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ON NONLINEAR SCHRÖDINGER EQUATIONS

by Jean BOURGAIN

1. Introduction.

Nonlinear Schrödinger equations of the form

$$iu_t + \Delta u + \partial_{\overline{u}} F(u, \overline{u}, x) = 0 \tag{0.1}$$

(or involving derivative nonlinearities) are well known models (or simplified models) for a number of physical issues such as electron plasma waves, laser propagation etc. But it is also a basic model mathematically speaking in the understanding of various aspects of Hamiltonian PDE's both on bounded and unbounded spatial domains. Such investigations have been pursued by many distinct groups of researchers, often from different perspectives and using different methods. Typical issues considered involve

(1) The Cauchy problem, local and global, taking into account the nature of the nonlinearity and the dimension.
(2) Behaviors of solutions blowing up in finite time, such as blowup speed and stability of the profile.
(3) Problems related to scattering.
(4) Existence of measures on phase space that are invariant under the dynamics.
(5) Existence and stability of periodic, quasi-periodic and almost periodic solutions in time (KAM theory for PDE's).

Our interest here is restricted to results that are mathematically completely rigorous. Despite the intensive research on this particular type of equations, it is safe to say that most of the natural questions are very far from fully resolved. This short paper is of course not the place for a survey. My only intention here is to report on some of my own research over the past years on a few of the items listed above. More precisely, we will discuss

(i) Problems and new results on global wellposedness on $\mathbb{R}^d$ for defocusing NLS with critical nonlinearity.

(ii) Construction of an invariant dynamics for (properly normalized) Gibbs measures in dimensions 1, 2, 3.

(iii) Construction and persistency of invariant tori.

In items (ii), (iii), we consider a bounded spatial domain (for instance periodic boundary conditions).

I. Global solutions for critical NLS

Consider the NLS on $\mathbb{R}^d$

$$iu_t + \Delta u \pm u|u|^{p-2} = iu_t + \frac{\partial H}{\partial \overline{u}} = 0 \tag{1.1}$$

where

$$\mathrm{H}(\phi) = \frac{1}{2}\int |\nabla\phi|^2 \mp \frac{1}{p}\int |\phi|^p \tag{1.2}$$

(the sign + (resp −) in (1.1) correspond to the focusing (resp. defocusing case). We will be mainly interested here in the defocusing case, in which situation the conservation of the Hamiltonian (1.2) gives an a priori bound on the H^1-norm

$$\|\phi\|_{\mathrm{H}^1} = \left(\int |\nabla\phi|^2\right)^{1/2}. \tag{1.3}$$

For equations of the form (1.1), recall that also the L^2-norm is conserved

$$\|\phi\|_2 = \left(\int |\phi|^2\right)^{1/2}. \tag{1.4}$$

Equation (1.1) is invariant under an appropriate scaling

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-2}}\, u(\lambda x, \lambda^2 t), \tag{1.5}$$

Defining the exponent s_0 by the equation

$$p - 2 = \frac{4}{d - 2s_0} \tag{1.6}$$

the (homogeneous) Sobolev space H^{s_0} is thus the corresponding scale invariant space. Considering the IVP

$$\begin{cases} iu_t + \Delta u \pm u|u|^{p-2} = 0 \\ u(0) = \phi \in \mathrm{H}^s \end{cases} \tag{1.7}$$

the case $s > s_0$ ($s = s_0$) is called subcritical (resp. critical).

There is the following local wellposedness statement

Theorem 1.8. *Assume in (1.7) that $s \geqslant 0$, $s \geqslant s_0$ and also $p - 2 > [s]$ if $p \notin 2\mathbb{Z}$. Then the Cauchy problem (1.7) is wellposed on a nontrivial time interval $[0, \mathrm{T}^*[$ and in particular*

$$u \in \mathcal{C}_{\mathrm{H}^s}([0,\mathrm{T}]), \mathrm{T} < \mathrm{T}^*. \tag{1.9}$$

In the subcritical case $s > s_0$, $\mathrm{T}^* > \mathrm{T}^*(s.\|\phi\|_{\mathrm{H}^s})$ and the flowmap is Lipschitz on a neighborhood of ϕ.

Remark. In the critical case, maximal existence time depends on φ, not only $\|\varphi\|_{H^s}$.

The next result gives a few cases in which the local solution given by Theorem 1 extends to a global one.

Theorem 1.10. *Under the assumptions of Theorem 1, there is moreover global wellposedness under each of the following assumptions.*

(i) $p < 2 + \frac{4}{d}$ (the problem is L^2-subcritical and use of the L^2-conservation (1.4));

(ii) $p \geqslant 2 + \frac{4}{d}$ and small H^s-data;

(iii) Defocusing NLS, $\varphi \in H^1$ and $p < 6$ for $d = 3$, $p < 4$ for $d = 4$, etc.

(the problem is H^1-subcritical and use of the Hamiltonian conservation).

Also, additional smoothness of the data is preserved under the flow (provided compatible with smoothness of nonlinearity).

For $p \geqslant 2 + \frac{4}{d}$, smooth solutions of focusing NLS may blowup in finite time.

Both Theorems 1.8, 1.10 are by now classical (cf. the work of Cazenave-Weissler [Ca-We], Ginibre-Velo [G-V] for instance).

In dimension $d \geqslant 3$, there is a H^1-critical nonlinearity. For $d = 3$, we get thus the equation

$$iu_t + \Delta u - u|u|^4 = 0. \tag{1.11}$$

For $d = 4$

$$iu_t + \Delta u - u|u|^2 = 0 \tag{1.12}$$

(considering the defocusing case). Thus H^1 is the scale-invariant space and the H^1-norm is a priori bounded. There is local wellposedness (both in the focusing and defocusing case) but it is an open problem whether the solution extends to a global one. The issue here is a possible concentration (in finite time) of the H^1-norm on small balls. In fact, the purely classical question of global existence of smooth solutions for (1.11) or (1.12) is unresolved. In a recent work, we have settled this for the radial case, proving the following (cf. [Bo1]).

Theorem 1.13. *Consider the 3D IVP*

$$\begin{cases} iu_t + \Delta u - u|u|^4 = 0 \\ u(0) = \phi \in H^s, s \geqslant 1 \end{cases} \tag{1.14}$$

and in 4D

$$\begin{cases} iu_t + \Delta u - u|u|^2 = 0 \\ u(0) = \phi \in H^s, s \geqslant 1. \end{cases} \tag{1.15}$$

Assume that ϕ is a radial data. Then both (1.14), (1.15) are globally wellposed,

$$u \in C_{H^s}(\mathbb{R}) \tag{1.16}$$

and there is moreover scattering in H^s-space, i.e.

$$\|u(t) - e^{it\Delta}\Omega_+\varphi\|_{H^s} \longrightarrow 0 \textit{ for } t \to \infty \tag{1.17}$$

where, in case (1.14) say, the wave map is given by

$$\Omega_+\varphi = \varphi + i\int_0^\infty e^{i(t-\tau)\Delta}(u|u|^4)(\tau)\,d\tau. \tag{1.18}$$

Remark. For the wave equation

$$y_{tt} - \Delta y + y^5 = 0 \qquad \text{(in3D)} \tag{1.19}$$

the analogue of Theorem 1.13 was established by M. Struwe for radial data [Str] and by M. Grillakis in general [Gr]. See also [Str-Sh]. On an heuristic level, the main difference between (1.14) and (1.19) is absence of finite speed propagation.

The general problem for (1.14), (1.15) seems still unresolved.

As in the wave equation case, a proof may be given that centers around the following complimentary ideas.

(i) In case of a finite existence time $T^* < \infty$, concentration effects for various norms in space and space-time need to occur. This fact results as a byproduct of the analysis local in time of the IVP (by means of harmonic analysis techniques such as Strichartz' inequality).

(ii) A Morawetz' type inequality that excludes certain concentrations.

In the wave equation case (1.19), those two facts do indeed produce directly the required result, cf. [Str-Sh]. The problem for NLS is that the Morawetz-Strauss inequality

$$\int_0^T\int \frac{|u|^6(x,t)}{|x|}\,dxdt \leqslant C \sup_{0<t<T} \|u(t)\|^2_{H^{1/2}} \leqslant C(\|\phi\|_2^2 + H(\phi)) \tag{1.20}$$

is too weak to prohibit the H^1-concentration phenomena (which is natural since (1.20) involves only $H^{1/2}$-norm). Localizing (1.20), one may show the following variant for any time interval I

$$\int_I\int_{|x|<|I|^{1/2}} \frac{|u|^6(x,t)}{|x|}\,dxdt \leqslant C|I|^{1/2}H(\phi) \tag{1.21}$$

($|I|$ denoting the size of I).

Theorem 1.13 is proven by "induction" on the size of $H(\phi)$; the result was known for small data. The main idea is, assuming concentration occurs, to decouple the solution u as

$$u = v + w \tag{1.22}$$

where v behaves as a small data solution and w satisfies the difference equation

$$iw_t + \Delta w - (w+v)|w+v|^4 + v|v|^4 = 0 \tag{1.23}$$

and has smaller (time dependent) Hamiltonian

$$\mathrm{H}(w) < \mathrm{H}(\phi) - \rho. \tag{1.24}$$

for some $\rho > 0$.

In order to invoke the "induction hypothesis", the point is that in the relevant time region $[b, \infty[$, the solution v has dispersed sufficiently to insure that w behaves, up to small perturbation, like the solution of

$$\begin{cases} i\mathrm{W}_t + \Delta \mathrm{W} - \mathrm{W}|\mathrm{W}|^4 = 0 \\ \mathrm{W}(b) = w(b) \end{cases} \tag{1.25}$$

to which the induction hypothesis may be applied by (1.24).

In this argument (1.21) is the main ingredient. In order to control v, we also use the "pseudo-conformal conservation law"

$$\|(x+2it\nabla)v(t)\|_2^2 + \frac{4}{3}t^2\|v(t)\|_6^6 = \|xv(0)\|_2^2 - \frac{16}{3}\int_o^t s\int_{\mathbb{R}^3}|(v(s,x)|^6 dxds \tag{1.26}$$

implying in particular an estimate

$$\|v(t)\|_6^6 < \mathrm{C}\,\frac{\|xv(0)\|_2^2}{t^2}. \tag{1.27}$$

The scheme explained above may have wider range of applicability.

II. Invariant measures

In the understanding of long time dynamics in bounded spatial domains, existence of invariant measures on various phase spaces is a natural thing to look for. From a classical point of view, the most desirable situation could be to construct such a measure on smooth functions. Except for the integrable model of the 1D cubic NLS $iu_t + u_{xx} \pm u|u|^2 = 0$, whether this may be done or not is unknown.

Essentially speaking, in the other cases, the only known examples of invariant measures are either living on spaces of rough data or fields (those produced from the Gibbs-measure) or live on finite dimensional or infinite dimensional tori with very strong compactness properties (obtained from KAM tori). Results along this line will be described in this and next section.

In finite dimensional phase space, it follows from Liouville's theorem that the Lebesgue measure on $\mathbb{R}^{2n}$ is invariant under the Hamiltonian flow

$$\dot{p}_i = -\frac{\partial \mathrm{H}}{\partial q_i} \qquad \dot{q}_i = \frac{\partial \mathrm{H}}{\partial p_i} \qquad (i = 1, \dots, n) \tag{2.1}$$

with Hamiltonian $H(p_1, ..., p_n, q_1, ..., q_n)$. Hence so is the Gibbs measure

$$d\nu = e^{-\beta H} \Pi_{i=1}^{n} dp_i dq_i. \tag{2.2}$$

Consider the case of NLS with periodic boundary conditions in dimension d

$$iu_t = \frac{\partial H}{\partial \bar{u}}. \tag{2.3}$$

Writing a Fourier expansion

$$u(t) = \phi = \sum_{n \in \mathbb{Z}^d} \widehat{\phi}(n) e^{in.x} \tag{2.4}$$

we choose

$$p_n = \operatorname{Re} \widehat{\phi}(n), \; q_n = \operatorname{Im} \widehat{\phi}(n) \tag{2.5}$$

as canonical coordinates for our infinite dimensional phase space. This choice (rather then (Re u, Im u)) has the advantage that it is easy to pass to finite dimensional models by projecting on finitely many Fourier modes. The formula (2.2) may be written as

$$d\nu = e^{\mp\frac{1}{p}\int|\phi|^p} (e^{-\frac{1}{2}\int|\nabla\phi|^2} \Pi d^2\phi) = e^{\mp\frac{1}{p}\int|\phi|^p} d\mu \tag{2.6}$$

which has the tentative interpretation of a weighted Wiener measure. This measure is formally invariant under the flow of (2.3). In order to develop this idea rigorously, the two problems to resolve are the normalization of (2.6) and next the construction of a welldefined dynamics on the support of that measure. This program was initiated in the papers [L-R-S, 1, 2]. The problem is strongly dimensional dependent and also reflects certain aspects of the Cauchy problem, especially in the focusing case (such as blowup behaviour).

In dimension $d = 1$, Wiener measure lives on functions of class $H^{\frac{1}{2}-}(\Pi)$ and hence $\int |\phi|^p dx$ is almost surely finite, for any p. Thus in the defocusing case, with weight $e^{-\frac{1}{p}\int|\phi|^p}$ in (2.6), the formula produces trivially a welldefined measure that is absolutely continuous wrt Wiener measure $d\mu$. One of the main results in [L-R-S, 1] is that in the focusing case, (2.6) may be normalized for $p \leqslant 6$ by restricting the L^2-norm. Thus $e^{\frac{1}{p}\int|\phi|^p}$ is replaced by

$$e^{\frac{1}{p}\int|\phi|^p} \chi_{[\|\phi\|_2 < B]} \tag{2.6'}$$

where B is an arbitrary cutoff for $p < 6$ and taken sufficiently small for $p = 6$. Observe that $p = 6 = 2 + \frac{4}{d}$ is critical in the sense of possible blowup behaviour for large data in the classical theory. It turns out indeed that the restrictions in the previous statement are necessary.

In [B2], the author established a unique dynamics. This amounts to proving global wellposedness of the Cauchy problem

$$\begin{cases} iu_t + u_{xx} \pm u|u|^{p-2} = 0 \\ u(0) = \phi \end{cases} \tag{2.7}$$

for almost all data ϕ in the support of the Wiener measure. This task in fact reduces to verifying the result local in time, since the measure invariance may be exploited similarly to a conservation law. Another (essentially equivalent) way of proceeding is to consider finite dimensional models

$$\begin{cases} iu_t^N + u_{xx}^N \pm P_N(u^N|u^N|^{p-2}) = 0 \\ u^N(0) = P_N\phi \end{cases} \tag{2.8}$$

by restriction of the Fourier modes, thus

$$\phi^N = P_N\phi = \sum_{|n|\leqslant N} \widehat{\phi}(n)\, e^{in.x}. \tag{2.9}$$

Since (2.8) is an ODE for each N which solution remains bounded

$$\|u^N(t)\|_2 = \|u^N(0)\|_2 \tag{2.10}$$

the existence of the invariant dynamics is clear in this case. Passing to the limit mainly amounts to show that

$$\lim_{N\to\infty} u^N \tag{2.11}$$

is uniquely defined, almost surely. This process is basically the PDE counterpart of the Gibbs-measure construction.

For $d = 2$ and $d = 3$, the expression $\int |\phi|^p$ diverges. The problem is resolved by the wellknown process of Wick ordering, assuming $|\phi|^p$ a polynomial in $\phi, \bar{\phi}$, hence ϕ an even integer. Thus $\int |\phi|^p$ is replaced by $\int :|\phi|^p:$ which is almost surely finite. Moreover, in the 2D defocusing case, the formula

$$e^{-\int :|\phi|^p:} d\mu \tag{2.12}$$

still defines a weighted Wiener measure with density in $\cap_{p<\infty} L^p(\mu)$.

Construction of the dynamics for $p = 4$ was performed in [B3].

Considering the 2D-focusing equation

$$iu_t + \Delta u + u|u|^2 = 0 \tag{2.13}$$

a natural construction would consist in considering measures

$$\chi_{[\int :|\phi|^2: < B]}(e^{\int :|\phi|^4:})\, d\mu \tag{2.14}$$

obtained by restriction of the Wick ordered L^2-norm,

$$\int :|\phi_\omega|^2 := \sum_n \frac{|g_n(\omega)|^2 - 1}{|n|^2 + \rho} \tag{2.15}$$

generating Wiener measure by the random Fourier series

$$\phi_\omega = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{|n|^2+\rho}}\, e^{inx}. \tag{2.16}$$

Here $\{g_n(\omega)\}$ refers to a system of standard, independent complex Gaussian random variables. Observe indeed that $\int|\phi|^2 = \infty$ a.s. if $d > 1$ while $\int :|\phi|^2: < \infty$ a.s. for $d = 2, 3$. Unfortunately, this construction barely fails for quartic nonlinearity (cf. [B-S]). The idea of restricted Wick-ordered L^2-norm was pointed out by A. Jaffe ([Ja]) who used the construction in the real case in normalizing Gibbs measure for cubic nonlinearity for instance, thus

$$\chi_{[\int :\phi^2: < B]}\, e^{\int :\phi^3:} \tag{2.17}$$

(of interest in wave equation context).

Coming back to NLS, one may replace (2.19) by an equation with convolution nonlinearity of the form (Hartree-equation)

$$iu_t + \Delta u + (|u|^2 * V)\, u = 0. \tag{2.18}$$

Assuming in 2D that the potential V satisfies any condition of the form

$$|\widehat{V}(u)| < |n|^{-\varepsilon} \text{ for } |n| \to \infty. \tag{2.19}$$

Wick ordering of the nonlinearity and restriction of the Wick ordered L^2-norm as in (2.4) produces then an invariant normalized measure for (2.18).

Similar constructions were performed in [B4] for the 3D case.

Coming back to the 2D normalization problem for focusing NLS, the preceding discussion leads naturally to the question how to Wick order (or develop a substitute for Wick ordering) in the case of powers $|\phi|^p$ where p is not an even integer (or $\phi^p, p \notin \mathbb{Z}_+$ in the real case). Those attempts seem only to produce the free measure on phase space however ([1]), which is physically not very interesting.

The study of invariant measures for Hamiltonian PDE's have been pursued by many authors. We mention for instance the works of Zhidkov, McKean, Vaninski, Malliavin of particular relevance in the preceding discussion.

III. Invariant KAM tori

Another direction that has been investigated is the persistency of invariant tori for Hamiltonian perturbations of linear or integrable PDE's. The basic model is that of a perturbed linear NLS of the form

$$iu_t + Au + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0 \tag{3.1}$$

([1]) T. Spencer (private communication) where A is self-adjoint second order elliptic (equivalent to the Laplacian) and containing parameters. Examples are

$$A = \text{Fourier multiplier with } A_n \sim |n|^2 \text{ for } n \to \infty \tag{3.2}$$

$$A = -\Delta + V \text{ with V a potential.} \tag{3.3}$$

Given a nonlinear equation, parameters may often be extracted from the nonlinearity using partial Birkhoff normal forms and amplitude-frequency modulation. This has been pursued for instance in [K-P] for 1D NLS of the form

$$iu_t + \Delta u + mu + u|u|^2 + \text{ (higher order) } = 0 \tag{3.4}$$

and in [B5] for the 2D-equation.

The basic difference with the corresponding topic in the theory of classical smooth dynamical systems are new resonance problems due to an infinite dimensional phase space. These difficulties are particularly severe in space dimension $D \geqslant 2$.

Several methods have been developed. In [K], the classical KAM and Melnikov scheme based on eliminating the perturbation by consecutive canonical transformations has been reworked in the context of finite dimensional tori in infinite dimensional phase space, provided there are no multiplicities or near-multiplicities in the normal frequencies. In the context of (3.1) this restriction limits the application to 1D-models with Dirichlet of Neumann bc's. Recall that the periodic spectrum (λ_n) of $-\frac{d^2}{dx^2} + V$ appears typically in pairs $\lambda_{2n-1}, \lambda_{2n}$, where roughly

$$|\lambda_{2n} - \lambda_{2n-1}| \sim |\widehat{V}(n)| \to 0 \tag{3.5}$$

rapidly for $n \to \infty$.

In higher dimension, unbounded multiplicities result from a possible large number of lattice points on a circle (or sphere) of radius R when $R \to \infty$.

One of the most interesting developments in this research is the emerging of new techniques to overcome this problem. The pioneering paper here is that of W. Craig and E. Wayne [C-Wa] in which the simplest case (presenting the multiplicity issue) of time periodic solutions of the 1D wave equations with periodic bc of the form

$$y_{tt} - y_{xx} + \rho y + y^3 + \text{ higher order } (x, y) = 0 \tag{3.6}$$

is investigated. The method used is a priori independent of Hamiltonian structure. It is based on a Liapounov-Schmidt decomposition scheme and a treatment of small divisor problems by multiscale analysis very reminiscent of the arguments used in localization theory for lattice Schrödinger operators. In particular in the context of a quasi-periodic potential, such as the almost Mathieu operator

$$\varepsilon\Delta + \cos(n\lambda + \sigma) \tag{3.7}$$

(cf. [F-S-Wi]).

This technique has lead to considerable progress. Time periodic solutions (i.e. 1 dim tori) for (3.1) may be produced in any dimension and the full theory is available for $D = 1, 2$ with periodic bc (cf. [B5]). In fact, the first new application relates to the classical Melnikov theorem in finite dimensional phase space which remains valid in the presence of multiple normal frequencies.

PDE's also provide a natural setting for problems of infinite-dimensional invariant tori. There has been research on this topic by various authors. Some of the models considered in mathematical physics only present finite interactions, cf. [F-S-Wa]. In other (more recent) investigations, frequencies are chosen sufficiently lacunary to avoid diophantine problems. My particular interest here goes to KAM tori constructed on the full set of frequencies (only of quadratic growth) for an NLS of the form

$$iu_t + u_{xx} = V(x)u + \varepsilon \frac{\partial H}{\partial \overline{u}} = 0. \tag{3.8}$$

Here the nonlinearity is fixed and V is taken to be a "typical" potential. Construction of such KAM tori was performed independently by J. Poschel and the author using different methods (see [P], [B6]). In both constructions, there is a very rapidly (hardly explicit) decay of the amplitudes, hence leading to strong compactness properties.

Imposing such fast decay appears here as trade-off for the severe small divisor issues. Refining the techniques to reach a more realistic decay condition is certainly a most interesting project. One of the applications of the present work we believe is to produce new examples of NLS in which (inf. dimensional) invariant tori fill a part of phase space of positive Gibbs measure, hence failing ergodicity properties. Roughly, such examples may be produced considering an equation

$$iu_t + Au + V * \left[\left(\frac{\partial H}{\partial \overline{u}}\right)(u * V)\right] = 0 \tag{3.9}$$

with A a typical multiplier as in (3.2) and where $\widehat{V}(n) \neq 0$ decays very rapidly.

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