldots, y_n, z\rangle + h\langle v, y_1, \ldots, y_n\rangle.$$

Namely, $\partial(\bar{\delta}) = \alpha' - \bar{\alpha}$.

It remains to define $\xi : P' \to \bar{K}$. Recall as above that if $U \to V$ in $\mathscr{P}_A$, then $U \underset{+}{\to} U \cap V \underset{-}{\leftarrow} V$. In particular, $U \to U \cap V \to V$. Also if $U \to X$ and $V \to Y$, then $U \cap V \to X \cap Y$. The simplex $\langle v, y_1, \ldots, y_n\rangle$ of C_v goes to $\langle V, Y_1, \ldots, Y_n\rangle$ where $V \underset{+}{\to} Y_j$. Let Z denote the intersection of all these Y_j's appearing for *all* the n-simplices in C_v. By (2.4) we know $Z \in \mathscr{P}_A$, and it is easy to verify that $V \underset{+}{\to} Z$ and $Y_j \underset{+}{\to} Z$. Extend $\tau_n : L' \to \bar{K}$ to the desired map $\xi : P' \to \bar{K}$ by sending

$$\langle v, y_1, \ldots, y_n, z\rangle \quad \text{to} \quad \langle V, Y_1, \ldots, Y_n, Z\rangle$$

vertex by vertex.

Case 2: There is at least one vertex x_j appearing in some simplex

$$R = \langle x_1, \ldots, x_a, v, y_1, \ldots, y_b\rangle$$

of C_v.

Define P' to be L' together with a face

$$T = \langle z, x_1, \ldots, x_a, v, y_1, \ldots, y_b\rangle$$

corresponding to each R where possibly there is no x_i and $a = 0$. Define M' to be

$$D \cup \tilde{C}_z \cup [\textstyle\bigcup_{u \neq v} C_u]$$

where $\tilde{C}_z$ consists of all faces of the $\langle z, x_1, \ldots, x_a, y_1, \ldots, y_b\rangle$.

To obtain $\bar{\alpha}$, let

$$S = (-1)^a \langle z, x_1, \ldots, x_a, y_1, \ldots, y_b\rangle$$

correspond to the simplex R of C_v. Let $\bar{\beta}_v$ be the sum of terms hS as hR runs over the terms in β_v. As in Case 1, let

$$\bar{\alpha} = \bar{\beta}_v + \Sigma_{u \neq v} \beta_u + \gamma.$$

To show $\bar{\alpha}$ and α' are homologous, let $\bar{\delta}$ be the sum of the terms hT as hR runs through β_v. Again, we must show

$$\partial(\bar{\delta}) = \alpha' - \bar{\alpha}.$$

Consider the formula

$$\begin{aligned}\partial(R) = {} & (-1)^a \langle x_1, \ldots, x_a, y_1, \ldots, y_b\rangle \\ & + \sum_{i=1}^{a} (-1)^{i-1} \langle x_1, \ldots, \hat{x}_i, \ldots, x_a, v, y_1, \ldots, y_b\rangle \\ & + \sum_{j=1}^{b} (-1)^{a+j} \langle x_1, \ldots, x_a, v, y_1, \ldots, \hat{y}_j, \ldots, y_b\rangle.\end{aligned}$$

As one sums over the terms $\partial(h\mathrm{R})$ in $\partial(\alpha')$, the last two sums in the above formula add up to zero as in Case 1. Now consider

$$\begin{aligned}\partial(\mathrm{T}) &= \langle x_1, \ldots, x_a, v, y_1, \ldots, y_b \rangle + (-1)^{a+1} \langle z, x_1, \ldots, x_a, y_1, \ldots, y_b \rangle \\ &+ \sum_{i=1}^{a} (-1)^i \langle z, x_1, \ldots, \hat{x}_i, \ldots, x_a, v, y_1, \ldots, y_b \rangle \\ &+ \sum_{j=1}^{b} (-1)^{a+j+1} \langle z, x_1, \ldots, x_a, v, y_1, \ldots, \hat{y}_j, \ldots, y_b \rangle.\end{aligned}$$

There is a one-to-one correspondence between the terms in the last two sums in this formula and the similar terms in $\partial\mathrm{R}$. The corresponding coefficients all differ by a factor of -1. Hence, upon summing the terms $\partial(h\mathrm{T})$, we see that $\partial(\bar{\delta})$ is the sum of the various expressions

$$h \langle x_1, \ldots, x_a, v, y_1, \ldots, y_b \rangle - h(-1)^a \langle z, x_1, \ldots, x_a, y_1, \ldots, y_b \rangle.$$

In other words, $\partial(\bar{\delta}) = \alpha' - \bar{\alpha}$.

Finally, we define $\xi : \mathrm{P}' \to \overline{\mathrm{K}}$. Let Z denote the intersection of all X_i of all the simplices R of C_v. Observe that $\mathrm{V} \underset{-}{\to} \mathrm{X}_j$ and therefore $\mathrm{V} \underset{-}{\to} \mathrm{Z}$ and $\mathrm{X}_j \underset{-}{\to} \mathrm{Z}$ for all X_j. Since $\mathrm{X}_i \underset{-}{\leftarrow} \mathrm{V} \underset{+}{\to} \mathrm{Y}_j$ for all X_i and Y_j regardless of whether X_i and Y_j belong to the same simplex in $\mathscr{P}_\mathrm{A}$, it is easy to verify that $\mathrm{X}_i \underset{+}{\to} \mathrm{X}_i \cap \mathrm{Y}_j \underset{-}{\leftarrow} \mathrm{Y}_j$. See (2.24) below. Therefore $\mathrm{X}_i \to \mathrm{Y}_j$. In particular $\mathrm{Z} \to \mathrm{Y}_j$ for all Y_j as well. Extend $\tau_n : \mathrm{L}' \to \overline{\mathrm{K}}$ to $\xi : \mathrm{P}' \to \overline{\mathrm{K}}$ by sending

$$\langle z, x_1, \ldots, x_a, v, y_1, \ldots, y_b \rangle \quad \text{to} \quad \langle \mathrm{Z}, \mathrm{X}_1, \ldots, \mathrm{X}_a, \mathrm{V}, \mathrm{Y}_1, \ldots, \mathrm{Y}_b \rangle$$

vertex by vertex. This completes the proof of the inductive step.

We are now finished proving $\mathscr{P}_\mathrm{A}$ is contractible.

Next we discuss certain boundedness properties of the matrices $\mathrm{M} = \mathrm{M}(\mathrm{U})$ where $\mathrm{U} \in \mathscr{P}_\mathrm{A}$. Recall that a non-negative matrix $\mathrm{M} = \{ \mathrm{M}(s, t) \}$ is *row finite* (resp. *column finite*) provided each row (resp. column) has at most finitely many non-zero entries. A matrix is *locally finite* provided it is both row and column finite. Let

$$\begin{aligned}\| \mathrm{M} \|_\infty &= \sup_s(\Sigma_t \mathrm{M}(s, t)), \\ \| \mathrm{M} \|_1 &= \sup_t(\Sigma_s \mathrm{M}(s, t)), \\ \| \mathrm{M} \| &= \max\{ \| \mathrm{M} \|_1, \| \mathrm{M} \|_\infty \}.\end{aligned}$$

Proposition **2.15.** — *Let* $\mathrm{M} = \mathrm{M}(\mathrm{U})$ *for* $\mathrm{U} \in \mathscr{P}_\mathrm{A}$. *If* A *is finite, then* M *is finite. If* A *is locally finite, then so is* M *and moreover* $\| \mathrm{A} \| < \infty$ *implies* $\| \mathrm{M} \| < \infty$.

If A is finite, then X_A is compact and therefore any cover by open and closed disjoint sets must be finite. Hence M is finite. So now assume A is locally finite. The proof that $\mathscr{P}_\mathrm{A}$ is connected shows that it is sufficient to consider the special cases

(I) $\mathrm{U} < \mathrm{V}$ and $\ell(\mathrm{U}, \mathrm{V}) \leqslant 1$

(II) $\mathrm{V} < \mathrm{U}$ and $\ell(\mathrm{V}, \mathrm{U}) \leqslant 1$

where we assume (2.15) has already been proved for $U \in \mathscr{P}_A$ and must show it then holds for $V \in \mathscr{P}_A$. The proof of these two cases is straightforward.

Next we consider Markov partitions from the viewpoint expounded in [F, p. 100] which uses canonical coordinates. Let $x \in X_A$ and $n \in \mathbf{Z}$. Let

$$W^s(x, n) = \{y \mid y_i = x_i, i \geqslant n\}$$
$$W^u(x, n) = \{y \mid y_i = x_i, i \leqslant n\}.$$

If $x, y \in X_A$ satisfy $x_0 = y_0$, then $W^u(x, 0) \cap W^s(y, 0)$ consists of the single point z where $z_i = x_i$ for $i \leqslant 0$ and $z_i = y_i$ for $i \geqslant 0$. Define $[x, y] = z$. A set $R \subset X_A$ is a *rectangle* provided it is open and closed and $x, y \in R$ implies $x_0 = y_0$ and $[x, y] \in R$. If R is a rectangle, let $W^s(x, R) = W^s(x, 0) \cap R$ and $W^u(x, R) = W^u(x, 0) \cap R$. Then for any $x \in R$ we have $W^u(x, R) \times W^s(x, R) \cong R$ via the correspondence $(a, b) \to [b, a]$.

Definition **2.16.** — *A topological Markov partition of rectangles* for σ_A on X_A is a covering $U = \{U_i\}$ by disjoint rectangles such that if $x \in U_i$ and $\sigma_A(x) \in U_j$, then

$$\sigma_A(W^u(x, U_i)) \supset W^u(\sigma_A(x), U_j)$$
$$\sigma_A(W^s(x, U_i)) \subset W^s(\sigma_A(x), U_j).$$

Moreover U is said to be *uniform* provided $U < U^A(-n, n)$ for some $n \geqslant 0$. Note that $U^A < U$ automatically because each U_i is a rectangle.

We let $\mathscr{R}_A$ denote the set of all topological Markov partitions by rectangles for σ_A. Clearly $U^A \in \mathscr{R}_A$. The following proposition extends to $\mathscr{R}_A$ several properties which are well known when A is finite.

Proposition **2.17.** *If* U *and* V *belong to* $\mathscr{R}_A$, *then so do* $U \cap V$, $U \cap \sigma_A(V)$, *and* $U \cap \sigma_A^{-1}(V)$. *Moreover, if* $U \in \mathscr{R}_A$, *then* $U \in \mathscr{P}_A$.

The proof is similar to the case when A is finite. Note also that $\mathscr{R}_A$ is closed under intersection while $\mathscr{P}_A$ is not.

Remark **2.18.** — The subcomplex $\mathscr{R}_A$ of $\mathscr{P}_A$ formed by only considering simplices with vertices in $\mathscr{R}_A$ is also contractible. The proof is the same as for $\mathscr{P}_A$. However, $\mathscr{R}_A$ is closed under intersection; so we automatically know, for example, that the vertex Z in Step III of (2.12) lies in $\mathscr{R}_A$. It is not necessary to invoke something like (2.4).

The set of uniform equivalences $\mathrm{Isom}(\sigma_A, \sigma_A)$ does not act on all of $\mathscr{R}_A$ but only on "small" elements of $\mathscr{R}_A$. For example, $U^A \in \mathscr{R}_A$ but $\sigma_A^{-1}(U^A) \notin \mathscr{R}_A$ even though $\sigma_A^{-1}(U^A) \in \mathscr{P}_A$. However, $\sigma_A^{-1}(U^A(-1, 0)) \in \mathscr{R}_A$. Here is the precise statement.

Lemma **2.19.** — *Let* $\alpha_1, \ldots, \alpha_k$ *be a finite collection of uniform equivalences from* X_A *to* X_B. *Let* $m \geqslant 0$. *Then there is an integer* $n \geqslant 0$ *such that for each* $j = 1, \ldots, k$ *if* $V \in \mathscr{R}_A$ *refines* $U^A(-n, n)$, *then* $\alpha_j(V) \in \mathscr{R}_B$ *and* $\alpha_j(V)$ *refines* $U^B(-m, m)$.

Definition **2.20.** — Any partition V as in (2.19) will be called (α, m, n)-*small*, or, for brevity, just α-*small*.

Proof of 2.19. — This is well known [F] for finite matrices. For completeness we give the argument in the general case.

Let $\mathcal{S}$ and $\mathcal{E}$ be the state spaces for A and B respectively. Recall from [H, W2] that a uniformly continuous $\alpha : X_A \to X_B$ can be written in the form $\alpha = \sigma_B^k h_\infty = h_\infty \sigma_A^k$ where h_∞ is a $(p+1)$-block map determined by $h : \mathcal{S}_A^{p+1} \to \mathcal{E}$. Here $\mathcal{S}_A^{p+1}$ consists of those $(p+1)$-tuples $[x_0, \ldots, x_p]$ with $A(x_i, x_{i+1}) = 1$ for $0 \leqslant i \leqslant p-1$, and h satisfies $B(h(x_0, \ldots, x_p), h(y_0, \ldots, y_p)) = 1$ whenever $x_i = y_{i-1}$ for $1 \leqslant i \leqslant p$.

Consider the special case of a single $\alpha : X_A \to X_B$ of the form $\alpha = h_\infty$. Suppose $U^A(0, p) < V$ and R is a rectangle of V with $x \in R$. We will show

$$\alpha(R) \text{ is a rectangle,}$$

$$\alpha(W^s(x, R)) = W^s(\alpha(x), \alpha(R)),$$

$$\alpha(W^u(x, R)) = W^u(\alpha(x), \alpha(R)).$$

Property (2.16) for $\alpha(V)$ is an immediate consequence of this. For brevity of notation, let $W_k^u(x) = W^u(x, k)$ and $W_k^s(x) = W^s(x, k)$.

(a) $\alpha(R)$ is a rectangle.

Let $x, y \in R$. Then $[\alpha(x), \alpha(y)] = \alpha[x, y] \in \alpha(R)$ because $x_i = y_i$ for $0 \leqslant i \leqslant p$.

(b) $\alpha(W^s(x, R)) = W^s(\alpha(x), \alpha(R))$.

Clearly the left-hand side $\alpha(W^s(x)) \cap \alpha(R)$ is contained in the right-hand side $W_0^s(\alpha(x)) \cap \alpha(R)$ because $\alpha(W_0^s(x)) \subset W_0^s(\alpha(x))$. To see the other way around, suppose that $z = \alpha(u)$ for some $u \in R$ and that $z_i = \alpha(x)_i$ for $i \geqslant 0$. Let $v = [u, x] \in W_0^s(x)$. Then $v \in R$ and $\alpha(v) = z$ because $u_i = x_i$ for $0 \leqslant i \leqslant p$.

(c) $\alpha(W^u(x, R)) = W^u(\alpha(x), \alpha(R))$.

The assumption on R implies that $W_0^u(x) \cap R = W_p^u(x) \cap R$. Since

$$\alpha(W_p^u(x)) \subset W_0^u(\alpha(x)),$$

we see that the left-hand side $\alpha(W_0^u(x) \cap R) = \alpha(W_p^u(x)) \cap \alpha(R)$ is contained in the right-hand side. Conversely, suppose z satisfies $z = \alpha(u)$ for $u \in R$ and $z_i = \alpha(x)_i$ for $i \leqslant 0$. Let $v = [x, u] \in W_0^u(x)$. Then $v \in R$ and $\alpha(v) = z$ because $x_i = u_i$ for $0 \leqslant i \leqslant p$.

It remains to see that $\alpha(V)$ is a uniform Markov partition. Suppose $V < U^A(-k, k)$. By uniform continuity of α^{-1} there is an $\ell \geqslant 0$ so that $U^A(-k, k) < \alpha^{-1}(U^B(-\ell, \ell))$. Then $\alpha(V) < \alpha(U^A(-k, k)) < U^B(-\ell, \ell)$.

It is now clear that given any $m \geqslant 0$ and a finite collection of block maps written in the form $\sigma_A^k h_\infty$, it is possible to choose n so large that the conclusion (2.19) holds. This completes the proof.

We now discuss the action (2.3) of $\mathrm{Aut}(\sigma_A)$ on $\mathcal{P}_A$. If G is any subgroup, let $\mathcal{P}_A^G$

denote the subcomplex of all simplices which are pointwise fixed by G. Assume, moreover, that A is *finite.*

Proposition **2.21.** — (A). *$\mathscr{P}_A^G$ is non-empty if and only if* G *is finite, in which case $\mathscr{P}_A^G$ is contractible.* (B) *The action of* $\mathrm{Aut}(\sigma_A)$ *on $\mathscr{P}_A$ is properly discontinuous.*

Proof of (A). — Assume $|G| < \infty$ and let $U \in \mathscr{R}_A$ be small. Since $\mathscr{R}_A$ is closed under intersections the partition $\bigcap_{\alpha \in G} \alpha(U)$ is again in $\mathscr{R}_A$ and is fixed by G. Thus $\mathscr{P}_A^G$ is non-empty. The proof that $\mathscr{P}_A$ is contractible carries over directly to $\mathscr{P}_A^G$ using the fact that if U and V are partitions fixed by G, then so is $U \cap V$. As in the proof of (2.12) these intersections are only to be taken under circumstances where they will still be Markov.

Conversely, we now show that if $U \in \mathscr{P}_A$, then the isotropy group H(U) of U in $\mathrm{Aut}(\sigma_A)$ is finite. Let $B = M(U)$ as in (2.5) and let $\theta : X_A \to X_B$ be as in (2.6). Observe $\theta(U) = U^B$. Therefore, if $\alpha(U) = U$, then $\beta = \theta\alpha\theta^{-1} \in \mathrm{Aut}(\sigma_B)$ fixes U^B and determines a permutation matrix $\mu(\beta)$ of the states corresponding to the sets in U such that $\mu(\beta)B = B\mu(\beta)$. Conversely, any such permutation gives a one-block automorphism of σ_B and hence an automorphism of σ_A. This procedure defines an isomorphism between H(U) and the finite group of such permutations.

Proof of (B). — We show that if K is a finite subcomplex of $\mathscr{P}_A$ and H is the set of symmetries α satisfying $K \cap \alpha(K) \neq \emptyset$, then H is finite. For each pair of vertices (U, V) of $K \times K$, let H(U, V) denote the set of those α for which $\alpha(U) = V$. A given H(U, V) may be empty, but H is the union of the finite collection of the H(U, V). Moreover, H(U, V) is a coset of the isotropy subgroup H(U) which is finite by (A).

Finally, in this section we present some facts about Markov partitions which will be used in § 4.

Lemma **2.22.** — *Let* $U, V \in \mathscr{P}_A$. *If* $U \underset{+}{\to} V$ *and* $U \underset{-}{\to} V$, *then* $U = V$.

Proof. — Let $U = \{U_i\}$ for $i \in I$ and let $B = M(U)$. According to (2.6) there is an isomorphism $\alpha : X_A \to X_B$ between σ_A and σ_B under which $\alpha(U_i) = U_i^B$, the standard cylinder set for $i \in I$. Thus $\alpha(U) = U^B$. Moreover, $\alpha(V) \in \mathscr{P}_B$ and we have $U^B \underset{+}{\to} \alpha(V)$ and $U^B \underset{-}{\to} \alpha(V)$. So it really suffices to consider the special case $U = U^A$. Let $V_k \in V$. Since $U < V < U(-1, 0)$, we can write

$$V_k = \bigcup_{\text{some } s} \sigma_A(U_s) \cap U_j$$

where $V_k \subset U_j$. Thus all $q \in \mathscr{S}$ for which $A(j, q) = 1$ can occur as $q = x_1$ for some $x = \{x_i\} \in V_k$. On the other hand, write

$$V_k = \bigcup_{\text{some } q} U_j \cap \sigma_A^{-1}(U_q)$$

using the hypothesis $U < V < U \cup \sigma_A^{-1}(U)$. From above all q with $A(j, q) = 1$ can occur. Thus

$$V_k = \bigcup_q U_j \cap \sigma_A^{-1}(U_q) = U,$$

where $A(j, q) = 1$.

Lemma **2.23.** — *If* $V, W, X \in \mathscr{P}_A$ *satisfy the condition* $V \underset{+}{\rightarrow} X \underset{-}{\leftarrow} W$, *then* $X = V \cap W$.

Proof. — Check that $V \cap W \underset{+}{\rightarrow} X \underset{-}{\leftarrow} V \cap W$ and apply (2.22):

$$\begin{array}{c} X < V \cap \sigma_A^{-1}(V) < (W \cap \sigma_A^{-1}(W)) \cap (V \cap \sigma_A^{-1}(V)) \\ \| \\ (V \cap W) \cap \sigma_A^{-1}(V \cap W) \end{array}$$

$$\begin{array}{c} X < W \cap \sigma_A(W) < (V \cap \sigma_A(V)) \cap (W \cap \sigma_A(W)) \\ \| \\ (V \cap W) \cap \sigma_A(V \cap W) \end{array}$$

Lemma **2.24.** — *Let* $U, V, W \in \mathscr{P}_A$ *satisfy* $V \underset{-}{\leftarrow} U \underset{+}{\rightarrow} W$. *Then* $V \underset{+}{\rightarrow} V \cap W \underset{-}{\leftarrow} W$.

Proof. — We have $U < V$, $\sigma_A^{-1}(U) < \sigma_A^{-1}(V)$, and $W < U \cap \sigma_A^{-1}(U) < V \cap \sigma_A^{-1}(V)$. Hence $V \cap W < V \cap (V \cap \sigma_A^{-1}(V)) = V \cap \sigma_A^{-1}(V)$. On the other hand, we have $U < W$, $\sigma_A(U) < \sigma_A(W)$, and $V < \sigma_A(U) \cap U < \sigma_A(W) \cap W$. Therefore

$$V \cap W < (\sigma_A(W) \cap W) \cap W = \sigma_A(W) \cap W.$$

3. The triangle identities

The main goal of this section is to prove the algebraic identities (3.3) arising from triangles in $\mathscr{P}_A$.

Let $U = \{ U_i \}$ and $V = \{ V_k \}$ be any two vertices of $\mathscr{P}_A$. As in [PW] define $R = R(U, V)$ and $S = S(V, U)$ to be

$$\text{(3.1)} \qquad \begin{aligned} R(i, k) &= \begin{cases} 1, & \text{if } U_i \cap V_k \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ S(k, i) &= \begin{cases} 1, & \text{if } V_k \cap \sigma_A^{-1}(U_i) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $P = M(U)$ and $Q = M(V)$ be as in (2.5).

Proposition **3.2.** — *If* $U \rightarrow V$, *then* $P = RS$ *and* $Q = SR$.

Remark. — We have not made any assumptions about finiteness of the matrices. So part of (3.2) asserts that $R(i, k)\, S(k, j) = 1$ for at most finitely many k. Similarly for $S(k, i)\, R(i, \ell)$. In fact, a step in the proof is to show RS and SR are indeed zero-one matrices.

Proof of 3.2.

Step I: $RS = P$. Fix a pair of indices (i, j). We must show

$$P(i, j) = \Sigma_k R(i, k)\, S(k, j).$$

First assume the right-hand side (RHS) is not zero. We will verify that the left-hand side (LHS) is non-zero and, moreover, only one term on the RHS is non-zero. Since $U < U \cap V < \sigma_A(V) \cap V$ we can write U_i as the disjoint union $U_i = \bigcup \sigma_A(V_a) \cap V_b$ of certain $\sigma_A(V_a) \cap V_b \neq \emptyset$. Similarly we write $U_j = \bigcup \sigma_A(V_c) \cap V_d$ where $\sigma_A(V_c) \cap V_d \neq \emptyset$. Suppose k is an index where $R(i, k)\, S(k, j) = 1$. Then $U_i \cap V_k \neq \emptyset$ and $V_k \cap \sigma_A^{-1}(U_j) \neq \emptyset$, and therefore some $b = k$ and some $c = k$. In particular,

$$U_i \cap \sigma_A^{-1}(U_j) \supset \sigma_A(V_a) \cap V_k \cap \sigma_A^{-1}(V_d) \neq \emptyset$$

for some pair of indices (a, d) where the triple intersection is non-empty by *(c)* of (2.1). In particular $P(i, j) \neq 0$. Recall that $V < U \cap V < U \cap \sigma_A^{-1}(U)$. Therefore, if a pair of indices (i, j) is given with $U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset$, then there is only one V_ℓ such that $V_\ell \supset U_i \cap \sigma_A^{-1}(U_j)$. Therefore, if $R(i, k)\, S(k, j) = 1$ we have

$$V_k \supset U_i \cap \sigma_A^{-1}(U_j) \supset \sigma_A(V_a) \cap V_k \cap \sigma_A^{-1}(V_d) \neq \emptyset$$

and k is determined by the pair (i, j). That is, there is only one k with $R(i, k)\, S(k, j) = 1$.

Conversely, suppose $P(i, j) = 1$. We show there is a k satisfying $R(i, k)\, S(k, j) = 1$. Let V_k be the unique k such that $V_k \supset U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset$. Then $U_i \cap V_k \neq \emptyset$ and $V_k \cap \sigma_A^{-1}(U_j) \neq \emptyset$. Hence $R(i, k)\, S(k, j) = 1$.

Step II: $SR = Q$. Fix a pair of indices (k, ℓ). We must show

$$Q(k, \ell) = \Sigma_i S(k, i)\, R(i, \ell).$$

Assume RHS is non-zero. We must show $Q(k, \ell) \neq 0$ and that there is exactly one i such that $S(k, i)\, R(i, \ell) = 1$. Write $V_k = \bigcup U_a \cap \sigma_A^{-1}(U_b)$ and $V_\ell = \bigcup U_c \cap \sigma_A^{-1}(U_d)$ using the condition $V < U \cap V < U \cap \sigma_A^{-1}(U)$. Let i satisfy $S(k, i)\, R(i, \ell) = 1$. Then some $b = i$ and some $c = i$. Hence

$$V_k \cap \sigma_A^{-1}(V_\ell) \supset U_a \cap \sigma_A^{-1}(U_i) \cap \sigma_A^{-2}(U_d) \neq \emptyset$$

for some pair of indices (a, d), where the triple intersection is non-empty by *(c)* of (2.1). As in Step I, we use the condition $U < U \cap V < \sigma_A(V) \cap V$ to show that i is determined by k and ℓ. Conversely, suppose $Q(k, \ell) = 1$. Let i be the unique index such that $U_i \supset \sigma_A(V_k) \cap V_\ell \neq \emptyset$. Then $S(k, i)\, R(i, \ell) = 1$. This completes the proof of (3.2).

Now consider a triangle

$$\begin{tikzcd} & V \arrow[dr, "\text{side 2}"] & \\ U \arrow[ur, "\text{side 1}"] \arrow[rr, "\text{side 3}"'] & & W \end{tikzcd}$$

in $\mathscr{P}_A$ where $U = \{ U_i \}$, $V = \{ V_k \}$, $W = \{ W_p \}$, and let

$$\begin{aligned} M &= M(U), & & \\ R_1 &= R(U, V), & S_1 &= S(V, U) \\ R_2 &= R(V, W), & S_2 &= S(W, V) \\ R_3 &= R(U, W), & S_3 &= S(W, U). \end{aligned}$$

Proposition **3.3.** — *Triangle identities*:

$$\begin{aligned} R_1 R_2 &= R_3 \\ S_2 S_1 &= S_3 M. \end{aligned}$$

Step I: *The triangle identity for the* R*-matrices.*

The argument proceeds by several special cases. Consider the triangle

$$\begin{tikzcd} U \arrow[r, "\varepsilon_1"] \arrow[dr, "\varepsilon_3"'] & V \arrow[d, "\varepsilon_2"] \\ & W \end{tikzcd}$$

where $\varepsilon_i = \pm 1$ for $i = 1, 2, 3$. If all $\varepsilon_i = +1$, then it follows easily from (3.1) that

(3.4) $$R(U, W) = R(U, V)\, R(V, W).$$

If all $\varepsilon_i = -1$, we similarly have

(3.5) $$R(W, U) = R(W, V)\, R(V, U).$$

Lemma **3.6.** — *If one* $\varepsilon_i = 1$ *and another* $\varepsilon_i = -1$, *then either* $U = V$ *or* $V = W$.

Proof. — By (2.22) it suffices to show that either $U \underset{\pm}{\rightarrow} V$ or $V \underset{\pm}{\rightarrow} W$. There are four cases to consider. For example, suppose $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, and $\varepsilon_3 = 1$. Then $W < U \cap \sigma_A^{-1}(U) < V \cap \sigma_A^{-1}(V)$ so that $V \underset{+}{\rightarrow} W$. Hence $V \underset{\pm}{\rightarrow} W$. The other cases are similar.

Lemma **3.7.** — *For any* $U, V \in \mathscr{P}_A$, *we have*

$$R(U, V) = R(U, U \cap V)\, R(U \cap V, V).$$

Proof. — Easy from (3.1).

Lemma **3.8.** — *If* $V \underset{-}{\leftarrow} U \underset{+}{\rightarrow} W$, *then*

$$R(V, W) = R(V, U)\, R(U, W).$$

Proof. — Let $R_1 = R(V, W)$, $R_2 = R(V, U)$ and $R_3 = R(U, W)$. Let $U = \{U_i\}$, $V = \{V_k\}$ and $W = \{W_\ell\}$. We must show $R_1 = R_2 R_3$. First observe that $R_2 R_3$ is a zero-one matrix. To see this, fix a pair of indices (k, ℓ) and write

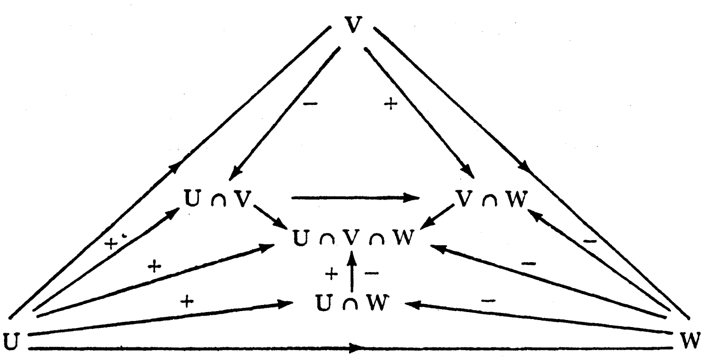
$$V_k = \bigcup_a \sigma_A(U_a) \cap U_b$$
$$W_\ell = \bigcup_d U_c \cap \sigma_A^{-1}(U_d).$$

Suppose a term $R_2(k, j)\, R_3(j, \ell) = 1$ in the sum $\Sigma_j R_2(k, j)\, R_3(j, \ell)$. Then $V_k \cap U_j \neq \emptyset$ and $U_j \cap W_\ell \neq \emptyset$. This implies $j = b = c$. Thus j is determined by (k, ℓ) and there is at most one non-zero term. Moreover, $V_k \cap W_\ell \supset \sigma_A(U_a) \cap U_j \cap \sigma_A^{-1}(U_d)$ which is non-empty by *(c)* of (2.1). Thus, $R_1(k, \ell) = 1$. Conversely, if $R_1(k, \ell) = 1$, then for some pair of indices (a, d)

$$\sigma_A(U_a) \cap U_b \cap U_c \cap \sigma_A^{-1}(U_d)$$

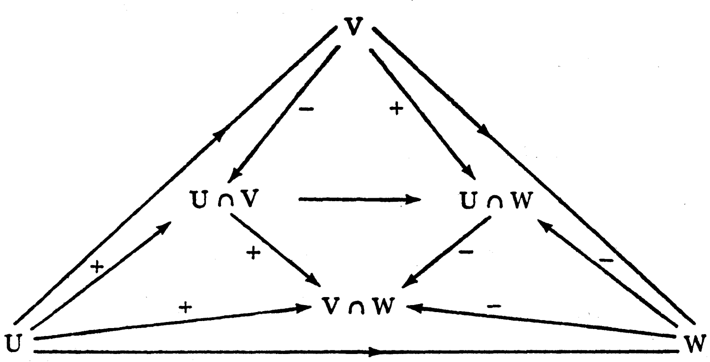
must be non-empty. Thus $b = c$ and the term $R_2(k, j)\, R_3(j, \ell) = 1$ for $j = b = c$.

Now observe that the triangle $\langle U, V, W \rangle$ gives rise to the following commutative diagram of Markov partitions:



From (2.22) we see that $U \cap W = U \cap V \cap W$. Hence

(3.9)



Note in particular that $V < U \cap W$. The required R-identity is now a consequence of several applications of (3.4), (3.5), (3.7) and (3.8).

Step II: *The triangle identity for the* S-*matrices.*

The diagram (3.9) yields the commutative diagram

$$\text{(3.10)} \qquad \begin{tikzcd} & V \arrow[dr] & \\ U \arrow[ur] \arrow[dr, "+"'] & & W \arrow[dl, "-"] \\ & U \cap W & \end{tikzcd}$$

Let $T_1 = S(U \cap W, U)$ and $T_2 = S(W, U \cap W)$. We now verify the identity

$$\text{(3.11)} \qquad S_2 S_1 = T_2 T_1.$$

Fix a pair of indices (p, i) and consider the two expressions

$$\text{(LHS)} \qquad \Sigma_k S_2(p, k)\, S_1(k, i)$$

and

$$\text{(RHS)} \qquad \Sigma_{(q,j)} T_2(p, (q, j))\, T_1((q, j), i).$$

In general, these sums will be non-negative integers and not just zero or one. We prove that LHS $=$ RHS by showing that for each pair (q, j) for which $T_2(p, (q, j))\, T_1((q, j), i) = 1$ there is exactly one corresponding index k such that $S_2(p, k)\, S_1(k, i) = 1$ and vice versa. Let $\sigma = \sigma_A$.

Let (q, j) be an index pair such that $T_2(p, (q, j))\, T_1((q, j), i) = 1$. Let V_k be the unique element of V such that $V_k \supset W_q \cap U_j \neq \emptyset$. Then

$$W_p \cap \sigma^{-1}(V_k) \supset W_p \cap \sigma^{-1}(W_q \cap U_j) \neq \emptyset,$$
$$V_k \cap \sigma^{-1}(U_i) \supset W_q \cap U_j \cap \sigma^{-1}(U_i) \neq \emptyset,$$

so that $S_2(p, k)\, S_1(k, i) = 1$. Now suppose (q', j') is another pair such that

$$T_2(p, (q', j'))\, T_1((q', j'), i) = 1$$

and let k' satisfy $V_{k'} \supset W_{q'} \cap U_{j'} \neq \emptyset$. If we can show that the condition $(q, j) \neq (q', j')$ implies $k \neq k'$, then we will know that RHS $\leqslant$ LHS. From (3.9) we can write

$$W_p = \bigcup_{a,b} V_a \cap \sigma^{-1}(V_b)$$
$$W_q = \bigcup_{c,d} V_c \cap \sigma^{-1}(V_d)$$
$$U_j = \bigcup_{e,f} \sigma(V_e) \cap V_f$$
$$U_i = \bigcup_{g,h} \sigma(V_g) \cap V_h.$$

Then we can write $W_p \cap \sigma^{-1}(W_q \cap U_j) \neq \emptyset$ more explicitly as

$$\bigcup_{a,b,c,d,e,f} (V_a \cap \sigma^{-1}(V_b)) \cap (\sigma^{-1}(V_c) \cap \sigma^{-2}(V_d)) \cap (V_e \cap \sigma^{-1}(V_f)).$$

Since $V < W \cap U$ and $V < \sigma(W) \cap W$, we have $\sigma^{-1}(V) < W \cap \sigma^{-1}(W) \cap \sigma^{-1}(U)$. The above expression then simplifies to

$$\bigcup_{a,d,e} (V_a \cap \sigma^{-1}(V_k)) \cap (\sigma^{-1}(V_k) \cap \sigma^{-2}(V_d)) \cap (V_e \cap \sigma^{-1}(V_k)).$$

Since $\sigma^{-1}(U) < W \cap \sigma^{-1}(W)$ we must have

$$W_p \cap \sigma^{-1}(W_q) \cap \sigma^{-1}(U_j) = W_p \cap \sigma^{-1}(W_q) \subset \sigma^{-1}(U_j)$$

and can therefore conclude

(3.12) every a for which $V_a \cap \sigma^{-1}(V_k) \neq \emptyset$ in the expression for W_p must occur as some e for which $\sigma(V_e) \cap V_k \neq \emptyset$ in the expression for U_j.

Next write $W_q \cap U_j \cap \sigma^{-1}(U_i) \neq 0$ more explicitly as the union

$$\bigcup_{c,d,e,f,g,h}(V_c \cap \sigma^{-1}(V_d)) \cap (\sigma(V_e) \cap V_f) \cap (V_g \cap \sigma^{-1}(V_h)).$$

Since $V < W \cap U$ and $V < U \cap \sigma^{-1}(U)$, we have $V < W \cap U \cap \sigma^{-1}(U)$. The above expression simplifies to

$$\bigcup_{d,e,h}(V_k \cap \sigma^{-1}(V_d)) \cap (\sigma(V_e) \cap V_k) \cap (V_k \cap \sigma^{-1}(V_h)).$$

Since $W < U \cap \sigma^{-1}(U)$, we see that $W_q \cap U_j \cap \sigma^{-1}(U_i) = U_j \cap \sigma^{-1}(U_i) \subset W_q$ and therefore have

(3.13) every h for which $\sigma(V_k) \cap V_h \neq \emptyset$ in the expression for U_i must occur as some d for which $V_k \cap \sigma^{-1}(V_d) \neq \emptyset$ in the expression for W_q.

Suppose now that $k = k'$. It then follows from (3.12) and (3.13) that

$$(W_q \cap U_j) \cap (W_{q'} \cap U_{j'}) \neq \emptyset,$$

contrary to the assumption that $(q, j) \neq (q', j')$. Hence $k \neq k'$ and the correspondence $(q, j) \to k$ sends no two (q, j) to the same k.

It remains to show that LHS $\leqslant$ RHS. Choose an index k such that $S_2(p, k)S_1(k, i) = 1$. This means $W_p \cap \sigma^{-1}(V_k) \neq \emptyset$ and $V_k \cap \sigma^{-1}(U_i) \neq \emptyset$. Use (3.9) to write

$$W_p = \bigcup_{a,b} V_a \cap \sigma^{-1}(V_b),$$
$$U_i = \bigcup_{c,d} \sigma(V_c) \cap V_d.$$

Then

$$W_p \cap \sigma^{-1}(V_k) = \bigcup_a V_a \cap \sigma^{-1}(V_k),$$
$$V_k \cap \sigma^{-1}(U_i) = \bigcup_d V_k \cap \sigma^{-1}(V_d).$$

Hence the Markov property (2.1) implies

$$W_p \cap \sigma^{-1}(V_k) \cap \sigma^{-2}(U_i) \neq \emptyset.$$

Now write

$$V_k = \bigcup_{x,y} W_x \cap U_y$$

and then

$$W_p \cap \sigma^{-1}(V_k) = \bigcup_{x,y} W_p \cap \sigma^{-1}(W_x) \cap \sigma^{-1}(U_y)$$
$$V_k \cap \sigma^{-1}(U_i) = \bigcup_{x,y} W_x \cap U_y \cap \sigma^{-1}(U_i).$$

We see that there must be some $W_q \cap U_j$ in the expression for V_k such that

$$W_p \cap \sigma^{-1}(W_q) \cap \sigma^{-1}(U_j) \cap \sigma^{-2}(U_i) \neq \emptyset.$$

In particular,

$$W_p \cap \sigma^{-1}(W_q \cap U_j) \neq \emptyset$$
$$(W_q \cap U_j) \cap \sigma^{-1}(U_i) \neq \emptyset$$

so that $T_2(p, (q, j))\, T_1((q, j), i) = 1$. We demonstrated above that for a given V_k satisfying $S_2(p, k)\, S_1(k, i) = 1$, there is exactly one pair (q, j) satisfying the above condition. Hence the correspondence $k \to (q, j)$ is well defined. It is clearly injective; because if $k \to (q, j)$ and $k' \to (q, j)$, then $V_k \cap V_{k'} \supset W_q \cap U_j \neq \emptyset$ which is contrary to the basic assumption that the V_k are disjoint. This completes the proof of (3.11).

The proof of the triangle identity for the S-matrices occurring for the triangle $\langle U, V, W \rangle$ now follows from two applications of (3.10): the first for (3.10) as is, and the second for (3.10) with $U = V$ because in this case $S_1 = S(V, U) = M$ and $S_2 = S(W, U) = S_3$.

This completes the proof of (3.3).

4. Invariants for $\mathrm{Aut}(\sigma_A)$

We first construct a homomorphism from $\mathrm{Aut}(\sigma_A)$ to the fundamental group $\pi_1(S(\bar{\mathscr{E}}), A)$ of the space $S(\bar{\mathscr{E}})$ of shift equivalences.

Throughout this section we let $\mathscr{E}$ denote the set of $\mathscr{S}$ by $\mathscr{I}$ zero-one matrices M on products $\mathscr{S} \times \mathscr{I}$ of various finite or countable state spaces $\mathscr{S}$ and $\mathscr{I}$ such that each row and each column has at least one non-zero entry. We will moreover assume that the matrices in $\mathscr{E}$ belong to one of the four following classes:

(4.1)
- *(a)* M is finite
- *(b)* M is infinite but locally finite
- *(c)* M is infinite and $\| M \| < \infty$
- *(d)* M is infinite.

In case *(d)* there is the tacit assumption that a product RS of matrices is written only when it is well defined. Thus the equation $P = RS$ assumes that even though we may have $R(i, k) > 0$ and $S(\ell, j) > 0$ for infinitely many k and ℓ, there are only finitely many k such that $R(i, k) > 0$ and $S(k, j) > 0$ simultaneously for a given pair of indices i and j. It is not hard to verify along the lines of (2.15) that if $U \to V$ in $\mathscr{P}_A$ and A is in $\mathscr{E}$, then the matrices R and S of (3.1) also belong to $\mathscr{E}$. We let $\bar{\mathscr{E}}$ be the category of matrices formed by taking the " closure " of $\mathscr{E}$; namely, all products of matrices in $\mathscr{E}$. A standard state splitting argument as in [K] shows that $\bar{\mathscr{E}}$ consists of all non-negative integral matrices satisfying the corresponding condition in (4.1) as do the matrices in $\mathscr{E}$.

Let A be an $\mathscr{S} \times \mathscr{S}$ matrix and B be a $\mathscr{I} \times \mathscr{I}$ matrix in $\bar{\mathscr{E}}$. Recall from, say, [E] that a *shift equivalence* $R : A \mapsto B$ in $\bar{\mathscr{E}}$ is an $\mathscr{S} \times \mathscr{I}$ matrix R in $\bar{\mathscr{E}}$ such that there exists a $\mathscr{I} \times \mathscr{S}$ matrix S in $\bar{\mathscr{E}}$ and an integer $n > 0$ satisfying

$$\textbf{(4.2)} \qquad \begin{aligned} AR &= RB, \quad & SA &= BS \\ RS &= A^n, \quad & SR &= B^n. \end{aligned}$$

Observe that if $P : A \mapsto B$ and $Q : B \mapsto C$ then $PQ : A \mapsto C$.

Definition **4.3.** — The *space* $S(\bar{\mathscr{E}})$ *of shift equivalences* in $\bar{\mathscr{E}}$ is the realization of the simplicial set where an n-simplex consists of

a) an $(n+1)$-tuple $\langle A_0, \ldots, A_n \rangle$ of square matrices A_i in $\bar{\mathscr{E}}$, and
b) a shift equivalence $R_i : A_{i-1} \mapsto A_i$ in $\bar{\mathscr{E}}$ for $1 \leqslant i \leqslant n$.

The face operators come from composition and the degeneracy operators insert the identity. See [S] or [Sp] for background on simplicial sets and simplicial complexes. It is immediate from the definitions that the set of path components $\pi_0(S(\bar{\mathscr{E}}))$ of $S(\bar{\mathscr{E}})$ are just the shift equivalence classes of matrices in $\bar{\mathscr{E}}$.

Take note of the following conventions. Composition will be read from left to right in $\mathscr{E}$ and $\bar{\mathscr{E}}$. In the category of sets and functions or of spaces and continuous maps, composition will be read from right to left. Thus if $f : X \to Y$ and $g : Y \to Z$ are functions, then the composition is $gf : X \to Z$. Also, in the category of right modules and module homomorphisms composition will be read from right to left. If $f : I \to J$ is a bijection of sets, let f also denote the $J \times I$ permutation matrix which is 1 in the (j, i) entry if and only if $j = f(i)$. If $f : I \to J$ and $g : J \to K$ are bijections then the $K \times I$ matrix associated to gf is the product of the $K \times J$ matrix for g followed on the right by the $J \times I$ matrix for f.

Let $\alpha : X_A \to X_B$ be a uniform equivalence from σ_A to σ_B. Let $U = \{ U_i \}$ be in $\mathscr{P}_A$ with $P = M(U)$ and let $U' = \{ U'_k \}$ be in $\mathscr{P}_B$ with $P' = M(U')$. Assume $\alpha(U) = \{ \alpha(U_i) \} = \{ U'_k \} = U'$. Then $U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset$ if and only if $\alpha(U_i) \cap \sigma_B^{-1}(\alpha(U_j)) \neq \emptyset$. Considering α as a bijection between the indexing sets I for U and K for U', we have the matrix identity

$$\textbf{(4.4)} \qquad P' = \alpha P \alpha^{-1}.$$

Hence

$$\alpha^{-1} : P \mapsto P' \quad \text{and} \quad \alpha : P' \mapsto P.$$

Suppose now that $U \to V$ in $\mathscr{P}_A$. Let $Q = M(V)$ and let R and S be as in (3.1). Let $V' = \alpha(V) \in \mathscr{P}_B$. Then $U' \to V'$ in $\mathscr{P}_B$ and we have the corresponding matrices R' and S'. These matrices satisfy the matrix equations

$$\begin{aligned} Q' &= \alpha Q \alpha^{-1} \\ R' &= \alpha R \alpha^{-1} \\ S' &= \alpha S \alpha^{-1} \end{aligned}$$

which translates into the following diagram

(4.5)
$$\begin{tikzcd} P \arrow[r, mapsto, "B"] \arrow[d, "\alpha^{-1}"'] \arrow[dr, mapsto, "B\alpha^{-1} = \alpha^{-1}B'" description] & Q \arrow[d, "\alpha^{-1}"] \\ P' \arrow[r, mapsto, "B'"'] & Q' \end{tikzcd}$$

of triangles in $S(\bar{\mathscr{E}})$.

Now let A and B be endomorphisms in $\bar{\mathscr{E}}$ which lie in the same path component of $S(\bar{\mathscr{E}})$. We let

$$\pi_1(S(\bar{\mathscr{E}}); A, B)$$

denote the set of homotopy classes of paths starting at A and ending at B. Concatenation of paths gives a pairing

$$\pi_1(S(\bar{\mathscr{E}}); A, B) \times \pi_1(S(\bar{\mathscr{E}}); B, C) \to \pi_1(S(\bar{\mathscr{E}}); A, C)$$

denoted by " $*$ ". When $A = B$ we just get the fundamental group $\pi_1(S(\bar{\mathscr{E}}), A)$. If $\gamma \in \pi_1(S(\bar{\mathscr{E}}); A, B)$, then γ^{-1} denotes the path in $\pi_1(S(\bar{\mathscr{E}}); B, A)$ which is the reverse of γ going back from B to A. If $R : A \mapsto B$ in $\bar{\mathscr{E}}$, let $\gamma(R) \in \pi_1(S(\bar{\mathscr{E}}); A, B)$ be the homotopy class of the corresponding edge from A to B in $S(\bar{\mathscr{E}})$.

Lemma **4.6.** — *(a)* $\gamma(\alpha)^{-1} = \gamma(\alpha^{-1})$.
(b) If $\gamma \in \pi_1(S(\bar{\mathscr{E}}); A, B)$, *then* $\gamma(A) * \gamma = \gamma * \gamma(B)$.

Proof. The proof of *(a)* is left as an exercise. To verify *(b)*, observe that since γ is a product of paths $\gamma(R)^{\pm 1}$, it suffices to consider the case $\gamma = \gamma(R)$. The formula then follows from the diagram

(4.7)
$$\begin{tikzcd} A \arrow[r, mapsto, "A"] \arrow[d, "R"'] \arrow[dr, mapsto, "AR = RB" description] & A \arrow[d, "R"] \\ B \arrow[r, mapsto, "B"'] & B \end{tikzcd}$$

Let $\mathrm{Isom}(\sigma_A, \sigma_B)$ denote the set of all uniform equivalences from (X_A, σ_A) to (X_B, σ_B).

Proposition **4.8.** — *There is a map* $\psi = \psi(A, B)$ *from* $\mathrm{Isom}(\sigma_A, \sigma_B)$ *to* $\pi_1(S(\bar{\mathscr{E}}); A, B)$ *such that if* $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ *and* $\beta \in \mathrm{Isom}(\sigma_B, \sigma_C)$, *then*

$$\psi(\beta\alpha) = \psi(\alpha) * \psi(\beta).$$

Considering $\sigma_A \in \mathrm{Isom}(\sigma_A, \sigma_A)$ *we have*

$$\psi(\sigma_A) = \gamma(A).$$

From this we get a homomorphism

$$\psi_A : \mathrm{Aut}(\sigma_A) \to \pi_1(S(\bar{\mathscr{E}}), A)$$

by taking $A = B$ and letting $\psi_A(\alpha) = \psi(\alpha^{-1})$.

The proof of (4.8) is based on the following more technical result.

Proposition **4.9.** — *Let* U *and* V *be in* $\mathscr{P}_A$ *with* $P = M(U)$ *and* $Q = M(V)$. *Then there is a well defined path* $\Gamma(U, V)$ *in* $\pi_1(S(\bar{\mathscr{E}}); P, Q)$ *such that*

$$\Gamma(U, U) = 1$$

$$\Gamma(U, W) = \Gamma(U, V) * \Gamma(V, W).$$

Moreover, if $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$, *then*

$$\Gamma(\alpha(U), \alpha(V)) = \gamma(\alpha) * \Gamma(U, V) * \gamma(\alpha)^{-1}.$$

Consider the special case $U \to V$ in $\mathscr{P}_A$. Then define $\Gamma(U, V) = \gamma(R)$. In general, choose a path from U to V in $\mathscr{P}_A$ which is a concatenation of edges $\langle U(i-1), U(i) \rangle^{\varepsilon(i)}$ for $i = 1, \ldots, n$ where $\varepsilon(i) = \pm 1$, $U(0) = U$, $U(n) = V$, and $U(i-1) \to U(i)$. Then define

$$\textbf{(4.10)} \qquad \Gamma(U, V) = \Gamma(U, U(1))^{\varepsilon(1)} * \Gamma(U(1), U(2))^{\varepsilon(2)} * \ldots * \Gamma(U(n-1), U(n))^{\varepsilon(n)}.$$

It must be shown that $\Gamma(U, V)$ is independent of the particular path chosen in $\mathscr{P}_A$ from U to V. Before completing (4.9) we show how to derive (4.8) from it.

Proof of Proposition 4.8. — Let $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$. As in (4.4), we have $\alpha^{-1} : M(U^A) \mapsto M(\alpha(U^A))$. Define

$$\textbf{(4.11)} \qquad \psi(\alpha) = \gamma(\alpha^{-1}) * \Gamma(\alpha(U^A), U^B).$$

Now let $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ and $\beta \in \mathrm{Isom}(\sigma_B, \sigma_C)$. Then

$$\begin{aligned} \psi(\beta\alpha) &= \gamma(\alpha^{-1}\beta^{-1}) * \Gamma(\beta\alpha(U^A), U^C) \\ &= \gamma(\alpha^{-1}) * \gamma(\beta^{-1}) * \Gamma(\beta\alpha(U^A), \beta(U^B)) * \Gamma(\beta(U^B), U^C). \end{aligned}$$

From (4.6) and (4.9) we see that

$$\Gamma(\beta\alpha(U^A), \beta(U^B)) = \gamma(\beta) * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}).$$

Substituting and then simplifying gives

$$\begin{aligned} \Psi(\beta\alpha) &= \gamma(\alpha^{-1}) * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}) * \Gamma(\beta(U^B), U^C) \\ &= \psi(\alpha) * \psi(\beta). \end{aligned}$$

To compute $\Psi(\sigma_A)$, let $U^A = \{ U_i^A \}$ be the standard partition and let $\sigma = \sigma_A$. Observe that $\sigma(U^A) \to U^A$ because $\sigma(U^A) \underset{+}{\to} \sigma(U^A) \cap U^A \underset{-}{\leftarrow} U^A$. Then we have the triangle

$$\begin{tikzcd}
A \arrow[rr, mapsto, "A"'] \arrow[dr, leftrightarrow, "\sigma^{-1}"'] & & A \\
& M(\sigma(U^A)) \arrow[ur, leftrightarrow, "\sigma A"'] &
\end{tikzcd}$$

in $S(\bar{\mathscr{E}})$ which shows

$$\psi(\sigma_A) = \gamma(\sigma^{-1}) * \Gamma(\sigma(U^A), U^A)) = \gamma(A).$$

Proof of Proposition 4.9. — From the definition (4.10) we have $\Gamma(U, U) = \gamma(1) = 1$. We also see that $\Gamma(U, V) * \Gamma(V, W) = \Gamma(U, W)$ provided Γ is independent of the path chosen from U to V in $\mathscr{P}_A$. But this follows immediately from the Triangle Identities (3.3) and simple connectivity of $\mathscr{P}_A$. The property

$$\Gamma(\alpha(U), \alpha(V)) = \gamma(\alpha) * \Gamma(U, V) * \gamma(\alpha)^{-1}$$

is a consequence of (4.5) and (4.6). This completes the proof.

In dynamical systems and operator algebras inverting functors provide a way to obtain invariants for the shift dynamical system (X_A, σ_A). Here is a framework for making these constructions natural enough to get invariants for $\mathrm{Aut}(\sigma_A)$. When A is finite it turns out that the dimension group is ubiquitous.

As usual let $\bar{\mathscr{E}}$ be one of the four classes (4.1). Let $\mathscr{D}$ be a category where composition reads from right to left. Assume that if f and g are morphisms and both fg and gf are isomorphisms, then so are f and g. For example, $\mathscr{D}$ could be an abelian category. An inverting functor F on $\bar{\mathscr{E}}$ first of all assigns to each endomorphism A of $\bar{\mathscr{E}}$ an object F(A) of $\mathscr{D}$. Next suppose A and B are endomorphisms of $\bar{\mathscr{E}}$ and X is a morphism of $\bar{\mathscr{E}}$ such that $AX = XB$. We are then given a morphism $f(X) : F(B) \to F(A)$ which must satisfy the composition rule $f(XY) = f(X) f(Y)$. Finally, observe this produces a morphism $f(A) : F(A) \to F(A)$ for each endomorphism A in $\mathscr{E}$. We say F is *inverting* provided $f(A)$ is an isomorphism. A wholesale method for manufacturing such F is to take $F(A) = \mathrm{coker}(I - Aq(A))$ or $F(A) = \ker(I - Aq(A))$ where q is a polynomial over a commutative ring Λ and $\mathscr{D}$ is the category of right Λ-modules and Λ-homomorphisms. A shift equivalence $R : A \mapsto B$ induces $f(R)$ via the Λ-homomorphism

$$R : \Lambda[T] \to \Lambda[S].$$

Some familiar examples are

(i) $F(A) = \mathrm{coker}(I - tA)$, $\Lambda = \mathbf{Z}[t, t^{-1}]$. This is the dimension group. See [BF], [Cu1], [CK], [K], [E], [W2].

(ii) $F(A) = \mathrm{coker}(I - A)$, $\Lambda = \mathbf{Z}$.

(iii) $F(A) = \ker(I - A)$, $\Lambda = \mathbf{Z}$.

(iv) $F(A) = \operatorname{coker}\begin{pmatrix} I - A & -A \\ 0 & I - A \end{pmatrix}$, $\Lambda = \mathbf{Z}$. See [BF], [Cu1], [Cu2], [F].

(v) $F(A) = \left\{\text{bounded solutions of } I - \frac{1}{n}A = 0\right\}$ where A is infinite. See [KV] and (4.25) below.

Observe that if F is an inverting functor, then F(A) is an invariant of shift equivalence. This is because $f(R)\,f(S) = f(RS) = f(P)^n$ and $f(S)\,f(R) = f(SR) = f(Q)^n$.

Let A, B be endomorphisms in $\bar{\mathscr{E}}$. $\operatorname{Isom}(f(A), f(B))$ will denote all the isomorphisms $g : F(A) \to F(B)$ in $\mathscr{D}$ such that $f(B)\,g = gf(A)$. When $A = B$, we let $\operatorname{Aut}(f(A)) = \operatorname{Isom}(f(A), f(A))$.

Proposition **4.12.** — *Let* A, B *be endomorphisms in* $\bar{\mathscr{E}}$. *There is a map* $\psi_F = \psi_F(A, B)$ *from* $\operatorname{Isom}(\sigma_A, \sigma_B)$ *to* $\operatorname{Isom}(f(B), f(A))$ *such that if* $\alpha \in \operatorname{Isom}(\sigma_A, \sigma_B)$ *and* $\beta \in \operatorname{Isom}(\sigma_B, \sigma_C)$, *then*

$$\psi_F(\beta\alpha) = \psi_F(\alpha)\,\psi_F(\beta).$$

If $A = B$, *then*

$$\psi_F(\sigma_A) = f(A).$$

From this we obtain a homomorphism

$$\textbf{(4.13)} \qquad \psi_{F,A} : \operatorname{Aut}(\sigma_A) \to \operatorname{Aut}(f(A))$$

by taking $A = B$ and letting $\psi_{F,A}(\alpha) = \psi_F(\alpha^{-1})$.

The homomorphism $\psi_{F,A}$ was first developed in connection with the algebraic K-theory group K_2. See (4.21) below. D. Lind observed that the method goes through for $F(A) = \operatorname{coker}(I - Aq(A))$ with basically no changes. This led to (4.12). There are entirely similar versions of (4.12) and (4.13) depending on whether F is covariant or contravariant and on whether composition in $\mathscr{D}$ is read from right to left or vice versa. The general character of (4.12) suggested that there should be a " universal version ". This turns out to be the case and involves the dimension group.

Proposition **4.14.** — *Let* $F : \bar{\mathscr{E}} \to \mathscr{D}$ *be an inverting functor and let* A, B *be endomorphisms in* $\bar{\mathscr{E}}$. *Then there is a map*

$$\eta_F = \eta_F(A, B) : \pi_1(S(\bar{\mathscr{E}}); A, B) \to \operatorname{Isom}(f(B), f(A))$$

such that

(i) *if* $\gamma \in \pi_1(S(\bar{\mathscr{E}}); A, B)$ *and* $\delta \in \pi_1(S(\bar{\mathscr{E}}); B, C)$, *then* $\eta_F(\gamma * \delta) = \eta_F(\gamma)\,\eta_F(\delta)$.

(ii) η_F *takes* $\gamma(A) \in \pi_1(S(\bar{\mathscr{E}}); A)$ *to* $f(A)$ *in* $\operatorname{Aut}(f(A))$.

The required map of (4.12) is then clearly just

$$\textbf{(4.15)} \qquad \psi_F = \eta_F\,\psi.$$

The proof of (4.14) is really " general nonsense ". See [Q]. But for the convenience of readers who find [Q] overly abstract, we give a more concrete formula for ψ_F in the spirit of (4.10) and (4.11).

Assume $U \to V$ in $\mathscr{P}_A$. Let $P = M(U)$, $Q = M(V)$ and $R = R(U, V)$. Then define

$$f(U, V) = f(R) \in \mathrm{Isom}(F(Q), F(P)).$$

In general, for any $U, V \in \mathscr{P}_A$, choose a path from U to V in $\mathscr{P}_A$ which is a concatenation of edges $\langle U(i-1), U(i) \rangle^{\varepsilon(i)}$ for $i = 1, \ldots, n$ where $\varepsilon(i) = \pm 1$, $U(0) = U$, $U(n) = V$, and $U(i-1) \to U(i)$. Let

$$\textbf{(4.16)} \qquad f(U, V) = \prod_1^n f(U(i-1), U(i))^{\varepsilon(i)}.$$

Remember we are reading composition from right to left in $\mathscr{D}$. If $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$, then

$$\textbf{(4.17)} \qquad \psi_F(\alpha) = f(\alpha^{-1})\, f(\alpha(U^A), U^B).$$

Now we discuss the unique role played by the dimension group G(A).

Let $\overline{\mathscr{E}}$ be as in *(a)*, *(b)* or *(c)* of (4.1). Let $A \in \overline{\mathscr{E}}$ and define

$$\begin{aligned} G(A) &= \varinjlim \mathbf{Z}[\mathscr{S}] \to \mathbf{Z}[\mathscr{S}] \\ &\cong \mathrm{coker}(I - tA) \end{aligned}$$

as in [E], [W2]. G(A) will be considered as a *right* $\mathbf{Z}[t, t^{-1}]$-module. Let $G(A)_+$ denote the set of positive elements and let $s_A = g(A)$. The homomorphism $\eta_G : \pi_1(S(\overline{\mathscr{E}}); A) \to \mathrm{Aut}(g(A))$ is constructed by sending a path in $S(\overline{\mathscr{E}})$ corresponding to $R : P \mapsto Q$ to the isomorphism $g(R) : G(Q) \to G(P)$ which takes $G(Q)_+$ to $G(P)_+$. In particular, for each loop $\gamma \in \pi_1(S(\overline{\mathscr{E}}), A)$, the isomorphism $\eta_G(\gamma)$ preserves the order structure of G(A). Let $\mathrm{Aut}(G(A), G(A)_+, s_A)$ denote all those automorphisms of G(A) which preserve the order structure and commute with s_A.

Proposition **4.18.** — *If* $A \in \overline{\mathscr{E}}$ *is finite, then*

$$\eta_G : \pi_1(S(\overline{\mathscr{E}}), A) \to \mathrm{Aut}(G(A), G(A)_+, s_A)$$

is an isomorphism.

Corollary **4.19.** — *If* $A \in \mathscr{E}$ *is finite, then* $\psi_{F, A} : \mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(f(A))$ *factors through* $\mathrm{Aut}(G(A), G(A)_+, s_A)$.

Proof of 4.18.

Surjectivity of $\eta = \eta_G$: This is another way of interpreting Krieger's argument proving that two finite non-negative integral matrices are shift equivalent if and only if the triples $(G(P), G(P)_+, s_P)$ and $(G(Q), G(Q)_+, s_Q)$ are isomorphic. See [K, 4.2] or [E, 6.4]. In fact, the argument shows that any element in $\mathrm{Aut}(G(A), G(A)_+, s_A)$

is the image under η_G of a path of the form $\gamma(R) * \gamma(A)^n$ where $R : A \mapsto A$ is a morphism in $S(\bar{\mathscr{E}})$ and $n \in \mathbf{Z}$.

Injectivity of $\eta = \eta_G$: Let $R : P \mapsto Q$ be a path in $S(\bar{\mathscr{E}})$ and choose $S : Q \mapsto P$ as in (4.2) so that $RS = A^k$. Then $\gamma(R) * \gamma(S) = \gamma(A)^k$ and hence $\gamma(R)^{-1} = \gamma(S) * \gamma(A)^{-k}$. Any loop γ in $\pi_1(S(\bar{\mathscr{E}}), A)$ is a product of paths $\gamma(R)^\varepsilon$ for $\varepsilon = \pm 1$. Hence it is a product of paths $\gamma(R) * \gamma(P)^k$ for $k \in \mathbf{Z}$ and various P. Since $\gamma(P) * \gamma = \gamma * \gamma(Q)$, the $\gamma(P)^k$ can be pushed to the end of the product. The relation $\gamma(R_1 R_2) = \gamma(R_1) * \gamma(R_2)$ can then be used to deform the loop to one of the form $\gamma(A)^n * \gamma(R)$ for some $n \in \mathbf{Z}$. Since all matrices are *assumed to be finite*, it is a consequence of the definition of G(A) as a direct limit that shift equivalences $P : A \mapsto A$ and $Q : A \mapsto A$ induce the same automorphisms of G(A) if and only if there is a non-negative integer k such that $A^k P = A^k Q$. We want to apply this under the assumption that $\eta_G(\gamma) = 1$.

Case 1: $\gamma = \gamma(A)^n * \gamma(R)$, $n \geqslant 0$. Then $\gamma = \gamma(A^n R)$. Since we assume $A^n R$ induces the identity on G(A), there is a $k \geqslant 1$ such that $A^k = A^k A^n R$. Hence, we have $\gamma(A)^k = \gamma(A^k) = \gamma(A^k A^n R) = \gamma(A)^k * \gamma(A^n R^n)$ and $\gamma(A^n R) = 1$.

Case 2: $\gamma = \gamma(A)^{-n} * \gamma(R)$, $n \geqslant 0$. Then R and A^n induce the same automorphism of G(A) and $A^k R = A^k A^n$ for some $k \geqslant 1$. We then have

$$\gamma(A)^k * \gamma(R) = \gamma(A^k R) = \gamma(A^k A^n) = \gamma(A^k) * \gamma(A^n).$$

Hence $\gamma(R) = \gamma(A)^n$, which gives $\gamma = 1$.

Product Formula

Let $F : \bar{\mathscr{E}} \to \mathscr{D}$ be an inverting covariant functor into a category $\mathscr{D}$ of right modules over a commutative right with identity 1. We say F is *compatible with tensor products* provided there is an isomorphism $F(A \otimes_{\mathbf{Z}} B) \cong F(A) \otimes_{\mathbf{Z}} F(B)$ of Λ-modules whenever A and B are endomorphisms in $\mathscr{E}$ such that if $R : A_1 \mapsto A_2$, then there is a commutative diagram

$$\begin{tikzcd} F(A_2 \otimes_{\mathbf{Z}} B) \arrow[r, phantom, "\cong"] \arrow[d, "f(R\otimes 1)"'] & F(A_2) \otimes_{\mathbf{Z}} F(B) \arrow[d, "f(R)\otimes 1"] \\ F(A_1 \otimes_{\mathbf{Z}} B) \arrow[r, phantom, "\cong"] & F(A_1) \otimes_{\mathbf{Z}} F(B) \end{tikzcd}$$

The prime example is the dimension group G(A), which is compatible because the tensor product commutes with direct limits.

Let (X_A, σ_A) and (X_B, σ_B) have standard partitions $U^A = \{ U_i^A \}$ and $U^B = \{ U_j^B \}$ respectively. Let $C = A \otimes B$. Then

$$C((i_1, j_1), (i_2, j_2)) = A(i_1, i_2)\, B(j_1, j_2)$$

and consequently there is an isomorphism

$$(X_{A \otimes B}, \sigma_{A \otimes B}) \cong (X_A \times X_B, \sigma_A \times \sigma_B)$$

under which $U^C = U^A \times U^B = \{ U_i^A \times U_j^B \}$. In particular $\sigma_A \times 1 \in \mathrm{Aut}(\sigma_{A \otimes B})$.

Proposition **4.20.** — *If* F *is compatible with tensor products, then*

$$\psi_{F, A\otimes B}(\sigma_A \times 1) = \psi_{F, A}(\sigma_A) \otimes 1.$$

Proof. — Direct computation using (4.17) similar to the proof that $\psi(\sigma_A) = \gamma(A)$ in (4.8).

Relation to K_2

Let F be a field and for each prime ideal $\mathscr{P} \subset F[t, t^{-1}]$, let $F_{\mathscr{P}}$ denote the field $F[t, t^{-1}]/\mathscr{P}$. Let K_2 be the algebraic K-theory group of [M1].

Proposition **4.21.** — *Assume* A *is finite. Then there is a commutative diagram*

$$\begin{tikzcd} & K_2(F(t)) \arrow[dd, "\partial"] \\ \mathrm{Aut}(\sigma_A) \arrow[ur, "K_A"] \arrow[dr, "\kappa_A"'] & \\ & \bigoplus_{\mathscr{P}} F^*_{\mathscr{P}} \end{tikzcd}$$

where ∂ *is the tame symbol. The image of* κ_A *is contained in the sum of those* $F^*_{\mathscr{P}}$ *where* $\mathscr{P}$ *divides* $\det(I - tA)$.

The first step is to define K_A and κ_A. Suppose A is an $m \times m$ matrix. Let $G(A; F) = G(A) \otimes F = \varinjlim F^m \xrightarrow[A]{} F^m \cong \operatorname{coker}(I - tA)$ where $I - tA$ is now viewed as an $m \times m$ matrix over $F[t, t^{-1}]$. Since $G(A; F) \cong \operatorname{Image}(A^k : F^m \to F^m)$ for k large enough, it is finite dimensional over F. If $\alpha \in \mathrm{Aut}(\sigma_A)$, then $\psi_G(\alpha)$ is a vector space automorphism and both $\psi_G(\alpha)$ and $I - tA$ are commuting automorphisms of $G(A; F) \otimes_F F(t)$ as a vector space over $F(t)$. We let

$$(\mathbf{4.22}) \qquad K_A(\alpha) = \psi_G(\alpha) \star (I - tA)$$

where the " $\star$ " product is defined in [M1, § 8].

Since $G(A; F)$ is finite dimensional over F, it is certainly a finitely generated torsion module over $F[t, t^{-1}]$. Let $G(A; F)_{\mathscr{P}}$ denote the p-primary part of $G(A; F)$, i.e., those elements killed by some power $\mathscr{P}^r$. Then $G(A; F)$ decomposes naturally as a direct sum

$$G(A; F) \cong \bigoplus_{\mathscr{P}} G(A; F)_{\mathscr{P}}.$$

Each $G(A; F)$ is filtered as

$$G(A; F)_{\mathscr{P}} \supset \mathscr{P} G(A; F)_{\mathscr{P}} \supset \mathscr{P}^2 G(A; F)_{\mathscr{P}} \supset \ldots \supset \mathscr{P}^{r-1} G(A; F)_{\mathscr{P}} \supset 0$$

so that $G^i_{\mathscr{P}} = \mathscr{P}^i G(A; F)_{\mathscr{P}} / \mathscr{P}^{i+1} G(A; F)_{\mathscr{P}}$ is a vector space over $F_{\mathscr{P}}$. Any automorphism α of $G(A; F)$ as an $F[t, t^{-1}]$-module takes each $G(A; F)_{\mathscr{P}}$ to itself and respects the fibration. Let

$$\Delta_{\mathscr{P}}(\alpha) = \prod_i \{ \det \text{ of } \alpha \text{ on } G^i_{\mathscr{P}} \} \in F^*_{\mathscr{P}}$$

and $$\Delta(\alpha) = \bigoplus_{\mathscr{P}} \Delta_{\mathscr{P}}(\alpha) \in \bigoplus_{\mathscr{P}} F^*_{\mathscr{P}}.$$

From (4.13) we have a homomorphism

$$\psi_{G,A} : \mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(s_A)$$

where $s_A = \psi_{G,A}(\sigma_A)$ is multiplication by t^{-1} on $G(A; F)$. We define

(4.23) $$\kappa_A(\alpha) = \Delta(\psi_{G,A}(\alpha))$$

for $\alpha \in \mathrm{Aut}(\sigma_A)$.

Proof of 4.21. — By naturality of the exact sequence involving the tame symbol as boundary map [M1] it suffices to consider the case where F is algebraically closed. Choose a basis for $G(A; F)$ over F for which $\psi_G(\alpha)$ and A are diagonal. Using the identities in [M1, § 9] one shows that $K_A(\alpha)$ can be computed just using the diagonal parts of $\psi_G(\alpha)$ and of A. Also, $\kappa_A(\alpha)$ can be computed from the diagonals. Lemma 8.3 of [M1] reduces the computation to symbols $\{\lambda, 1 - \mu t\}$ where it follows directly from the definition of the tame symbol [M1, § 11]. Cramer's Rule, as expressed by Proposition 6.6 of [B; IX, § 6] shows that the prime factors of $G(A; F)$ can only be those involving prime polynomials which divide $\det(I - tA)$. So κ_A only brings in those primes as well.

The Dual Dimension Group

Let A be an endomorphism in $\bar{\mathscr{E}}$ with $\| A \|$ finite. Define the ℓ^∞ *dual dimension* group to be the inverse limit

$$G^\infty(A) = \lim \ell^\infty(\mathscr{S}) \underset{A}{\leftarrow} \ell^\infty(\mathscr{S}).$$

This is an inverting functor into the category of Banach spaces, so we obtain a homomorphism

(4.24) $$\mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(s_A)$$

where $s_A = g^\infty(A)$ is induced on $G^\infty(A)$ by the standard shift to the left by one step.

Random Walk on an Infinite Group

For general background about a random walk on a countably infinite discrete group G see [KV]. Here we consider the very special case of a measure μ on G with finite support and satisfying the condition that $\mu(g) = 1/n$ whenever $\mu(g) \neq 0$ where n is the number of those g for which $\mu(g) \neq 0$. Let $\mathscr{H} \subset \ell^\infty(G)$ denote the space of μ-harmonic functions on G. Namely, those bounded functions $\kappa : G \to \mathbf{C}$ satisfying

$$\kappa(g) = (P\kappa)(g) = \Sigma_h P(g, h)\, \kappa(h) = \Sigma_h \mu(g^{-1} h)\, \kappa(h) = \Sigma_h \kappa(gh)\, \mu(h)$$

where $P = \dfrac{1}{n} A$ and

$$A(g, h) = 1 \quad \text{if and only if } \mu(g^{-1} h) \neq 0.$$

Then $\mathscr{H}$ is given as the inverting functor

$$\mathscr{H} = \mathscr{H}(A) = \ker\left(1 - \frac{1}{n}A\right) \text{ on } \ell^\infty(G)$$

and we can apply the preceding machinery to obtain a homomorphism

(4.25) $h : \mathrm{Aut}(\sigma_A) \to \text{Isomorphisms of } \mathscr{H}(A).$

There is a version of this for a general Markov measure μ and μ-preserving symmetries of σ_A.

Finally, we discuss strong shift equivalence. Let $\mathscr{E}$ be one of the classes of zero-one matrices in (4.1). First there is the generalization of Williams' strong shift equivalence criterion for topological conjugacy as given in, say, [Wi] or [PT].

Proposition **4.26.** — *Let* $A \in \mathscr{E}$ *and let* B *be a zero-one matrix. If there is an isomorphism* $(X_A, \sigma_A) \cong (X_B, \sigma_B)$, *then* $B \in \mathscr{E}$. *Moreover, if* $A, B \in \mathscr{E}$ *there exists an isomorphism* $(X_A, \sigma_A) \cong (X_B, \sigma_B)$ *if and only if* A *and* B *are strong shift equivalent in* $\mathscr{E}$.

The proof is basically the same as in [Wi], [PT], [PW] with the key ingredient being the connectedness of $\mathscr{P}_A$.

Now let A and B be two endomorphisms (square matrices) in $\mathscr{E}$. As in [Wi], [PT], [PW] we say a pair (R, S) of matrices in $\mathscr{E}$ is a *strong shift equivalence in* $\mathscr{E}$ *from* A *to* B provided

(4.27) $RS = A \quad \text{and} \quad SR = B.$

We denote this by $(R, S) : A \longrightarrow B$ or $A \xrightarrow[(R,S)]{} B$.

Definition **4.28.** — The *space* $SS(\mathscr{E})$ *of strong shift equivalences in* $\mathscr{E}$ is the realization of the simplicial set where an n-simplex consists of the following data:

(a) an $(n+1)$-tuple $\langle A_0, A_1, \ldots, A_n \rangle$ of endomorphisms $A_i \in \mathscr{E}$, and
(b) for each $i < j$ a strong shift equivalence (R_{ij}, S_{ji}) from A_i to A_j such that whenever $i < j < k$, the triangle identities hold; that is,

$$R_{ij} R_{jk} = R_{ik}, \qquad S_{kj} S_{ji} = S_{ki} A_i.$$

As with $S(\bar{\mathscr{E}})$, it follows directly from the definition that the set of path components of $SS(\mathscr{E})$ is just the set of strong shift equivalence classes in $\mathscr{E}$.

If $(R, S) : A \to B$ in $\mathscr{E}$, then $R : A \mapsto B$. The correspondence $(R, S) \to R$ induces a map of simplicial sets and a continuous map

(4.29) $SS(\mathscr{E}) \to S(\bar{\mathscr{E}}).$

In fact, there is a commutative diagram

(4.30)
$$\begin{tikzcd}
 & \pi_1(SS(\mathscr{E}), A) \arrow[d] \\
\mathrm{Aut}(\sigma_A) \arrow[ur, "\Psi'_A"] \arrow[r, "\psi_A"'] & \pi_1(S(\bar{\mathscr{E}}), A)
\end{tikzcd}$$

The homomorphism Ψ'_A is obtained by proving (4.8) and (4.9) with $S(\bar{\mathscr{E}})$ replaced by $SS(\mathscr{E})$. The proofs are similar and the key observations are as follows. The equation (4.4) shows that

(4.31) $\qquad (\alpha^{-1}, \alpha P) : P \to P' \quad$ and $\quad (\alpha, P\alpha^{-1}) : P' \to P$

and the analogue for (4.5) is the diagram

$$\begin{tikzcd}
P \arrow[r, "{(R, S)}"] \arrow[d, "{(\alpha^{-1}, \alpha P)}"'] \arrow[dr, "{(R\alpha^{-1}, \alpha S)}" description] & Q \arrow[d, "{(\alpha^{-1}, \alpha Q)}"] \\
P' \arrow[r, "{(R', S')}"'] & Q'
\end{tikzcd}$$

Next, let $\gamma(R, S)$ be the path from A to B in $\pi_1(SS(\mathscr{E}); A, B)$.

Lemma **4.32.** — *(a)* $\gamma(\alpha^{-1}, \alpha P) = \gamma(\alpha, P\alpha^{-1})^{-1}$.

(b) $\gamma(1, A) = 1$.

(c) If $\gamma \in \pi_1(SS(\mathscr{E}); A, B)$, *then* $\gamma(A, 1) * \gamma = \gamma * \gamma(B, 1)$.

(d) $\gamma(R, S)\, \gamma(S, R) = \gamma(A, 1)$.

Verification of Lemma 4.32 uses the diagram above and the diagram

$$\begin{tikzcd}
A \arrow[r, "{(A, 1)}"] \arrow[d, "{(R, S)}"'] & A \arrow[d, "{(R, S)}"] \\
B \arrow[r, "{(B, 1)}"'] \arrow[ur, "{(S, R)}" description] & B
\end{tikzcd}$$

The formulas (4.10) and (4.11) are virtually the same with $\gamma(R)$ replaced by $\gamma(R, S)$ and $\gamma(\alpha^{-1})$ by $\gamma(\alpha^{-1}, \alpha P)$. That Γ is well defined uses the Triangle Identities plus the fact that $\mathscr{P}_A$ is simply connected.

Williams' problem of " strong shift equivalence vs. shift equivalence " [Wi], [E] for the category $\bar{\mathscr{E}}$ of non-negative integral matrices can be rephrased as asking whether

$$\pi_0(SS(\mathscr{E})) \to \pi_0(S(\bar{\mathscr{E}}))$$

is a bijection. This mere reformulation is heuristic and does not help in solving the problem. But we do note that if A and B are strong shift equivalent, then

$$\pi_1(SS(\mathscr{E}), A) \cong \pi_1(SS(\mathscr{E}), B)$$

because A and B lie in the same path component of $SS(\mathscr{E})$. The groups $\pi_1(S(\bar{\mathscr{E}}), A)$ are clearly invariants of shift equivalence. However, it is not known and, at any rate, certainly not obvious, that $\pi_1(SS(\mathscr{E}), A)$ is an invariant of shift equivalence. An open problem is to obtain more information about $\pi_1(SS(\mathscr{E}), A)$ or, for that matter, about $\pi_i(SS(\mathscr{E}), A)$ for $i \geq 2$.

REFERENCES

[B] H. Bass, *Algebraic K-Theory*, W. A. Benjamin, 1968.

[BF] R. Bowen and J. Franks, Homology for zero-dimensional non-wandering sets, *Ann. of Math.*, **106** (1977), 73-92.

[BK] M. Boyle and W. Krieger, Periodic points and automorphisms of the shift, to appear in *Trans. Amer. Math. Soc.*

[BLR] M. Boyle, D. Lind and D. Rudolph, *The automorphism group of a subshift of finite type*, preprint, University of Washington / University of Maryland (1986).

[C] A. Connes, Outer conjugacy of classes of automorphisms of factors, *Ann. Sci. Ec. Norm. Sup.*, 4e série, **8** (1975), 383-420.

[CS] A. Connes and E. Störmer, Entropy for automorphisms of II_1 von Neumann algebras, *Acta Mathematica*, **134** (1975), 289-306.

[Cu1] J. Cuntz, A class of C*-algebras and topological Markov chains, II: Reducible chains and the Ext-functor for C*-algebras, *Invent. Math.*, **63** (1981), 25-40.

[Cu2] J. Cuntz, On the homotopy groups of the space of endomorphisms of a C*-algebra (with applications to topological Markov chains), in *Operator Algebras and Group Representations*, Vols. 1 and 2, Monographs and Studies in Mathematics, Nos. 17 and 18, Conference at Neptun, Romania, 1980, pp. 124-137.

[CK] J. Cuntz and W. Krieger, A class of C*-algebras and topological Markov chains, *Invent. Math.*, **56** (1980), 251-268.

[E] E. G. Effros, *Dimensions and C*-algebras*, CBMS, no. 46 (American Mathematical Society, 1981).

[F] J. Franks, *Homology and dynamical systems*, CBMS, no. 49 (American Mathematical Society, 1982).

[H] G. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, *Mathematical Systems Theory*, **3**, no. 4 (1969), 320-375.

[HW] A. Hatcher and J. Wagoner, Pseudo-isotopies of compact manifolds, *Astérisque*, **6**, Société Mathématique de France (1973).

[K] W. Krieger, On dimension functions and topological Markov chains, *Invent. Math.*, **56** (1980), 239-250.

[KV] V. A. Kaimanovich and A. M. Vershik, Random walks on discrete groups: boundary and entropy, *Annals of Probability*, **11**, no. 3 (1983), 457-490.

[M1] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math. Studies No. 72, Princeton University Press, 1971.

[M2] J. Milnor, Infinite cyclic coverings, *Conference on Topology of Manifolds*, ed. J. G. Hocking, Vol. 13, Prindle, Weber & Schmidt, Inc., 1968, 115-133.

[M3] J. Milnor, Whitehead torsion, *BAMS*, **72** (1966), 358-426.

[PT] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, LMS Lecture Notes, **67**, Cambridge University Press, 1982.

[PW] W. Parry and R. F. Williams, Block coding and a zeta function for finite Markov chains, *Proc. London Math. Soc.* (3), **35** (1977), 483-495.

[Q] D. Quillen, Higher algebraic K-theory, I, *Springer Lecture Notes in Mathematics*, No. 341 (1973), 85-147.

[S] G. Segal, Classifying spaces and spectral sequences, *Publ. Math., I.H.E.S.*, **34** (1968), 105-112.

[Sp] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.

[T] S. Tuncel, A dimension, dimension modules, and Markov chains, *Proc. London Math. Soc.*, **36** (1983), 100-116.

[W1] J. Wagoner, K_2 and diffeomorphisms of two and three dimensional manifolds, *Geometric Topology*, ed. J. C. Cantrell, Academic Press, 1979, 557-580.

[W2] J. Wagoner, Topological Markov chains, C^*-algebras, and K_2, preprint, to appear in *Advances in Mathematics*.

[Wi] R. F. Williams, Classification of subshifts of finite type, *Ann. of Math.*, **98** (1973), 120-153; Errata, *ibid.*, **99** (1974), 380-381.

[Z] F. Zizza, *Automorphisms of hyperbolic dynamical systems and* K_2, Ph.D. Thesis, University of California, Berkeley (1985). Also a preprint from University of Washington, Seattle.

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