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HYPERBOLIC 4-MANIFOLDS AND CONFORMALLY FLAT 3-MANIFOLDS

by M. GROMOV, H. B. LAWSON, Jr. *and* W. THURSTON

Dedicated to René Thom.

0. Introduction

It is a simple consequence of Chern-Weil theory that for a compact manifold of constant sectional curvature all the Pontrjagin numbers are zero. In particular, the signature must be zero. One might speculate that somehow this continues to be true for complete non-compact manifolds. The first cases to consider would be those of 4-manifolds which are constructed as 2-plane bundles over compact surfaces. Here the speculation would be that only the trivial bundles carry complete, constant curvature metrics. For curvature $\geqq 0$ this is essentially true. However, in the negative case it fails. We present here a construction of a family of complete hyperbolic 4-manifolds, some of which are diffeomorphic to non-trivial vector bundles over compact surfaces. These bundles all satisfy the inequality

$$(+) \qquad | \chi(E) | \leqq | \chi(\Sigma) |$$

(where $E \to \Sigma$ denotes the bundle over the surface).

It is intriguing to conjecture that such an inequality is in fact a necessary condition for the existence of a complete hyperbolic metric on E. Intriguing also is the fact that this inequality is precisely the necessary and sufficient condition for the existence of a reduction of the structure group of E to a discrete group (or, more generally, for the existence of a 2-dimensional foliation of E transverse to the fibres [W] [T]). Similar conditions have also arisen in the work of Massey [Ma].

Each example M coming from our general construction has the property that there is a compact surface Σ and a PL-embedding $\Sigma \to M$ which is a homotopy equivalence. When M is a 2-plane bundle over Σ, this embedding is PL-equivalent to the smooth zero-section. However, in the general case the embedding is *not locally flat.*

A second feature of our methods is that they simultaneously produce compact conformally flat 3-manifolds. In particular we obtain conformally flat structures on a large portion of the circle bundles over compact surfaces whose Euler classes satisfy the inequality $(+)$. This inequality is known to be necessary when $\chi(\Sigma) = 0$ [G_1].

The conformally flat 3-manifolds which correspond to the cases above where $\Sigma \subset M$ is not locally flat, are interesting. Topologically they can be obtained from circle bundles over surfaces by removing disjoint tubular neighborhoods of a finite number of fibres and replacing them with certain knot complements in S^3.

Each of the conformally flat 3-manifolds X constructed here has a non-surjective developing map $\delta : \widetilde{X} \to S^3$ of its universal covering space $\widetilde{X}$ into S^3. The image $\Omega \equiv \delta(\widetilde{X}) \subsetneq S^3$ is preserved by a discrete subgroup $\Gamma \subset SO_{4,1}$ which is abstractly isomorphic to the fundamental group of a compact surface. The maps $\delta : \widetilde{X} \to \Omega$ and $\pi : \Omega \to \Omega/\Gamma \cong X$ are covering maps which factor the universal one. They correspond to the short exact sequence

$$(++) \qquad 0 \to \pi_1 \Omega \to \pi_1 X \to \Gamma \to 0$$

(All of this can be seen directly; however, the fact that δ is a covering map is required by $\delta : \widetilde{X} \to S^3$ not being surjective. This is due to Kulkarni and Pinkall [KP].)

In the case where X is diffeomorphic to a circle bundle over a surface Σ, the set Ω is the complement of an unknotted circle $\gamma \subset S^3$. This circle is the limit set of Γ, and it has the " self-similarity " property that any closed segment of γ contains an image under some $g \in \Gamma$ of any other closed segment. It is nowhere differentiable and could be called a " Julia " curve. Since γ is unknotted, its complement $\Omega = S^3 - \gamma \cong D^2 \times S^1$ is homotopy equivalent to S^1, and the sequence $(++)$ is just the one coming from the fibration $S^1 \to X \to \Sigma$. It is an interesting fact that the geometry of the curve γ contributes in a subtle way to the topology of this fibration.

In each of the remaining cases (those in which the embedding $\Sigma \to M$ is not locally flat), the set Ω is the complement of an infinitely compounded " Julia " knot $\gamma \subset S^3$ which is everywhere non-tame. The associated discrete group, considered as a group of isometries of $\mathbf{H}^4$, has this wild knot as its limit set in $\partial \mathbf{H}^4 = S^3$. Hence, this construction yields particularly contorted representations of compact surface groups as discrete subgroups of $SO_{4,1}$.

In each of these remaining cases, the resulting compact, conformally flat 3-manifold $X \cong \Omega/\Gamma$ can be described topologically as a circle bundle over a compact surface which has been modified by removing tubular neighborhoods of a finite number of fibres and replacing each neighborhood with the complement of a torus knot in S^3. Examples of conformally flat 3-manifolds with incompressible tori were first given by R. Kulkarni [Ku]. These had surjective developing maps.

The specific ideas examined here suggest a general construction of discrete groups $\Gamma \subset SO_{4,1}$. While we were preparing this manuscript, B. Maskit told us that he had known a related construction for some time (cf. [A], [M]). The idea of this related construction is roughly as follows. Consider a smooth knot κ in S^3. Suppose that κ is covered by a cyclic sequence of metric balls $\mathscr{C} = \{ B_1, \ldots, B_m \}$ with the property that

$B_i \cap B_j \neq \emptyset$ if and only if $i - j \equiv \pm 1 \pmod m$ (i.e. if and only if they are adjacent). Suppose that each pair of adjacent " beads " B_i, B_{i+1} meets in an exterior angle of the form π/n_i where n_i is a positive integer. (Such configurations are easily arranged for any κ.) Then the group $\Gamma_{\kappa,\mathscr{C}}$ generated by the inversions in each of the spheres ∂B_i, $i = 1, \ldots, m$, is a discrete subgroup of $SO_{4,1}$ whose limit set $\lim \Gamma_{\kappa,\mathscr{C}} \subset S^3$ is a certain infinite compounding of the original knot κ.

If one begins with an " unknot " κ and passes to a subgroup $\Gamma \subset \Gamma_{\kappa,\mathscr{C}}$ of finite index which acts freely on $\Omega = S^3 - \lim \Gamma_{\kappa,\mathscr{C}}$, then the manifold Ω/Γ is a circle bundle over a compact surface. Furthermore, the reflection symmetries force the Euler class of this bundle to vanish, that is, this bundle is always topologically trivial. In particular therefore, the construction which we shall present here cannot be reformulated in this way.

However, a nice generalization of our construction has been found by N. Kuiper and will appear in a sequel to this work. His method yields our examples as the special " homogeneous " ones.

Recently the authors received a preprint by M. Kapovich [Kap] who independently proved the existence of a flat conformal structure on the total space of the circle bundle X associated to $E \to \Sigma$, for

$$1 \leqslant |\chi(E)| \leqslant |\chi(\Sigma)|/22.$$

(Kapovich's argument is similar to our basic construction in § 1.) At this point Kapovich additionally observes that the natural circle action on X lifts to a group of *uniformly quasiconformal* transformations of the sphere S^3 which are *not topologically conjugate to conformal* transformations.

Kapovich also states the following

Conjecture. — Let N be a Haken manifold whose fundamental group contains no solvable subgroup of finite index. Then there exists a finite covering $\tilde{N}$ of N which admits a conformally flat structure.

Using a version of the cusp closing method he proves this conjecture for those N where the Seifert part of the Haken decomposition contains no component homeomorphic to $T^2 \times [0, 1]$ and where there is no gluing between the hyperbolic components of N. This result applies, for example, to an N obtained by attaching the above X to a cusp of a hyperbolic manifold N_1 (compare § 7). On the other hand Kapovich's theorem does not apply if N is obtained by gluing together two hyperbolic manifolds N_1 and N_2 along the cuspidal tori. (If the gluing diffeomorphism f between these tori is close to being conformal, then by the cusp closing argument the manifold $N = N_1 \cup_f N_2$ admits a conformally flat structure.)

The paper of Kapovich contains several other interesting results including the solution of Goldman's problem on variations of conformally flat structures with fixed holonomy and an example of a manifold N admitting no conformally flat structure, while some finite covering $\tilde{N}$ of N does admit such a structure.

Finally we would like to thank W. Goldman and Y. Kamishima for some useful remarks. We want to express particular appreciation to Nico Kuiper for indicating a number of errors in the first draft of the manuscript and for his generous help in shaping the final version of the paper.

1. The basic construction

In this section we set the notational conventions for the paper and we present the main idea of the construction.

Let $P = P(n, \ell, a)$ be a regular polygon in $\mathbf{H}^2$ where:

$n \equiv$ the number of vertices of P,

$\ell \equiv$ the length of each edge of P,

$a \equiv$ the interior angle at each vertex of P.

Any two of the parameters (n, ℓ, a) are independent and determine the third (by hyperbolic trigonometry). For each n, the parameters ℓ and a can be varied continuously, and $\ell \to \infty$ as $a \to 0$. As $n, \ell \to \infty$, the distance between opposite edges of P also goes to ∞.

At each vertex we shall join together copies of P according to a closed, oriented polygonal curve $\gamma \subset S^3$ which we call the *template*. We shall assume γ is *regular*, i.e. that the orientation-preserving self-congruences of γ are transitive on the vertices. Then $\gamma = \gamma(\nu, \lambda, \alpha, \tau)$ has dependent parameters:

$\nu \equiv$ the number of vertices of γ,

$\lambda \equiv$ the length of each edge of γ,

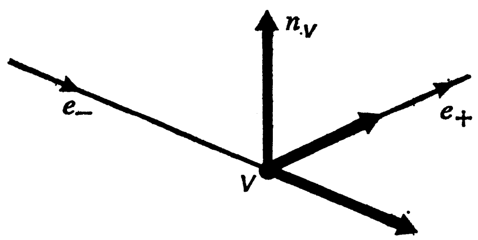
$\alpha \equiv$ the interior angle at each vertex of γ,

$\tau \equiv$ the *torsion* along each edge of γ.

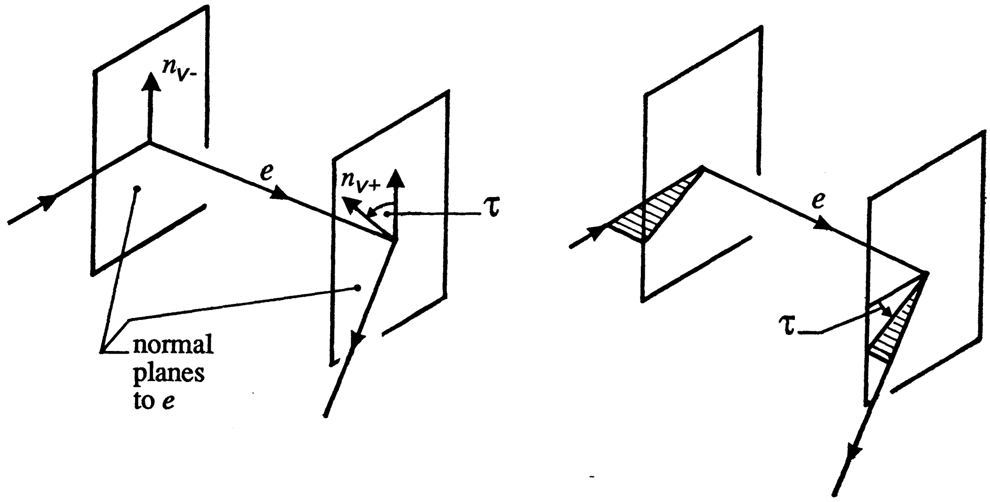
The torsion τ is defined as follows. Assume an orientation is given for S^3. Along each edge e of γ, let $N_x(e)$, $x \in e$, denote the field of oriented normal planes to e. These planes are mutually identified by parallel translation along e. We now define at each vertex v of γ, a distinguished unit normal vector

(1.1) $$n_v \in N_v(e_-) \cap N_v(e_+)$$

where e_+ and e_- are the edges which meet at v, and where e_+ follows e_- in the orientation of γ. This vector is determined up to sign by (1.1). The sign is chosen by taking the direction of the cross-product $\dot{e}_- \times \dot{e}_+$ of the forward pointing tangent vectors $e_\pm$ of $\dot{e}_\pm$ at v.

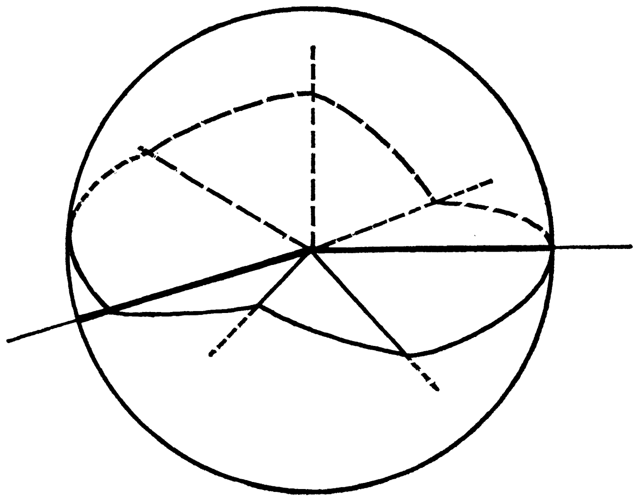


Suppose now that e is an edge of γ with endpoints v_+ and v_- where v_+ follows v_- in the orientation. Then the *torsion* of e is the unique angle τ, with $|\tau| < \pi$ formed in passing from n_{v_-} to n_{v_+} in the normal plane (field) to e.



This is also the angle change measured between the orthogonal projections at the endpoints of the " arriving " and " departing " edges of γ. (Again we measure *from* the arriving *to* the departing edge.)

At any point $x \in \mathbf{H}^4$, we can consider $\gamma \subset S^3_x \equiv \{ V \in T_x \mathbf{H}^4 : \| V \| = 1 \}$ and take the *geodesic cone* $C_x(\gamma) \equiv \{ \exp_x(tV) \in \mathbf{H}^4 : V \in \gamma \text{ and } t \geqslant 0 \}$. This cone is a union of geodesic planar wedges meeting at x. Each wedge has interior angle λ.



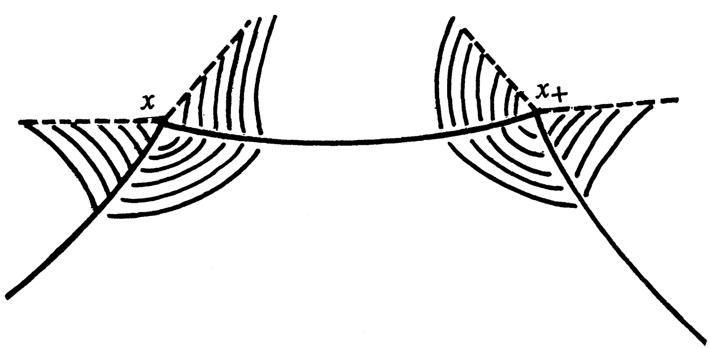
The basic idea of the construction now is the following. We fix a copy of P in $\mathbf{H}^4$ and at each vertex x of P we adjoin more copies of P so that in a neighborhood of x the resulting surface coincides with $C_x(\gamma)$. The process is then repeated at each exposed vertex of the resulting polyhedral surface, and so on... For this process to work we must require two *compatibility conditions*:

(1.2) $$a = \lambda,$$

(1.3) $$n\tau \equiv 0 \pmod{2\pi}.$$

The necessity of (1.2) is obvious. Condition (1.3) arises by considering what happens when we apply the construction to each vertex successively as we pass around the boundary of the polygon P.

We proceed formally as follows. Fix a copy of P in $\mathbf{H}^4$ and fix a vertex x of P. Choose an isometric embedding $\gamma \cong \gamma_x \cap S^3_x$ so that $P \subset C_x(\gamma_x)$. (Near x, P determines one of the wedges of the cone. It is then contained in the infinite wedge.) For this we need condition (1.2). Note that there is an SO_2-family of such embeddings possible for γ. These correspond to rotations of the flat normal plane field to P. From this point on the process is determined. At x we now adjoin $(\nu - 1)$ more copies of P so that in a small neighborhood of x the resulting surface coincides with the cone $C_x(\gamma_x)$. We now pass to the adjacent vertex x_+ in the positive direction. The pair of wedges in $C_x(\gamma_x)$ along the edge $\widehat{xx_+}$ completely determines the placement of γ in the tangent sphere $S^3_{x_+}$ at x_+.



We now fill out the surface at x_+ by attaching polygons as we did before. We then pass on to the next vertex and continue the process around the boundary of P until we return to the original vertex x. Upon our return to x a new embedding of γ in S^3_x is determined. It should agree with the original one, so that the surface " closes ". Using the flat normal plane field to P one checks that this closure occurs if and only if condition (1.3) is satisfied.

We have now constructed a polyhedral immersion of a 2-disk into $\mathbf{H}^4$. We choose a vertex on the boundary of the disk where two polygons come together. Here the template is fixed. We fill out at this vertex and then continue around the boundary of the disk as before. Periodically, in fact at every n-th vertex, we encounter a template from the previous construction because we have encircled a polygon. Compatibility is guaranteed as before. Global compatibility around the boundary of the disk is now obvious. Proceeding in this way by " concentric annuli " we generate an infinite surface

$$(1.4) \qquad \Sigma_{P,\gamma} \to \mathbf{H}^4.$$

This surface is given as a polygonal immersion of the open disk D^2. The immersion induces a complete metric with *isolated* singularities and with $K \equiv -1$ at all regular points on D^2. It also induces a topological tiling of D^2 by n-gons, ν of which meet at each vertex. Note that the induced metric is singular only at vertex points.

Let $\Gamma_{P,\gamma}$ be the group of orientation preserving isometries of the surface in its induced metric. This coincides with the group of orientation preserving automorphisms of the tiling. (Recall that the template is never a great circle: the angles are not 0 or π.) From the action on the template cones, it is clear that the construction determines a homomorphism

$$(1.5) \qquad \Gamma_{P,\gamma} \to \mathrm{Isom}(\mathbf{H}^4)$$

with respect to which the map (1.4) is equivariant.

The main result of the next section is the following.

Theorem **1.6.** — *There is a constant* $L = L(\gamma)$ *such that for any compatible polygon* P *with side length* $\ell \geqslant L(\gamma)$, *this resulting immersion* (1.4) *is a proper embedding.*

Suppose we are in the range $\ell \geqslant L(\gamma)$, so that we have a proper embedding

$$(1.4)' \qquad \Sigma_{P,\gamma} \hookrightarrow \mathbf{H}^4.$$

Evidently there are freely acting subgroups $\Gamma \subset \Gamma_{P,\gamma}$ of finite index such that $\Sigma_{P,\gamma}/\Gamma$ is a compact oriented surface. Since (1.4)′ is equivariant we get an embedding

$$(1.7) \qquad \Sigma_{P,\gamma}/\Gamma \hookrightarrow \mathbf{H}^4/\Gamma$$

which is a homotopy equivalence.

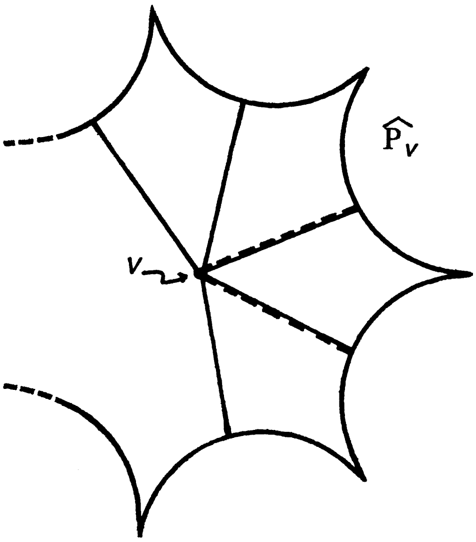
Remark **1.8.** — If the template γ is unknotted in S^3, then the embedding (1.7) is locally flat and the manifold $\mathbf{H}^4/\Gamma$ is PL-homeomorphic to the PL normal bundle of the embedding. In fact, an elementary-argument shows that $\mathbf{H}^4/\Gamma$ is diffeomorphic to the smooth normal bundle N of a smooth approximation to the embedding.

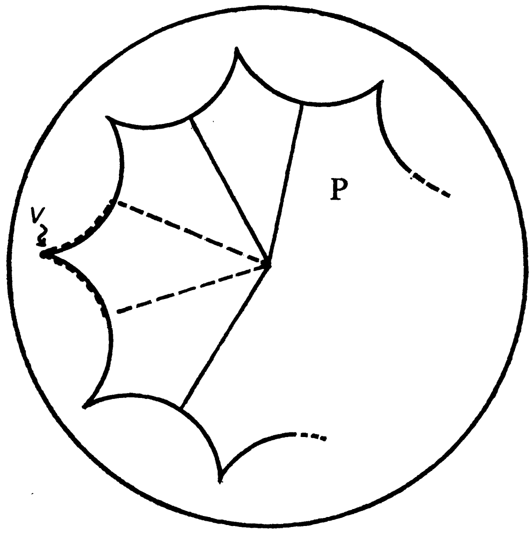
In § 4 we shall compute the Euler class of N in terms of $\chi(\Gamma)$ and the geometry of P and γ.

2. A criterion for properness

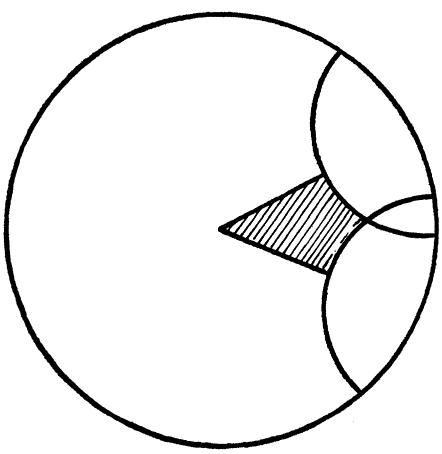
The point of this section is to prove Theorem 1.6. Consider the immersed polygonal surface $\Sigma_{P,\gamma} \to \mathbf{H}^4$ constructed above and fix one of its vertices, say v. For convenience we imagine v to be located at the origin in the Poincaré model for $\mathbf{H}^4$, so that $C_v(\gamma_v)$ actually becomes a linear cone on an isometric embedding of γ in the unit 3-sphere $S^3 = \partial B^4 \approx \partial_\infty \mathbf{H}^4$.

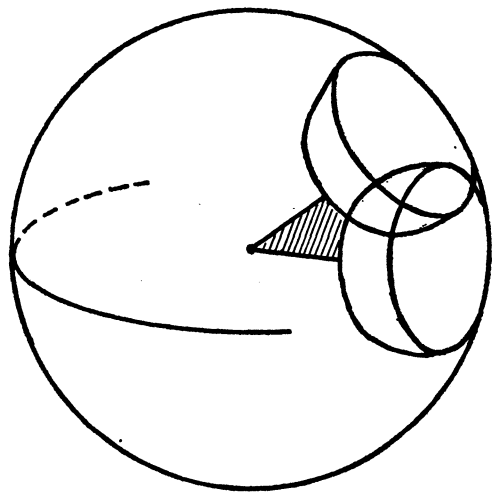
We shall now study the star of this vertex v in $\Sigma_{P,\gamma}$. To do this we divide our basic polygon P into n congruent regions by cutting along each geodesic which joins the center of P to the midpoint of an edge. These components are called *pie slices.* The *star of the vertex v*, denoted $\hat{P}_v$, is then defined to be the union of the ν pie slices with vertex v.



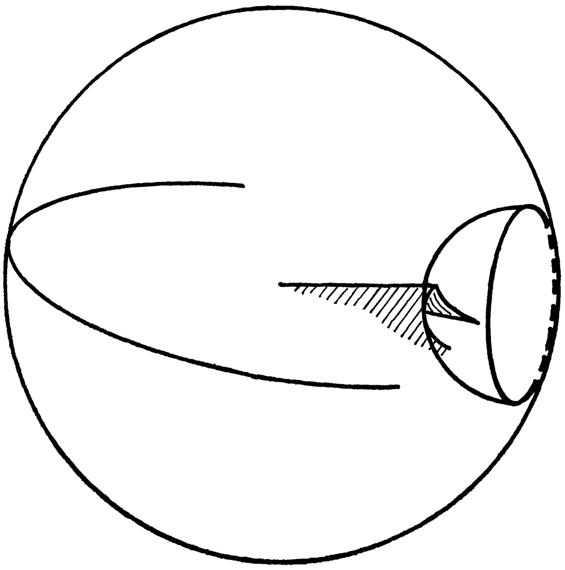


Note that we have $\hat{P}_v \subset C_v(\gamma_v)$. Note also that the geodesics from the center of P to the midpoints of the edges meet these edges orthogonally. Hence, in $\mathbf{H}^4$ each such geodesic lies on the hyperplane orthogonal to the edge at that point. Hence, each pie slice at v lies in the intersection of the two half-spaces determined by the hyperplanes perpendicular to the two edges of the slice emanating from v.





Keep in mind that the surface $\hat{P}_v$ bends along these edges.



Consider now the following. For each vertex ω of γ we let $\mathscr{H}_\omega \subset \mathbf{H}^4$ denote the geodesic hyperplane orthogonal to the ray $e_\omega(t) = \exp_v(t\omega)$ at the point $e_\omega(\tilde{\ell}/2)$ = the midpoint of the corresponding edge of the polygon. (So $\tilde{\ell}$ = the length of an edge of P.) Let $\mathbf{H}^-_\omega$ be that open half-space of $\mathbf{H}^4$ which contains v and has

$$\partial \mathbf{H}^-_\omega = \mathscr{H}_\omega .$$

Recall that the curve γ is regular, and note that as $\tilde{\ell} \to \infty$ (considered as a free parameter), we have $\mathscr{H}_\omega \cap \mathscr{H}_{\omega'} = \emptyset$ for all vertices of γ. We shall assume that there is a number $L = L(\gamma)$ such that whenever $\ell \geqslant L$, we have

(2.1) $$\mathscr{H}_\omega \cap \mathscr{H}_{\omega'} \neq \emptyset$$

only if ω and ω' are adjacent vertices of γ. This will be true if, say, γ is constructed by taking a sufficiently fine subdivision of a smooth curve.

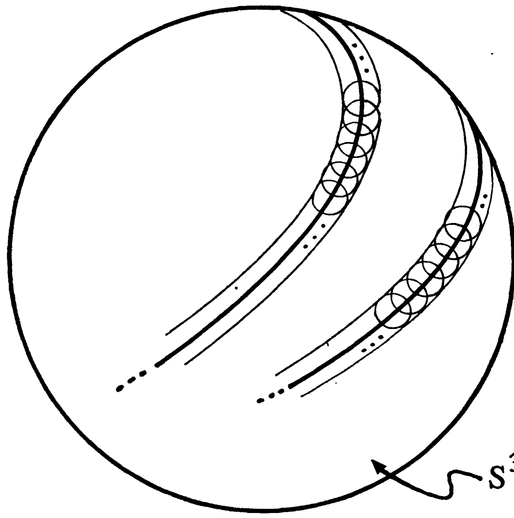
Assume now that $\ell \geqslant L(\gamma)$ and that ℓ is the side length of P in our basic construction. Then (2.1) does hold whenever ω and ω' are adjacent vertices. Furthermore these hyperplanes meet with "interior" angle π/n.

Consider the geodesically convex set

$$\mathscr{B}_v \equiv \cap \mathbf{H}^-_\omega$$

where the intersection is over the vertices ω of γ. This set is diffeomorphic to $\mathbf{H}^4$. It can be visualized as the result of digging a trench in $\mathbf{H}^4$ along the curve $\gamma \subset S^3 = \partial_\infty \mathbf{H}^4$ by chopping out a series of spherical pieces (one at each vertex of γ). The complement $\tilde{\mathscr{B}}_v \equiv \mathbf{H}^4 - \mathscr{B}_v$ is homeomorphic to $S^1 \times D^3$ and retracts down onto a tubular neighborhood of γ in S^3. Note that

(2.2) $$\hat{P}_v = \mathscr{B}_v \cap C_v(\gamma_v).$$



The region $\mathscr{B}_v$ has boundary consisting of geodesic hyperplanes which meet in dihedral angles π/n. We can now form an abstract complete hyperbolic 4-manifold M^4 by gluing together $2n$-copies of $\mathscr{B}_v$ along each " edge " $\mathscr{H}_\omega \cap \mathscr{H}_{\omega'}$ and then continuing the process indefinitely in the same manner as that of § 1. We shall perform this gluing process in the following very specific way. Consider $\mathscr{B}_v \subset \mathbf{H}^4$ with its oriented template tile $\hat{P}_v \subset \mathscr{B}_v$. Fix a codimension-2 face $\mathscr{E} \equiv \mathscr{H}_\omega \cap \mathscr{H}_{\omega'}$ and let $c = \hat{P}_v \cap \mathscr{E}$. (Note that c will become the center of one of the basic polygons in our extended surface.) We now apply to $\mathscr{B}_v$ the unique isometry g of $\mathbf{H}^4$ which fixes c, rotates $T_c \hat{P}_v$ by π/n in the positive direction, and rotates $(T_c \hat{P}_v)^\perp$ by τ (the torsion). This map is just rotation by π/n at the codimension-2 face $\mathscr{E}$ followed by a normal twist about c (which leaves both $\mathscr{B}_v$ and its rotated image invariant). The union $\mathscr{B}_v \cup g(\mathscr{B}_v) \cup \ldots \cup g^{2n-1}(\mathscr{B}_v)$ gives a tiling of a neighborhood of the codimension-2 face $\mathscr{E}$. The union $\mathscr{P} \equiv \hat{P}_v \cup g(\hat{P}_v) \cup \ldots \cup g^{2n-1}(\hat{P}_v)$ is exactly the continuation of our surface in this neighborhood. (Note that rotation by π/n at $\mathscr{E}$ corresponds exactly to rotation of the basic polygon $P \subset \mathscr{P}$ about its center c. However, simple rotation by π/n does not preserve the array of template curves attached to the vertices of P. To achieve this we must apply a normal twist by τ.)

We have now described the local gluing process. Using it we form a complete hyperbolic 4-manifold $M^4_{P,\gamma}$ following exactly the same formal procedure invoked to build the complete polygonal surface $\Sigma_{P,\gamma}$. Note that by its construction, $M^4_{P,\gamma}$ comes equipped with a locally isometric immersion $M^4_{P,\gamma} \to \mathbf{H}^4$ which, since $M^4_{P,\gamma}$ is compete and $\pi_1(M^4_{P,\gamma}) = 0$, must be a global isometry. We conclude the following.

Lemma **2.3.** — *Applying the τ-twisted rotations described above at the codimension-2 faces of $\mathscr{B}_v$ generates a tiling of* $\mathbf{H}^4$.

It is clear from the construction above that there exists a proper embedding $\Sigma_{P,\gamma} \hookrightarrow M^4_{P,\gamma}$ with exactly one template tile lying in each fundamental region. In fact there is an action of the group $\Gamma_{P,\gamma}$ by isometries on $M^4_{P,\gamma}$ with respect to which this embedding is equivariant. The isometry $M^4_{P,\gamma} \to \mathbf{H}^4$ carries the tesselation of $M^4_{P,\gamma}$ to the tiling of $\mathbf{H}^4$ and is $\Gamma_{P,\gamma}$-equivariant. Its restriction to $\Sigma_{P,\gamma} \subset M^4_{P,\gamma}$ gives the immersion constructed in § 1. It is now clear that this immersion is a proper embedding and that the representation $\Gamma_{P,\gamma} \to SO_{4,1}$ is discrete. This completes the proof of Theorem 1.6.

3. Examples of templates

Let p and q be positive integers with $(p, q) = 1$, and fix ε with $0 < \varepsilon < 1$. Consider the homogeneous curve $\Gamma = \Gamma_{q,p,\varepsilon} \subset S^3 = \{ (z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 = 1 \}$ parameterized by

(3.1) $$\Gamma(t) = \left(\sqrt{1-\varepsilon^2}\, e^{iqt}, \varepsilon e^{ipt}\right) \quad \text{for } 0 \leqslant t \leqslant 2\pi.$$

For each integer $\nu \geqslant 3$, we have the associated polygonal curve $\gamma = \gamma_{\varepsilon,q,p,\nu}$ obtained by subdividing Γ into ν equal parts. The vertices of γ are the points

(3.2) $$X_k = \left(\sqrt{1-\varepsilon^2}\, \omega^{qk}, \varepsilon\omega^{pk}\right) \quad k = 0, \dots, \nu - 1,$$

where $\omega \equiv e^{2\pi i/\nu}$. Note that

(3.3) $$X_k = \varphi^k(X_0)$$

where $\varphi : S^3 \to S^3$ is the isometry defined by $\varphi(z, w) \equiv (\omega^q z, \omega^p q)$. The polygonal curves $\gamma_{\varepsilon,q,p,\nu}$ are clearly regular.

Let $\tau = \tau_{\varepsilon,q,p,\nu}$ be the torsion at an edge of this curve. Then direct computation shows that

(3.4) $$\cos\tau = \frac{\varepsilon^2 \sin^2\left(\frac{2\pi p}{\nu}\right)\cos\left(\frac{2\pi q}{\nu}\right) + (1-\varepsilon^2)\sin^2\left(\frac{2\pi q}{\nu}\right)\cos\left(\frac{2\pi p}{\nu}\right)}{\varepsilon^2\sin^2\left(\frac{2\pi p}{\nu}\right) + (1-\varepsilon^2)\sin^2\left(\frac{2\pi q}{\nu}\right)}.$$

Note that

(3.5) $$\lim_{\varepsilon\to 0}\tau = \frac{2\pi p}{\nu} \quad \text{and} \quad \lim_{\varepsilon\to 1}\tau = \frac{2\pi q}{\nu}.$$

For our construction however the template must have the following additional property:

(3.6) The minimum distance between distinct vertices of γ is realized only by adjacent pairs.

This property will guarantee condition (2.1). Note that for each smooth curve Γ as above, there is a number $\nu(\Gamma)$ such that the polygonal curve $\gamma = \Gamma_\nu$, obtained by subdividing Γ into ν equal parts, satisfies property (3.6) for every $\nu \geq \nu(\Gamma)$. Elementary calculations show for example that we could take

$$\nu(\Gamma) = \frac{\pi(p^2 \varepsilon^2 + q^2(1 - \varepsilon^2))}{\varepsilon\sqrt{1 - \varepsilon^2}}. \tag{3.7}$$

4. Computation of the Euler class

Consider now a template of type $\gamma = \gamma_{\varepsilon, 1, p, \nu}$ and choose a compatible polygon $P = P(n, \ell, a)$ where $\ell \geq L(\gamma)$. This generates a properly embedded surface $\Sigma_{P,\gamma} \subset \mathbf{H}^4$ which, since γ is unknotted, is locally flat. For each subgroup $\Gamma \subset \Gamma_{P,\gamma}$ of finite index which acts freely on $\Sigma_{P,\gamma}$ we get a compact quotient $\Sigma \equiv \Sigma_{P,\gamma}/\Gamma$ and an embedding

$$\Sigma \subset \mathbf{H}^4/\Gamma.$$

It is straightforward to see that

$$\Sigma \subset \mathbf{H}^4/\Gamma \overset{\text{diffeo}}{\cong} N$$

when N is the normal bundle to a smooth approximation to this embedding.

The main aim of this section is to compute the Euler class of this bundle N. To do this we keep Σ in its original polygonal form. The tiling of Σ by copies of P gives a cell decomposition of Σ with, say, V vertices, E edges and F faces. Since each face has n edges and each vertex meets ν edges, we have the relations:

$$E = \frac{1}{2} nF = \frac{1}{2} \nu V. \tag{4.1}$$

Observe now that the normal bundle to Σ has a natural flat connection along each face. Furthermore, by making the obvious rotation we get a canonical identification of the normal spaces to the faces along each edge. This produces a flat connection on N over all of Σ-{ vertices }.

We can use this connection to compute $\chi(N)$ as follows. In a neighborhood of each vertex v, consider a small disk $D_r(v) = \{ x \in \Sigma : \text{dist}(x, v) \leq r \} \subset \hat{P}_v$. Let F_v be the 0-framing of the normal bundle N along the circle $\partial D_r(v)$. This is the framing with self-linking zero in the 3-sphere $S^3_r(v) \equiv \{ x \in \mathbf{H}^4/\Gamma : \text{dist}(x, v) = r \}$. It is also the framing which extends to a framing of N over the disk $D_r(v)$. Define h_v to be the *total* rotation with respect to this framing induced by parallel translation around $\partial D_r(v)$ in the flat connection on N. This number is the same for all vertices, since they are mutually congruent.

The standard curvature formulas yield the relation

$$\chi(N) = \frac{1}{2\pi} \Sigma h_v = \frac{V}{2\pi} h_v. \tag{4.2}$$

To see this, choose a smoothing of $D_r(V)$ near v and extend F_v to a smooth framing of the normal bundle. Extend the flat connection on $N|_{\partial D_r(v)}$ smoothly over the disk. In the framing the connection is given by a 1-form σ, and the curvature is $\Omega = d\sigma$. Do this at each vertex. Since $\Omega = 0$ outside the disks $D_\varepsilon(v)$ we have

$$\chi(N) = \frac{1}{2\pi}\int_\Sigma \Omega = \frac{1}{2\pi}\sum_v \int_{D_\varepsilon(v)} d\sigma$$
$$= \frac{1}{2\pi}\sum_v \int_{\partial D_\varepsilon(v)} \sigma = \frac{1}{2\pi}\sum_v h_v$$

as claimed.

At each vertex v we have defined the *total torsion* — $\nu\tau$ (where ν = the number of vertices of γ). Examination of the definition shows that this number represents the total rotation induced by parallel translation *with respect to the "Frenet framing"* which is defined as follows. Let $e = v_- v_+$ be an edge of $\partial D_r(v) \cong \gamma_v \cong \gamma$ with vertices v_+ and v_-. Let n_{v_+} and n_{v_-} be the distinguished normals to the curve at these vertices (cf. (1.1)). Define a normal vector field n along e by rotating n_{v_-} to n_{v_+} linearly through the smallest possible angle. This defines a global unit normal field to $D_r(v)$, and hence a normal framing. (The second field is just the $(\pi/2)$-rotation of n.) This is the *Frenet framing* of $\partial D_r(v) \cong \gamma$.

The self-linking of the Frenet framing of γ is *not* zero. It is equal to the linking number $LK(\gamma, \gamma_n)$ in S^3, where γ_n is obtained by pushing γ off itself in the direction n.

Proposition **4.3.** — *If* $\gamma = \gamma_{\varepsilon,1,p,\nu}$ *for* $0 < \varepsilon < 1$, $p \geqslant 0$ *and for* $\nu > 2p$, *then the self-linking of the Frenet framing of* γ *is* p.

Proof. — We compare γ to the smooth curve Γ given in (3.1) with $q = 1$. For Γ the Frenet framing is given as usual by the normal and binormal vectors, and one easily sees that, for $0 < \varepsilon < 1$, its self-linking is p. Using homogeneity one also easily checks that if $\nu > 2p$, then γ with its Frenet framing is isotopic to Γ.

Corollary **4.4.** — *If* $\gamma \cong \gamma_v$ *is as in* (4.3), *then* $h_v = \nu\tau - 2\pi p$, *where* τ *is the torsion at an edge of* γ.

Equation (4.2) now implies that

$$\chi(N) = \frac{1}{2\pi} V\nu\tau - Vp. \tag{4.5}$$

This of course should be an integer. To see that it is, note from (4.1) that we have $V\nu\tau = Fn\tau$, and that from the compatibility condition (1.3) we have $n\tau \equiv 0 \pmod{2\pi}$.

We now assume that $\nu \geqslant \nu(\Gamma)$ where $\nu(\Gamma)$ is given, say, by (3.7) with $q = 1$. Combining (4.1) and (4.5) then gives the following result.

Theorem **4.6.** — *Fix a template* $\gamma = \gamma_{\varepsilon,1,p,\nu}$ *with* $0 < \varepsilon < 1$ *and* $\nu > 2p$, *and choose any compatible polygon* $P = P(n, \ell, a)$ *with* $\ell \geqslant L(\gamma)$. *Let* $\Sigma \hookrightarrow \mathbf{H}^4/\Gamma$ *be any quotient of* $\Sigma_{P,\gamma} \hookrightarrow \mathbf{H}^4$

by a subgroup of finite index $\Gamma \subset \Gamma_{P,\gamma}$ *which acts freely on* $\Sigma_{P,\gamma}$. *Then* $\mathbf{H}^4/\Gamma$ *is diffeomorphic to a 2-plane bundle* N *over the compact surface* Σ, *where*

$$\textbf{(4.7)} \qquad \frac{\chi(N)}{\chi(\Sigma)} = \frac{(p/\nu) - (\tau/2\pi)}{(1/2) - (1/\nu) - (1/n)}$$

and where τ *is given by* (3.4).

Set $T = (\nu/2\pi)\,\tau$ and note from (3.4) that T takes on all values in the interval $1 < T < p$ as ε ranges from 0 to 1. From above we have

$$\textbf{(4.8)} \qquad \left|\frac{\chi(N)}{\chi(\Sigma)}\right| = \left(\frac{1}{2} - \frac{1}{n} - \frac{1}{\nu}\right)^{-1} \frac{1}{\nu}(p - T).$$

Hence, this quotient takes values in the range

$$\left|\frac{\chi(N)}{\chi(\Sigma)}\right| < \left(\frac{1}{2} - \frac{1}{n} - \frac{1}{\nu}\right)^{-1} \frac{(p-1)}{\nu}.$$

Since $\nu > 2p$ we conclude that

$$\left|\frac{\chi(N)}{\chi(\Sigma)}\right| < \left(\frac{1}{2} - \frac{1}{n} - \frac{1}{\nu}\right)^{-1} \left(\frac{1}{2} - \frac{1}{\nu}\right)$$

which implies that for large values of n and ν the bundles N obtained by this construction lie essentially in the range

$$|\chi(N)| \leqslant |\chi(\Sigma)|.$$

Note that, since $T \to 1$ as $\varepsilon \to 1$ (see (3.5)), we have that

$$\lim_{\substack{n,\nu\to\infty \\ \varepsilon\to 1}} \left|\frac{\chi(N)}{\chi(\Sigma)}\right| = 1.$$

However, we have restricted ν and ε by the inequality

$$\nu \geqslant \nu(\Gamma) = \pi(1 + (p^2 - 1)\,\varepsilon^2)/\varepsilon\sqrt{1 - \varepsilon^2},$$

so we cannot fully saturate this range of possibilities.

5. Knots and limit sets

The templates $\gamma = \gamma_{\varepsilon,q,p,\nu}$ with $q > 1$, $p > 1$ and $\nu > 2\max(p, q)$ are torus knots of type (q, p). Choosing a compatible polygon $P = P(n, \ell, a)$ with $\ell \geqslant L(\gamma)$, we again generate a surface $\Sigma_{P,\gamma}$ and a group $\Gamma_{P,\gamma}$, and we can pass as before to a subgroup $\Gamma \subset \Gamma_{P,\gamma}$ of finite index acting freely on $\Sigma_{P,\gamma}$. The quotient $\Sigma = \Sigma_{P,\gamma}/\Gamma$ is a compact oriented surface and the embedding

$$\textbf{(5.1)} \qquad \Sigma \hookrightarrow \mathbf{H}^4/\Gamma$$

is a homotopy equivalence. However, this embedding is not locally flat.

The manifold $\mathbf{H}^4/\Gamma$ is diffeomorphic to a 2-plane bundle E over Σ which has been modified at a finite number of points as follows. At a point $p \in \Sigma$, remove a local trivialization $E|_{D^2} \cong D^2 \times \mathbf{R}^2$ from E and glue in a copy of the set $\mathscr{B}_v$ along the boundary $S^1 \times \mathbf{R}^2$. Recall that $\mathscr{B}_v$ is $\mathbf{H}^4$ with a knotted trench removed at infinity. The knot $\gamma \subset \partial_\infty \mathbf{H}^4$ is a torus knot of type (q, p).

These examples are interesting because they give discrete groups $\Gamma \subset \mathrm{Isom}(\mathbf{H}^4)$ whose limit sets are infinitely compounded knots in $S^3 = \partial_\infty \mathbf{H}^4$. Recall that the *limit set* of Γ is the set

$$\lim \Gamma = (\overline{\Gamma \cdot p}) \cap S^3$$

where p is any point of $\mathbf{H}^4$ and where the closure of the orbit $\Gamma \cdot p$ is taken in $\overline{\mathbf{H}^4} \equiv \mathbf{H}^4 \cup \partial_\infty \mathbf{H}^4$. The limit set is unchanged by passing to subgroups of finite index.

The limit set for the groups $\Gamma_{P,\gamma}$ constructed above can be seen directly. Let $\mathscr{B}_v$ be the fundamental domain constructed in § 2, and let $\hat{\mathscr{B}}_v$ denote the closure in $\overline{\mathbf{H}^4}$ of its complement $\mathbf{H}^4 - \mathscr{B}_v$. Note that $\hat{\mathscr{B}}_v$ is the closure of the *trench*. The set

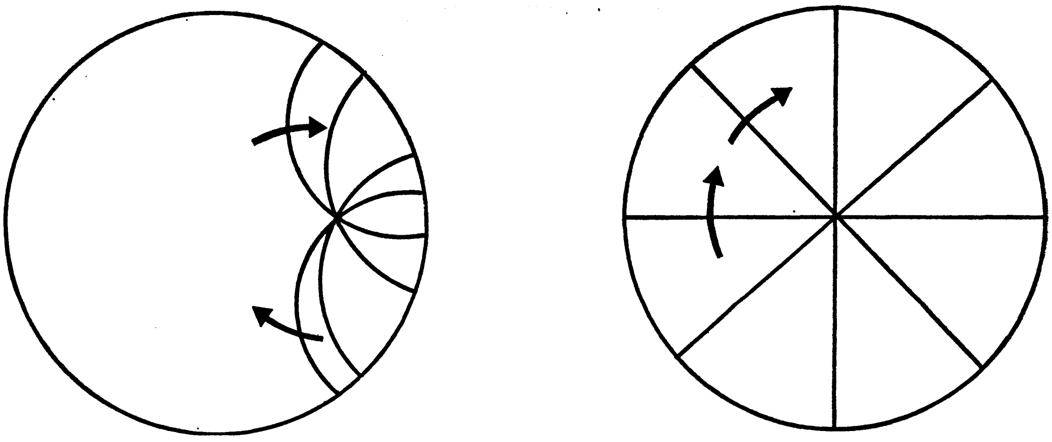
$$U_v \equiv \hat{\mathscr{B}}_v \cap S^3$$

is a closed tubular neighborhood of the knot $\gamma \subset S^3$.

Now it is evident from the construction that

$$\lim \Gamma = \bigcap_{g \in \Gamma} U_{gv}$$

and we shall examine these intersections at finite stages. Let g denote the element generated by a (π/n)-rotation at a codimension-2 face of $\mathscr{B}_v$ (together with a normal twist as in § 2). We consider what the effect of this isometry is on $\mathscr{B}_v$. Consider the n hyperplanes meeting at equal angles π/n in this codimension-2 face, two of which form part of the boundary of $\mathscr{B}_v$. These hyperplanes divide $\mathbf{H}^4$ into $2n$ congruent regions and g performs a cyclic permutation of these regions. At infinity, these hyperplanes divide the sphere into conformally equivalent regions, say $R_1, \ldots, R_{2n}$, which are cyclically permuted



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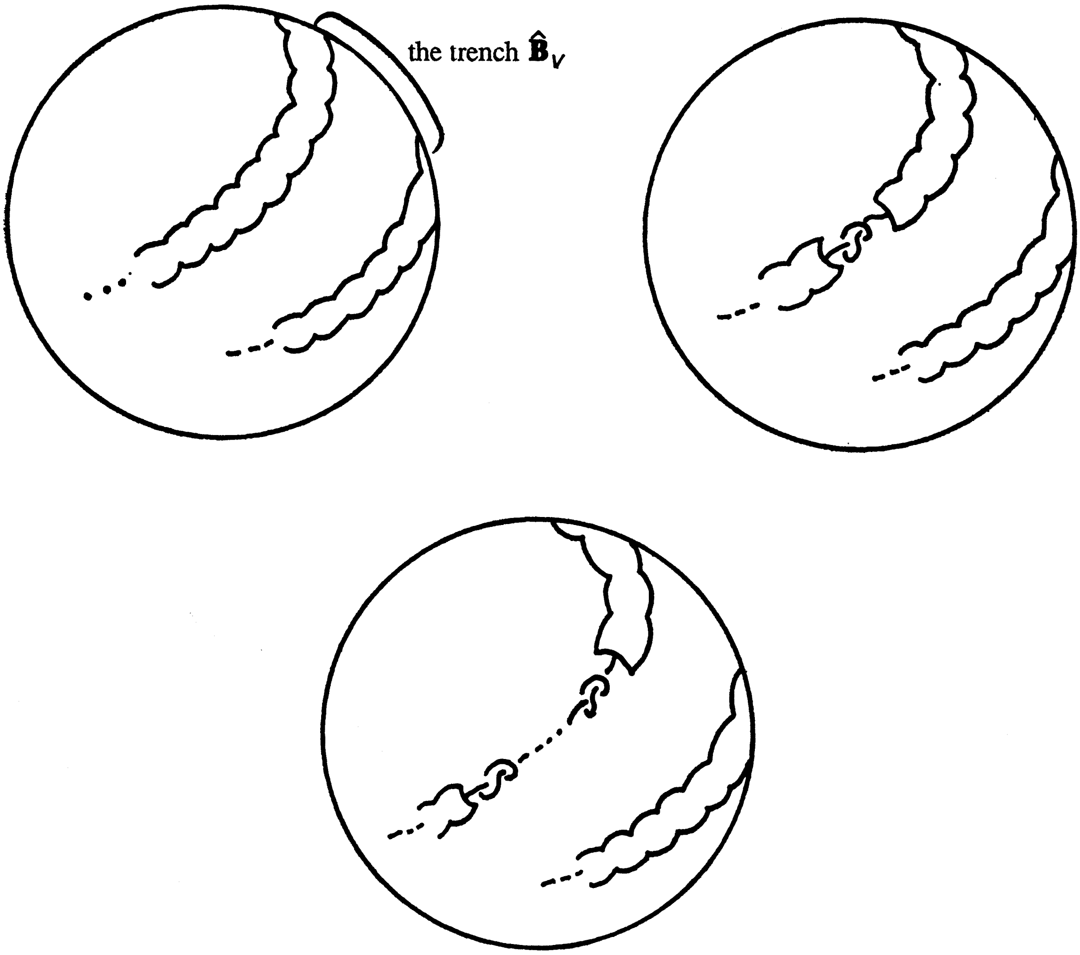
by g. Note however that all but one of these regions, say R_1, is contained in the thickened knot U_v, while R_1 contains all of the knot but a local segment. Hence



the intersection $U_v \cap gU_v = U_v \cap U_{gv}$ is a thickening of the connected sum of the knot with itself, and $\bigcap_{k=1}^{2n} U_{g^k(v)}$ is a $2n$-fold connected sum of the knot.

It is now clear that $\lim \Gamma = \cap \{ U_{gv} : g \in \Gamma \}$ is an infinite compounding of the knot as claimed.

6. The construction generalized

Certain of the ideas presented above can be generalized in the following way (cf. [A], [M]). Let

$$L = \gamma_1 \amalg \dots \amalg \gamma_k \subset S^3$$

be any differentiable link in S^3, and consider S^3 to be the sphere at infinity for $\mathbf{H}^4$. Send a caterpillar along the link to eat out a trench from $\mathbf{H}^4$. As it goes along, the caterpillar takes a sequence of spherical bights, i.e. each bight is a geodesic half-space. Two bights (i.e. two such half-spaces) overlap if and only if they are adjacent. Each pair of adjacent bights forms an interior angle (" interior " to $\mathbf{H}^4$) of the form π/n where n is an integer > 1 which varies from bight to bight. Such sequences of bights are easy to arrange for any given link.

We now consider the group generated by reflections at each codimension-1 face. This generates a discrete group $\Gamma \subset SO_{4,1}$. The set

$$\mathscr{B} \equiv \mathbf{H}^4 - \bigcup(\text{bights})$$

is a fundamental domain for Γ and generates a tiling of $\mathbf{H}^4$. This can be seen by applying exactly the arguments given in the end of § 2. Furthermore, reasoning as in § 5, we see that $\lim \Gamma$ is an infinitely compounded link which is everywhere wild.

This construction is interesting but does not recapture the examples above. To get these we must replace reflections in the faces of $\mathscr{B}$ with certain twisted rotations at the edges. To make such a scheme work it is necessary to impose certain compatibility conditions extending condition (1.3) above. The details of a general construction of this type will appear in the sequel by N. Kuiper [Kui].

7. Conformally flat 3-manifolds

Each of the discrete groups Γ that we constructed above gives rise to a compact conformally flat 3-manifold. To be specific, let $\Gamma \subset \Gamma_{P,\gamma}$ be a subgroup of finite index which acts freely on $\Sigma_{P,\gamma}$. Then Γ acts freely and properly discontinuously on the open set

$$\mathscr{O}_\Gamma = S^3 - \lim \Gamma$$

and the quotient

$$M^3_\Gamma \equiv \mathscr{O}_\Gamma/\Gamma$$

is a compact conformally flat 3-manifold. When $\gamma = \gamma_{\varepsilon, 1, p, \nu}$, we know from § 4 that $\mathbf{H}^4/\Gamma$ is diffeomorphic to a 2-plane bundle N over a compact surface Σ. The manifold M^3_Γ is then diffeomorphic to the unit circle bundle in N.

Theorem **7.1.** — *Let* M^3 *be the total space of a circle bundle over a compact surface arising from one of the constructions above. Then* M^3 *carries a conformally flat structure.*

Remark **7.2.** — These manifolds include many of the circle bundles $N_1 \to \Sigma$ where $|\chi(N_1)| < |\chi(\Sigma)|$. (See [Kap] and [Kui].)

Even more interesting are the examples coming from groups generated with a knotted template γ. These manifolds are obtained from a circle bundle over a surface by removing sets of the form $D^2 \times S^1$ (i.e. parts of the bundle which lie over disjoint disks in the base) and gluing back in manifolds of the form $\partial_\infty \mathscr{B}_v \cong S^3 - U(\gamma)$ where $U(\gamma)$ denotes a small tubular neighborhood of γ in S^3.

Theorem **7.3.** — *There exist conformally flat structures on compact 3-manifolds of the form*

$$M_0 \,\#_{T^2}(KC) \,\#_{T^2} \dots \#_{T^2}(KC)$$

where M_0 *is a circle bundle over a surface and where* KC *is the complement of any prescribed knot in* S^3.

Note that none of these conformally flat manifolds has a surjective developing map. In particular therefore, their developing maps must be coverings by [KP].

Remark **7.4.** — In the constructions above one also replaces knots by arbitrary links. The result is a certain joining together of the manifolds which result from making the construction with each component of the link separately.

This "joining process" can be axiomatized and generalized as follows. Let A be any subset of S^n and assume that both A and $\widetilde{A} = S^n - A$ have non-empty interiors. Then it is elementary that both A and $\widetilde{A}$ can be squeezed by global conformal transformations of S^n, into arbitrarily small open subsets. Hence, both A and $\widetilde{A}$ can be embedded conformally into any conformally flat n-manifold. Given two such embeddings $A \subset M$ and $\widetilde{A} \subset M'$ into conformally flat manifolds M and M′, we can form the A-*connected sum* (after Kulkarni and Goldman)

$$M \,\#_A\, M' = (M - A) \cup (M' - \widetilde{A}) \tag{7.5}$$

with the obvious inherited structure of a conformally flat manifold. (Note that now the developing map $(M \,\#_A\, M')^\sim \to S^n$ can be surjective and not a covering.)

For example, let $L \subset S^3$ be a link with components $\gamma_1, \dots, \gamma_k$, and let $A = U(\gamma_1) \amalg \dots \amalg U(\gamma_k)$ where $U(\gamma_j)$ is a tubular neighborhood of γ_j in S^3. Consider conformally flat 3-manifolds $M_1, \dots, M_k$ and choose conformal embeddings $U(\gamma_j) \subset M_j$ for each j. We then define the *conformal bouquet of* $M_1, \dots, M_k$ *along the link* L to be the conformally flat 3-manifold

$$S^3 \,\#_A(M_1 \amalg \dots \amalg M_k).$$

One can also connect a manifold to itself by this process. The variations are enormous.

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