PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

SIMON K. DONALDSON

Complex curves and surgery

Publications mathématiques de l'I.H.É.S., tome 68 (1988), p. 91-97
<http://www.numdam.org/item?id=PMIHES_1988__68__91_0>

COMPLEX CURVES AND SURGERY

by Simon K. Donaldson

Dedicated to Professor R. Thom.

1. Introduction

The genus of a smooth algebraic curve C of degree d in the complex projective plane is given by the well-known formula:

$$\text{genus}(C) = \frac{1}{2}(d-1)(d-2).$$

The degree can be viewed as the fundamental homology class [C] of C, expressed as a multiple of the standard generator h for $H_2(\mathbf{CP}^2)$. An entrancing problem in Geometric Topology, usually ascribed to R. Thom, asks wether C minimises the genus among all C^∞ representatives for the homology class. We formulate this as:

Conjecture **1.** — *For any smoothly embedded, oriented surface* Σ *in* $\mathbf{CP}^2$, *with homology class* $[\Sigma] = d.h$, $d > 0$, *we have*

$$\text{genus}(\Sigma) \geqslant \frac{1}{2}(d-1)(d-2).$$

One can extend this conjecture by considering, in place of $\mathbf{CP}^2$, an arbitrary smooth complex projective surface S (a C^∞ four manifold).

Conjecture **2** (cf. [11], Problem 4.36). — *For any smooth complex algebraic curve* C *in* S *and embedded* C^∞ *surface* Σ *homologous to* C, *we have genus*$(\Sigma) \geqslant$ *genus*(C).

The nature of these conjectures is clarified by the following elementary remarks. Let Σ be a surface embedded in a smooth 4-manifold X and p, q be distinct points of Σ. For each embedded arc γ in X, starting at p and ending at q but otherwise disjoint from Σ, we can form a new embedded surface $\Sigma^{\#(\gamma)}$ by removing small discs about p and q from Σ and adding a tube $S^1 \times [0, 1]$ in the boundary of a regular neighbourhood of γ. The new surface is homologous to Σ but its genus is increased by 1. A given homology class can thus be represented by surfaces of arbitrarily large genus (there always exists some connected embedded representative) but it is interesting to ask for the minimal genus of a representative. We define a function

$$g_X : H_2(X; \mathbf{Z}) \to \mathbf{N}$$

by $g_X(\alpha) = \min\{\text{genus}(\Sigma) \mid [\Sigma] = \alpha\}$, where the minimum runs over smoothly embedded, connected, oriented surfaces Σ.

The obvious approach to this minimising problem is to attempt to perform the surgical procedure inverse to that above yielding $\Sigma^{\#\gamma}$. Let Ξ be a surface embedded in a 4-manifold X and δ be a loop in Ξ with $[\delta]$ non-zero in $H_1(\Xi)$. Suppose we can find an embedding of the 2-disc:

$$i : D^2 \to X$$

with $i(\partial D^2) = \delta$ but $i(D^2)$ otherwise disjoint from Ξ. The tangent bundle of Ξ defines a non-zero section of the normal bundle to $i(D^2)$ over δ; suppose we can extend this to a non-zero section of the normal bundle over the disc. Then there is a pair of disjoint " parallel " discs $i_+, i_- : D^2 \to X$ with boundaries parallel curves δ_+, δ_- in Ξ. We cut out the cylinder in Ξ bounded by δ_+, δ_- and replace it by the discs $i_+(D^2)$, $i_-(D^2)$ to get a new surface Σ with $\text{genus}(\Sigma) = \text{genus}(\Xi) - 1$, and such that $\Xi = \Sigma^{\#(\gamma)}$ for a suitable arc γ. In this case we say that Σ has been obtained from Ξ by surgery within X along δ. If we could always find surgering discs the problem of representing homology classes would become trivial, since the genus could always be decreased to zero. But in fact this cannot always be done, and the search for these discs (closely akin to " Whitney discs ") leads to the fundamental special difficulties of 4-manifold topology (see [13], for example).

A number of results related to Conjectures 1 and 2 have been proved. Kervaire and Milnor [10] showed that, in $\mathbf{CP}^2$, $3h$ is not represented by a sphere, thus verifying Conjecture 1 for $d \leqslant 3$. Hsiang and Sczarba [9] and Rohlin [14] used cyclic branched covers to produce lower bounds on g_X. For example we have:

$$g_{\mathbf{CP}^2}(d.h) \geqslant \frac{1}{4} d^2 - 1$$

when d is even. For large d this is roughly half the bound asked for in Conjecture 1. More recently the results and techniques of Yang-Mills theory have been applied to these problems by Kuga [12], Fintushel and Stern [3] and Furuta [5]. These authors have shown that certain homology classes cannot be represented by embedded spheres, exploiting the fact that the result of collapsing a 2-sphere Σ in a 4-manifold to a point is an orbifold if $\Sigma . \Sigma \neq 0$ and a smooth manifold if $\Sigma . \Sigma = \pm 1$. Unlike those of Rohlin, Hsiang and Sczarba these techniques make essential use of the assumption that the surfaces are smoothly embedded. In this regard Rudolph [15] has shown that Conjecture 1 is false if one replaces " smoothly embedded " with " topologically locally flat " surfaces.

In this note we wish to point out a general result which gives further evidence for the truth of Conjectures 1 and 2. Recall that to an algebraic curve C in a projective surface S one associates a holomorphic line bundle L_C and that L_C is said to be ample if the sections of some large positive power $L_C^{\otimes n}$ define a projective embedding of S. (If we can take $n = 1$ then C is a hyperplane section of S.)

Theorem. — Let S *be a simply connected complex projective surface and* C *a smooth complex curve in* S *with* L_C *ample. Then it is impossible to perform smooth surgery within* S *on any loop* δ *in* C *(not homologous to zero).*

This result says that, with the stated hypotheses, the obvious approach for seeking a counterexample to Conjecture 2 will always fail. It says nothing about the possibility of finding a counterexample by some other procedure.

2. The proof

The theorem follows quite easily from a general result proved in [2]. While the arguments used are probably routine for experts, it seems worth writing down the proof in view of the interest of the Thom conjecture and its generalisations. The result we need is this:

Proposition (see [2]). — *Let* Z *be a simply connected complex projective surface. If* Z *can be decomposed as a smooth connected sum* $Z = X \# (S^2 \times S^2)$, *then the 4-manifold* X *has a negative definite intersection form.*

This is proved using the invariants of smooth 4-manifolds provided by Yang-Mills theory; we shall apply it to our problem by considering branched covers so our approach here is in some repects an amalgam of those mentioned above.

Suppose that S and C are as in the statement of the theorem, and that there is, on the contrary, a loop δ in C which can be surgered. We will derive a contradiction to the proposition above. We are given that $C = \Sigma^{\#(\gamma)}$ for some arc γ in S. Now the isotopy class of the surface $\Sigma^{\#(\gamma)}$ in S depends only upon the isotopy class of γ. More precisely, if arcs γ_0, γ_1 can be joined by an isotopy γ_s with $\gamma_s(0)$ and $\gamma_s(1)$ in Σ for all s and $\gamma_s(t)$ not in Σ for $t \neq 0, 1$, then $\Sigma^{\#(\gamma_0)}, \Sigma^{\#(\gamma_1)}$ are isotopic. We have in particular the isotopy class of a " trivial " arc, wholly contained in a small ball in S, and we denote the corresponding surface by $\Sigma^{\#}$.

Lemma. — If surgery can be performed on a loop in C *then* C *is isotopic to* $\Sigma^{\#}$ *for some embedded surface* Σ *in* S.

To prove this lemma we consider the fundamental group of the complement $S \backslash \Sigma$, where $C = \Sigma^{\#(\gamma)}$. By an application of Van Kampen's theorem we see that $\pi_1(S \backslash \Sigma)$

is isomorphic to $\pi_1(S\setminus C)$. The latter group is however well understood. The homotopy version of the Lefschetz hyperplane theorem ([8], p. 156) asserts that S is obtained from C by the addition of 2-, 3- and 4-cells. It follows that any loop in $S\setminus C$ can be deformed in the complement onto the boundary ∂N of a tubular neighbourhood N of C. Hence $\pi_1(S\setminus C)$ is a quotient of $\pi_1(\partial N)$, which is a central extension of $\pi_1(C)$ by the class of a small circle ν linking C. On the other hand, since S is simply connected, any class in $\pi_1(S\setminus C)$ can be expressed as a product of elements of the form $\alpha\nu\alpha^{-1}$, for α in $\pi_1(S\setminus C)$. But ν is in the centre of $\pi_1(\partial N)$, therefore also in the centre of $\pi_1(S\setminus C)$. So we deduce that $\pi_1(S\setminus C)$ is generated by ν. Therefore the fundamental group of $S\setminus\Sigma$ is also generated by a loop linking Σ and it follows that the arc γ in S can be deformed, with end points fixed, into a small neighbourhood of Σ. If the deformation is generic it will be an isotopy. Finally we deform γ into the trivial arc by sliding the end points on Σ. This shows that C is isotopic to $\Sigma^{\#}$ as stated.

We now introduce branched covers. In general if Ξ is a surface in a 4-manifold X such that $[\Xi]$ is divisible by 2 in $H_2(X; Z)$, so we have $[\Xi] = 2\alpha$ say, the corresponding double cover $\widetilde{X}_{\Xi}$ of X branched over Ξ is defined as follows. Let L be the complex line bundle with $c_1(L)$ Poincaré dual to α, so that $L^{\otimes 2}$ has a section s cutting out Ξ. Then $\widetilde{S}_{\Xi}$ is the subspace of the total space of L:

$$\widetilde{S}_{\Xi} = \{\, l \in L \mid l^{\otimes 2} \in \Gamma_s \}$$

where $\Gamma_s \subset L^{\otimes 2}$ is the "graph" of s. Strictly, the double cover depends on the choice of α, but for simply connected 4-manifolds X this is uniquely determined by Ξ.

Lemma. — If Σ is an embedded surface in a 4-manifold X and $\Sigma^{\#}$ is obtained from Σ by adjoining a tube around a trivial arc as above, then the double cover $\widetilde{X}_{\Sigma^{\#}}$ is diffeomorphic to the connected sum of $\widetilde{X}_{\Sigma}$ and $S^2 \times S^2$.

Here we are assuming that the same homology class α is used to construct the cover in each case.

To prove this lemma we choose a small ball B in X containing the arc used to construct $\Sigma^{\#}$. The boundary of this ball is a 3-sphere S which, we can suppose, meets both Σ and $\Sigma^{\#}$ in a circle S^1. The circle is unknotted in S^3: it is the boundary of the portion of Σ in B, which is the standard disc. Now the double cover of S^3 branched over an unknotted circle is again a 3-sphere. In terms of complex co-ordinates (z, w) we have the explicit description of the covering map:

$$(z, w) \to (| z |^4 + | w |^2)^{-1} (z^2, w).$$

So the preimages of this sphere in $\widetilde{X}_{\Sigma}$ and $\widetilde{X}_{\Sigma^{\#}}$ are again 3-spheres. It is easy to see that the double cover of the ball over the disc $\Sigma \cap B$ is likewise a ball in $\widetilde{X}_{\Sigma}$, so $\widetilde{X}_{\Sigma^{\#}}$ is the connected sum of $\widetilde{X}_{\Sigma}$ with another manifold Y, where $Y\setminus\{ pt \}$ is the double cover of B branched over $B \cap \Sigma^{\#}$. We now glue another standard 4-ball and disc pair to $(B, B \cap \Sigma^{\#})$

to see that Y is the double cover of the 4-sphere branched over the standard image of $S^1 \times S^1$. It is then easy to recognise Y as $S^2 \times S^2$: we can write down a covering map $f: S^2 \times S^2 \to S^4$ explicitly in the form:

$$f[(x_1, y_1, z_1), (x_2, y_2, z_2)] = [(x_1^2 - 1)(x_2^2 - 1)]^{-1/2}(x_1 x_2, y_1, z_1, y_2, z_2),$$

where $x_i^2 + y_i^2 + z_i^2 = 1/2$. (More generally the double cover of S^{p+q+2} branched over the standard image of $S^p \times S^q$ can be identified with $S^{p+1} \times S^{q+1}$.)

These two lemmas provide the main ingredients in our proof. Suppose for the moment that the homology class of the complex curve C in S is even. Then the double cover $\widetilde{S}_C$ of S branched over C is again a projective surface ([1], p. 128) and it follows easily from the Lefschetz hyperplane theorem, as used above, that $\widetilde{S}_C$ is simply connected. If C can be surgered then it has the form $\Sigma^\#$ by the first lemma, and this gives a connected sum decomposition of $\widetilde{S}_C$ into $\widetilde{S}_\Sigma$ and $S^2 \times S^2$ by the second lemma. If we also assume that the rank, $b^+(\widetilde{S}_C)$, of a maximal positive subspace for the intersection form of $\widetilde{S}_C$ is strictly greater than 1 we obtain the desired contradiction to the Proposition (since $b^+(\widetilde{S}_\Sigma) = b^+(\widetilde{S}_C) - 1 > 0$).

It remains only to extend the argument to remove the two extra assumptions —that [C] is even and $b^+(\widetilde{S}_C) > 1$. Let σ be the holomorphic section of L_C cutting out C, so for $n > 0$, σ^{2n} is a section of $L_C^{\otimes 2n}$ vanishing with multiplicity $2n$ on C. For large n the linear system $|L_C^{\otimes 2n}|$ has no base points and it follows from Bertini's Theorem ([8], p. 137) that we can choose another section τ of $L_C^{\otimes 2n}$ such that for all sufficiently small non-zero complex numbers ε, $\sigma^{2n} + \varepsilon\tau$ vanishes transversely on a complex curve $C^*(\varepsilon)$. The homology class of $C^*(\varepsilon)$ is $2n[C]$ and as ε tends to zero $C^*(\varepsilon)$ degenerates into C counted with multiplicity $2n$. For small ε, $C^*(\varepsilon)$ is contained in a tubular neighbourhood of C and by choosing a (non-holomorphic) projection map in the tubular neighbourhood we express $C^*(\varepsilon)$ as a $2n$-fold cyclic cover

$$\pi : C^*(\varepsilon) \to C,$$

branched over the zeros of τ on C. Suppose δ is a loop in C which admits a surgering disc $i: D^2 \to S$. The same is true for any loop isotopic to δ on C. Now by moving the loop across a suitable collection of branch points of π we can suppose that the restriction of π to δ is trivial; that is, $\pi^{-1}(\delta)$ consists of $2n$ disjoint "parallel" loops in $C^*(\varepsilon)$. A small stretching of i then gives a surgering disc for one of these loops. Moreover the lifted loops are non-trivial in $H_1(C^*(\varepsilon))$ if δ is in $H_1(C)$. In short, if C can be surgered then so can $C^*(\varepsilon)$.

The homology class of $C^*(\varepsilon)$ is even, by construction, so the proof of the theorem is now completed by observing that $b^+(\widetilde{S}_{C^*(\varepsilon)})$ tends to infinity with n. To see this, one can either do a calculation with characteristic classes or use the Hodge Index formula:

$$b^+(\widetilde{S}_{C^*(\varepsilon)}) = 1 + 2p_g(\widetilde{S}_{C^*(\varepsilon)}).$$

A meromorphic 2-form on S with only a simple pole along $C^*(\varepsilon)$ lifts to a holomorphic form on the double cover, so

$$p_g(\widetilde{S}_{C^*(\varepsilon)}) \geqslant \dim H^0(K_S \otimes L_C^{2n})$$

and the latter tends to infinity with n. Thus, by our previous argument $C^*(\varepsilon)$ cannot be surgered for large n, and we deduce that C cannot be surgered.

3. Further remarks

In the last few year strong connections between the geometry and differential topology of complex surfaces have been established, and this trend provides a general reason for hoping that Conjectures like Thom's may be true. The formula for the genus of a complex curve C in a surface S is:

$$\text{genus}(C) = \frac{1}{2}(C.C + K_S.C) + 1.$$

The right hand side represents a polynomial function of the homology class [C] of C. If Conjecture 2 is true the differential topological function g_S is given, at least on the classes represented by complex curves, by this polynomial expression and this would lead to a differential topological interpretation of the canonical class $[K_S]$ of the surface. Now the invariants introduced in [2] are also polynomial functions on the homology of 4-manifolds: in the case of algebraic surfaces it is quite likely that the canonical class can be extracted from them, and this has been proved for many surfaces by Friedman and Morgan [7]. In that case we would get a differential topological interpretation of the canonical class through Yang-Mills theory. It is thus possible that if Conjecture 2 is true it will provide a means for understanding the differential topological significance of the Yang-Mills invariants more directly, via the functions g_X.

The prospects of proving the conjectures by a refinement of the approach in this paper do not seem to be very good. The difficulty is comparable to that with the "11/8ths. conjecture"—that for a simply connected, spin, 4-manifold X one always has

$$b_2(X) \geqslant (11/8)\,|\,\text{sign}(X)|.$$

In that case too one knows that one cannot obtain a counterexample by performing surgery on an algebraic surface [6], but the conjecture has only been proved so far for small values of the signature.

Returning to the proof of this paper: the basic scheme is to use information about the Yang-Mills invariants of 4-manifolds of the form $\widetilde{X}_{\Sigma^*(\varepsilon)}$, where $\Sigma^*(\varepsilon)$ is a surface obtained by smoothing the "divisor" $2n\Sigma$ as above. It would be interesting to know if one could say more about the invariants in this situation. A possible approach would be to use the natural degeneration of the branched cover, as ε tends to zero, into $2n$ copies of X glued along Σ. In other contexts one can use degeneration arguments to give formulae for the Yang-Mills invariants involving the "instanton homology" groups for 3-manifolds defined by Floer [4].

REFERENCES

[1] BARTH W., PETERS C. and VAN DE VEN A., *Compact Complex surfaces*, Berlin-Heidelberg, Springer, 1984.

[2] DONALDSON S. K., Polynomial Invariants for smooth 4-manifolds, to appear in *Topology*.

[3] FINTUSHEL R. and STERN R., Pseudo-free Orbifolds, *Annals of Math.*, **2,** 122 (1985), 523-529.

[4] FLOER A., *An instanton invariant for 3-manifolds*, Courant Institute Preprint, 1987.

[5] FURUTA M., *On self-dual pseudo-connections on some orbiofolds*, Tokyo University Preprint, 1985.

[6] FRIEDMAN R. and MORGAN J. W., Algebraic surfaces and 4-manifolds: some conjectures and speculations, *Bull. A.M.S.*, **18** (1) (1988), 1-19.

[7] FRIEDMAN R. and MORGAN J. W., *Complex versus differentiable classification of algebraic surfaces*, Columbia Univ. Preprint, 1988.

[8] GRIFFITHS P. A. and HARRIS J., *Principles of algebraic geometry*, New York, John Wiley, 1978.

[9] HSIANG W. C. and SCZARBA R., On embedding surfaces in 4-manifolds, *Proc. Symp. Pure Math.*, **22** (1970), 97-103.

[10] KERVAIRE M. and MILNOR J. W., On 2-spheres in 4-manifolds, *Proc. Nat. Acad. Sci. U.S.A.*, **47** (1961), 1651-1657.

[11] KIRBY R., Problems in low-dimensional topology, *Proc. Symp. Pure. Math.*, **32** (1970), 273-322.

[12] KUGA K., Representing homology classes in $S^2 \times S^2$, *Topology*, **23** (1984), 133-137.

[13] MANDELBAUM R., Four-dimensional topology: an introduction, *Bull. A.M.S.*, **2** (1) (1980), 1-160.

[14] ROHLIN V., Two-dimensional submanifolds of four-dimensional manifolds, *Functional Analysis Appl.*, **6** (1972), 136-138.

[15] RUDOLPH L., Some topologically locally flat surfaces in the complex projective plane, *Comment. Math. Helvetici.*, **59** (1984), 592-599.

The Mathematical Institute
24-29 St. Giles
Oxford OX13LB, England

Manuscrit reçu le 5 octobre 1988.