

SPECTRAL CURVES, OPERS AND INTEGRABLE SYSTEMS

by DAVID BEN-ZVI *and* EDWARD FRENKEL *

ABSTRACT

We establish a general link between integrable systems in algebraic geometry (expressed as Jacobian flows on spectral curves) and soliton equations (expressed as evolution equations on flat connections). Our main result is a natural isomorphism between a moduli space of spectral data and a moduli space of differential data, each equipped with an infinite collection of commuting flows. The spectral data are principal G -bundles on an algebraic curve, equipped with an abelian reduction near one point. The flows on the spectral side come from the action of a Heisenberg subgroup of the loop group. The differential data are flat connections known as opers. The flows on the differential side come from a generalized Drinfeld-Sokolov hierarchy. Our isomorphism between the two sides provides a geometric description of the entire phase space of the hierarchy. It extends the Krichever construction of special algebro-geometric solutions of the n -th KdV hierarchy, corresponding to $G = \mathrm{SL}_n$.

An interesting feature is the appearance of formal spectral curves, replacing the projective spectral curves of the classical approach. The geometry of these (usually singular) curves reflects the fine structure of loop groups, in particular the detailed classification of their Cartan subgroups. To each such curve corresponds a homogeneous space of the loop group and a soliton system. Moreover the flows of the system have interpretations in terms of Jacobians of formal curves.

CONTENTS

1. Introduction	88
1.1. Background	88
1.2. The Present Work	90
1.3. Summary of Contents	93
1.4. Schemes, Stacks, etc.	94
2. A Construction of Connections	95
2.1. Types of Connections	95
2.2. Harish-Chandra structures	97
2.3. Double Quotients	98
2.4. Alternative Formulation	101
3. The Abelian Story	102
3.1. GL_n -opers and Differential Operators	102
3.2. The Krichever Construction	103
3.3. Other Perspectives	106
4. Loop Groups and Moduli Spaces	107
4.1. Loop Groups	108
4.2. The Big Cell	111
5. Heisenbergs and Spectral Curves	113
5.1. Basic Properties	113
5.2. The Principal Heisenberg	115
5.3. Singularities of Heisenbergs	118
5.4. Regular Centralizers	120
5.5. Examples	121
5.6. Filtrations	122

* Partially supported by grants from the Packard Foundation and the NSF.

6. Abelianization	124
6.1. Abelianized Grassmannians	125
6.2. The Principal Grassmannian	127
6.3. General Case	128
6.4. Higgs Fields and Formal Jacobians	128
6.5. Krichever Construction, Revisited	132
7. Opers	133
7.1. Introducing Opers	134
7.2. Affine Opers	135
7.3. Drinfeld-Sokolov Gauge	138
8. Integrable Systems	141
8.1. The Principal Case: KdV	141
8.2. Zero Curvature Representation	146
8.3. Flag Manifolds and the mKdV Hierarchy	147
8.4. Generalized Affine Opers	152
8.5. Generalized Drinfeld-Sokolov Hierarchies	154
Acknowledgments	157
References	157

1. Introduction

1.1. Background

The Korteweg-deVries hierarchy is an infinite family of commuting flows on the space of second-order differential operators $L = \partial_t^2 + q$ in one variable. It has long been known that this hierarchy has close ties to the geometry of algebraic curves. The Krichever construction explains how to obtain such an operator L from a line bundle \mathcal{L} on a hyperelliptic curve Y , equipped with some local data near a point $\infty \in Y$. Changing \mathcal{L} by the action of the Jacobian of Y changes L by the KdV flows. This picture was extended to the n -th KdV hierarchy, in which the second-order operator L is replaced by an n -th order operator. By replacing hyperelliptic curves with n -fold branched coverings Y of \mathbf{P}^1 , one finds a relation between Jacobian flows on line bundles on Y and KdV flows on associated differential operators. The resulting special “algebraic-geometric” solutions to KdV may be understood in great detail.

This picture illustrates a general phenomenon: an integrable system, naturally expressed in terms of differential data (differential operators, flat connections etc.), may be characterized using spectral data, on which the flows become linear and which have group-theoretic, and sometimes geometric, significance. However, most differential operators L do not arise from the geometry of curves in this way. Instead, the full phase space may be described using the beautiful algebraic formalism of the Sato Grassmannian and pseudodifferential operators (see [DJKM, SW, M1, M3]).

Drinfeld and Sokolov [DS] generalized the differential side of KdV by replacing n -th order differential operators by connections on rank n vector bundles, that is by translating from n -th order equations to first-order systems. This enabled them to

associate a generalized KdV hierarchy to an arbitrary semisimple Lie group G . These hierarchies live on spaces of connections on principal G -bundles on the line. Recently, these connections were given new importance, a coordinate-free formulation, and the name “opers” by Beilinson and Drinfeld [BD1] in the course of their work on the geometric Langlands correspondence [BD2]. An oper is a G -bundle on a complex curve with a (flat) holomorphic connection and a flag, which is not flat but obeys a strict form of Griffiths transversality with respect to the connection.

Drinfeld and Sokolov also incorporated a spectral parameter into their connections – thereby providing a generalization of the eigenvalue problem for a differential operator. They showed that the resulting connections (a loop group version of opers, which we name affine opers) may be brought into a canonical gauge, where an infinite-dimensional abelian group of symmetries becomes apparent. Using these symmetries, it is easy to write a commuting hierarchy of flows as “zero-curvature equations” – constraints expressing the flatness of the connection, when extended to new variables using the symmetries. Algebraic generalizations of the Drinfeld-Sokolov hierarchies have been introduced (see [dGHM, Fe]) in which the abelian group underlying the Drinfeld-Sokolov equations is replaced by more general abelian subgroups of loop groups.

The spectral side of KdV has also been greatly developed and generalized (see [M2, AB, DM, LM]). Classically, one studies line bundles \mathcal{L} on a curve Y which is an n -fold branched cover of \mathbf{P}^1 , or more generally of some projective curve X . Taking the pushforward of \mathcal{L} down to X produces a rank n vector bundle \mathcal{E} . Away from the branch points, the bundle \mathcal{E} decomposes into a direct sum of lines, while at the branch points this decomposition degenerates, producing a flag. This additional structure makes \mathcal{E} into a Higgs bundle. Conversely, from this Higgs data on \mathcal{E} we may recover the “spectral curve” Y and the line bundle \mathcal{L} on Y . (Usually one defines Higgs fields on \mathcal{E} as one-form valued endomorphisms of \mathcal{E} . The decomposition of \mathcal{E} is then achieved by considering the eigenspaces of the endomorphism, and the spectral curve Y parameterizing the eigenvalues is naturally embedded in T^*X . We will only be interested in “abstract” Higgs fields, where we retain the decomposition structure on E but forget the endomorphism which induced it.)

By reformulating decompositions into lines as reductions to maximal tori, one can extend this picture from vector bundles to principal G -bundles, following the general formalism developed by Donagi [D, DM, DG] (see also [Fa]). One considers reductions of a G -bundle \mathcal{E} to a family of Cartan subgroups of G , which is allowed to degenerate at certain points. This gives rise to the definition of a (regular) principal Higgs field as a sub-bundle of regular centralizers in the adjoint bundle of a G -bundle. Moduli spaces of principal Higgs bundles provide natural models for completely integrable systems in algebraic geometry. This was probably first realized by Hitchin [Hi]. Similar ideas have been used by Cherednik [Ch1, Ch2, Ch3] in his study of algebro-geometric solutions

of generalized soliton hierarchies. However, apparently no attempts have been made to *identify* the spectral and differential sides of soliton equations.

1.2. *The Present Work*

In the papers [FF1, FF2, FF3, EF1, EF2] a new approach to the study of KdV equations was introduced by Feigin, Enriquez and one of the authors (see [F] for an overview). This approach is based on the study of certain homogeneous spaces for (subgroups of) the loop groups (which also arose in [W] from a different point of view). These spaces come with an obvious action of an infinite-dimensional abelian group, and carry simple systems of coordinates in which the flows are easily understood. Using these coordinates, it is shown that these spaces are isomorphic to the KdV phase spaces and that the flows agree with the Drinfeld-Sokolov hierarchies.

Our original motivation for the current work was to understand geometrically how opers arise from homogeneous spaces for loop groups. This involved three main steps:

- (1) Identifying a moduli space interpretation for the homogeneous spaces (in particular thinking of them as schemes rather than as sets).
- (2) Finding a natural morphism between the moduli spaces describing the spectral side and the differential side (opers), explaining the explicit construction of [FF2].
- (3) Establishing an intrinsic reason for this morphism to be an isomorphism.

It is well known that homogeneous spaces for loop groups correspond to moduli spaces of bundles on a curve, with some extra structure. In the present case, we found this extra structure to be a formal generalization of the Krichever data. Specifically, the relevant moduli space is the “abelianized Grassmannian” $\mathcal{G}_\Lambda^{\mathbf{P}^1}$. This is the moduli space of G -bundles on \mathbf{P}^1 , equipped with a spectral curve description on the formal neighborhood D of a point $\infty \in \mathbf{P}^1$. This spectral datum may be formulated as a principal Higgs field on D with prescribed branching, or as a reduction of the structure group of the bundle to a twisted family of Cartan subgroups of G . (In the case of SL_n , the appropriate spectral curve is an n -fold cover of D , fully branched over ∞ .)

Suppose we are given such a G -bundle \mathcal{E} on \mathbf{P}^1 with a Higgs field on D . If the Higgs field extends to all of \mathbf{P}^1 , it will do so uniquely. Thus our moduli space contains a distinguished subspace, consisting of Higgs bundles on \mathbf{P}^1 whose spectral curve is a global n -fold branched cover Y — in other words a classical spectral curve for KdV. Thus we have embedded the Krichever data into a much bigger space of formal spectral data. This explains why the homogeneous spaces in [FF2] have a chance to be isomorphic to the *entire* phase space of KdV hierarchy (the space of all opers on the disc), while the global spectral data only recover special “finite-gap” differential

operators. In particular, these formal spectral curves serve as an algebro-geometric substitute for the analytic theory of infinite-genus spectral curves, [McK].

The most important aspect of the abelianized Grassmannians $\mathcal{G}_A^{\mathbf{P}^1}$ is that they come with a canonical action of an infinite-dimensional abelian Lie algebra. The formal group A/A_+ of this Lie algebra can be interpreted as the formal Jacobian (or Prym) variety of the formal spectral curve. Its action is the natural generalization (and extension) of the Jacobian flows appearing in the Krichever construction.

Now that both sides of the isomorphism from [FF2, F] have geometric interpretations, corresponding to the differential and spectral sides of KdV, the second step is a natural construction of flat connections from moduli spaces of bundles. In the Krichever setting of line bundles on global branched covers, such a construction was explained in the classic works of Drinfeld and Mumford [Dr, Mum]. We discovered that the calculations in [F] can be interpreted as a generalization of this idea, where we work not with line bundles on a projective curve, but with G -bundles on \mathbf{P}^1 with formal spectral data. The main idea is that the flows on the moduli space $\mathcal{G}_A^{\mathbf{P}^1}$ lift to tautological bundles, and that this lifting leads to the construction of flat connections. These connections naturally live on the formal group A/A_+ itself – i.e. on the formal Prym variety. Their restrictions to distinguished one-parameter formal subgroups are identified as opers. The zero-curvature equations relating the different one-parameter flows (i.e. the condition of the flatness of the connection on A/A_+) translate precisely into the zero-curvature formulation of the KdV hierarchy.

We have thus found a natural morphism from $\mathcal{G}_A^{\mathbf{P}^1}$ to the space of opers, under which the action of A/A_+ on $\mathcal{G}_A^{\mathbf{P}^1}$ translates into the KdV flows on the space of opers. The final step is to find out why this morphism is an isomorphism. As in [DS], one gains more insight by replacing opers by their loop group cousins, the affine opers (in other words, by incorporating the spectral parameter into the connection). Hence we explain how to go back and forth between opers and affine opers. Using the Drinfeld-Sokolov gauge for affine opers, we obtain a simple inverse to our map from $\mathcal{G}_A^{\mathbf{P}^1}$ to opers, in particular proving that it is an isomorphism.

Thus, our main result is that when suitably generalized, the Krichever construction can actually be made into an *isomorphism* between a moduli space of bundles with formal spectral data and the phase space of a soliton hierarchy. In the abelian setting, the Krichever construction has been explained by Rothstein in [Ro1, Ro2] and Nakayashiki [N1, N2] in the language of the (generalized) Fourier-Mukai transform ([Lau1]). Thus our isomorphism should perhaps be thought of in the context of a non-abelian Fourier transform.

On closer examination, the construction of connections we use turns out to be independent of the specifics of the problem, but rather an application of a very general construction. The underlying structure is an isomorphism between any double quotient

of an algebraic group with an appropriate space of flat connections on a subgroup. In fact, these connections reflect a certain remnant of the connections coming from the trivial Harish-Chandra structures on homogeneous spaces. When this construction is applied to the spaces $\mathcal{S}_A^{\mathbf{P}^1}$, one naturally obtains affine opers in the Drinfeld-Sokolov gauge.

We find several interesting contexts in which to apply these abstractions. Our isomorphism between formal spectral data and differential data not only specializes to the Krichever construction and extends it to principal bundles, but

- (1) We may replace the base curve \mathbf{P}^1 by an arbitrary curve.
- (2) We may allow arbitrary monodromy of the spectral curve, obtaining geometric descriptions of all of the generalized Drinfeld-Sokolov hierarchies of [dGHM, Fe] (labeled by conjugacy classes in the Weyl group).
- (3) We may allow arbitrary singularities of the spectral curve (replacing the smooth spectral data appearing above), obtaining continuous families of new integrable systems.

In forthcoming work, we apply this approach as follows:

- (1) The description of (generalized) Drinfeld-Sokolov hierarchies as flows on spectral data automatically implies a strong compatibility with the Hamiltonian structure of the Hitchin system and its meromorphic or formal generalizations (recovering in particular results of [DM]). This is closely related to the geometry of the affine Springer fibration.
- (2) In the case of the principal Heisenberg A , we generalize the isomorphism of Theorem 8.1.1 (3) between the open subspace of the moduli space $\mathcal{S}_A^{\mathbf{P}^1}$ and the moduli space of (ordinary) opers on the disc to the case of an arbitrary curve X . Namely, we obtain a map from a subspace of G_A corresponding to a “generic” G -bundle on X to the space of opers on the disc.
- (3) We extend the ideas of this paper to GL_∞ , providing a similar point of view on the KP hierarchy and pseudo-differential operators (allowing more “natural” modifications of the Krichever construction in the case of line bundles).

Some areas for future work include understanding the behavior of the exotic new integrable systems and their relations with the geometry of singular spectral curves; the interpretation of tau functions as theta functions for formal Jacobians, and its application to explicit formulas for solutions; relations with conformal blocks, vertex algebras and Virasoro actions; identifying the “spectral” meaning of the Gelfand-Dickey hamiltonian structure; and analogs where we replace differential operators by q -difference operators or polynomials in Frobenius, relating to the q -KdV equations and elliptic sheaves respectively.

1.3. *Summary of Contents*

The paper proceeds in the opposite direction from the introduction: we start with the most general notions, and step by step specialize them, until we end with the calculations which motivated the work. This simplifies the exposition because the proofs become elementary in the appropriate light, and we hope this will help clarify the underlying ideas. We refer the reader desiring a more concrete and explicit picture to our descriptions in the most important case (the principal Heisenberg algebra and the usual KdV hierarchy) and to the survey [F] for the origins of our approach.

In § 2 we explain a general group-theoretic construction of isomorphisms between moduli spaces of bundles and moduli spaces of connections, which is responsible for the spectral-differential equivalence for KdV. The connections arise from pulling back equivariant bundles on a space with a group action, to the group itself. When the space is a double quotient and the bundle tautological, one easily characterizes precisely which connections are obtained. Roughly speaking, we identify double quotients $H \backslash G / K$ with moduli of certain connections on the normalizer of K . This characterization is phrased in terms of the relative position of a reduction of a bundle with respect to a connection, a notion we describe in § 2.1. The ideas behind this are that of a period map and the localization for Harish-Chandra pairs. We also present a formulation in terms of differential schemes, which is closer in spirit to the theory of elliptic sheaves in characteristic p .

In § 3, we summarize the Krichever construction, which relates line bundles on an algebraic curve with differential operators in one variable. The exposition is inspired by [Dr, Mum] and [Ro1, Ro2], and informed by our general approach.

In § 4 we introduce loop groups and some of their homogeneous and double quotient spaces, which are interpreted in a standard way as moduli spaces of bundles on a curve. This gives a context in which to apply the general constructions. To find interesting connections, however, we need interesting group actions. So in § 5, we study the Heisenberg subgroups of the loop group at some length. In particular, we describe their fine classification and explain their relation with the geometry of formal spectral curves. This section may be read independently of the remainder of the paper.

This leads us in § 6 to the study of the main objects of interest, the abelianized Grassmannians, the moduli spaces of bundles equipped with a reduction to a Heisenberg subgroup A . Alternatively, they can be described as the moduli of G -bundles with formal Higgs field and fixed spectral curve. The abelianized Grassmannians come with the action of an infinite-dimensional abelian formal group A/A_+ , which is naturally interpreted as the Jacobian (or Prym) variety of the formal spectral curve. We apply the abstract construction of § 2 to obtain an isomorphism between the abelianized Grassmannian and a certain moduli space of flat bundles.

These flat bundles can be recast in the more tangible form of affine opers, discussed in § 7. Affine opers are LG-bundles with a flat connection and a reduction

having a distinguished relative position. The concept of affine opers is modeled on that of G -opers introduced by Beilinson and Drinfeld [BD1] following [DS]. For classical groups G -opers are identified with special differential operators. Although the concepts of G -opers and affine opers turn out to be essentially equivalent, the latter is more suitable in our context. The most important property of affine opers is a canonical abelian structure identified by the Drinfeld-Sokolov gauge (Proposition 7.3.7). The notion of affine opers and the Drinfeld-Sokolov gauge may be extended to arbitrary Heisenberg with good regular elements.

Our main results are presented in § 8. There we use the Drinfeld-Sokolov gauge to establish an isomorphism between the moduli of affine opers and the abelianized Grassmannians, thus establishing a general differential-spectral equivalence for a wide range of integrable systems.

From the point of view of the theory of integrable systems, our main result in § 8 is a natural and coordinate independent construction of an integrable hierarchy of flows on the appropriate space of affine opers, associated to an arbitrary Heisenberg subalgebra of the loop algebra $L\mathfrak{g}$, and a strongly regular element. In the special case when the Heisenberg is smooth we recover the generalized Drinfeld-Sokolov hierarchies introduced in [dGHM, Fe].

1.4. Schemes, Stacks, etc.

This paper is concerned with moduli spaces of bundles and of flat connections, from the viewpoint of algebraic geometry. Since these “spaces” are rarely varieties, this necessitates the use of some less familiar objects, namely algebraic stacks and ind-schemes. We refer the reader to [BL, LS, Tel] for a detailed description of moduli spaces in this language, and to [LMB, Sor] for a general treatment of stacks. Our main results of interest to experts in integrable systems are formulated in § 8 in terms of varieties, differential polynomials and evolutionary derivations. All of our stacks will be algebraic (in the Artin sense, so that automorphism groups of an object may be infinite). We will sometimes abuse notations and write $x \in M$ for a stack M , signifying an S -point of M for some scheme S .

All schemes, groups, sheaves, representations etc. will be defined over \mathbf{C} .

Throughout the paper we will refer to group schemes, for which the underlying scheme structure is obvious, simply as groups. By a G -torsor over a scheme X we will understand a scheme $E \rightarrow X$ equipped with a right action of G , such that locally in the flat topology it is isomorphic to $G \times X$. The term G -torsor is of course synonymous to the term principal G -bundle.

An ind-scheme is by definition an inductive limit of schemes (in the category of *spaces*, namely sheaves of sets in the *fpqc* topology). Ind-schemes can be very pathological in general; however, the ones we will encounter owe their inductive nature primarily

to infinite-dimensionality. In particular, they are unions of closed subschemes and are formally smooth. We will refer to group ind-schemes simply as ind-groups.

The typical example of an ind-scheme in our setting is the loop group $\mathcal{L}\mathcal{G}$ of an algebraic group G , whose \mathbf{R} -points are the points of G over the formal Laurent power series $\mathbf{R}((t))$. It is an inductive limit of schemes corresponding to Laurent series with bounded poles. Perhaps a more familiar class of ind-schemes is that of formal groups, which are group ind-schemes having points over rings with nilpotents, but no non-trivial points over any field. They arise from formally exponentiating the action of a Lie algebra.

2. A Construction of Connections

In this section we describe a general construction, which allows us to identify double quotient spaces with moduli spaces of connections. In our applications, the double quotient will be a moduli space of bundles, while the connections will be opers and their generalizations, which appear in soliton theory. We first describe the notion of *relative position* for a connection and a reduction of a bundle, and its relation to period maps. We then characterize the connections arising on homogeneous spaces from the theory of Harish-Chandra pairs [BB]. In this way our construction is related to the localization theory of representations. We identify an aspect of this picture which can be generalized to double quotient spaces, namely we obtain a map from the double quotient to a space of connections of a particular type. Finally using period maps we show that this map is an isomorphism (Proposition 2.3.12): the double quotient space itself classifies all connections of the prescribed type.

2.1. Types of Connections

Let G be a group scheme and $K \subset G$ a subgroup, with Lie algebras \mathfrak{g} and \mathfrak{k} , respectively. Given a G -torsor \mathcal{E} on a scheme X , and a scheme M equipped with an action of G (e.g., a representation of G), we define the \mathcal{E} -twist of M as $\mathcal{E} \times_G M$, and denote it by $(M)_{\mathcal{E}}$. This is a bundle over X , whose fibers are isomorphic to M .

Suppose \mathcal{E} is a G -torsor on a smooth scheme X , with a connection ∇ and a reduction \mathcal{E}_K to K . Then we may describe the failure of ∇ to preserve \mathcal{E}_K in terms of a one-form with values in $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$. Locally, choose any flat connection ∇' on \mathcal{E} preserving \mathcal{E}_K , and take the difference $\nabla' - \nabla$.

2.1.1. Lemma. — The local $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$ -valued one-forms $[\nabla' - \nabla]$ are independent of ∇' . They define a global section ∇/\mathcal{E}_K of $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K} \otimes \Omega^1$.

2.1.2. Remark. — More abstractly, this construction can be phrased as follows. Let $\mathcal{A}_{\mathcal{E}}$ be the Lie algebroid of infinitesimal symmetries of \mathcal{E} , and $\mathcal{A}_{\mathcal{E}_K}$ be the

subalgebroid preserving \mathcal{E}_K . Thus ∇ is a splitting of the anchor map of $\mathcal{A}_{\mathcal{E}}$ to the tangent sheaf, and ∇' does the same for $\mathcal{A}_{\mathcal{E}_K}$. The difference $\nabla' - \nabla$ is therefore a map from the tangent sheaf to the quotient $\mathcal{A}_{\mathcal{E}}/\mathcal{A}_{\mathcal{E}_K}$. But the latter sheaf is canonically isomorphic to $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$. Thus we obtain a $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$ -valued one-form, which is ∇/\mathcal{E}_K .

2.1.3. We can also view ∇/\mathcal{E}_K as a map from the tangent bundle TX of X to $(\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$ (denoted in the same way). This map can be realized as follows. Consider the G/K -bundle of \mathcal{E} , $(\mathrm{G}/\mathrm{K})_{\mathcal{E}}$. The K -reduction \mathcal{E}_K gives a section $s : \mathrm{X} \rightarrow (\mathrm{G}/\mathrm{K})_{\mathcal{E}}$, with differential $ds : \mathrm{TX} \rightarrow \mathrm{T}((\mathrm{G}/\mathrm{K})_{\mathcal{E}})$. The connection ∇ gives rise to a horizontal subbundle of the tangent bundle of $(\mathrm{G}/\mathrm{K})_{\mathcal{E}}$, and hence a canonical projection $p_{\nabla} : \mathrm{T}((\mathrm{G}/\mathrm{K})_{\mathcal{E}}) \rightarrow (\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$. The following simple fact will be useful in the proof of Proposition 2.3.12 below.

2.1.4. Lemma. — The map $\nabla/\mathcal{E}_K : \mathrm{TX} \rightarrow (\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$ coincides with the composition

$$\mathrm{TX} \xrightarrow{ds} \mathrm{T}((\mathrm{G}/\mathrm{K})_{\mathcal{E}}) \xrightarrow{p_{\nabla}} (\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}.$$

2.1.5. Definition: Relative Position. — Let \mathbf{O} be an orbit for the adjoint action of K on $\mathfrak{g}/\mathfrak{k}$, and ξ a vector field on X . The reduction \mathcal{E}_K is said to have *relative position* \mathbf{O} with respect to ∇_{ξ} if the image of ξ under the map $\nabla/\mathcal{E}_K : \mathrm{TX} \rightarrow (\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$ takes values in $(\mathbf{O})_{\mathcal{E}_K} \subset (\mathfrak{g}/\mathfrak{k})_{\mathcal{E}_K}$.

2.1.6. Period Maps. — The action of K on the quotient $\mathfrak{g}/\mathfrak{k}$ may be identified with its action, as the stabilizer subgroup of the identity coset $[1] \in \mathrm{G}/\mathrm{K}$, on the tangent space at that point. Thus there is a one-to-one correspondence between the K -orbits on $\mathfrak{g}/\mathfrak{k}$ and the G -orbits in the tangent bundle $\mathrm{T}(\mathrm{G}/\mathrm{K})$ to G/K (G -invariant distributions on G/K). This leads to a period map interpretation of relative position. Fix a point $x \in \mathrm{X}$ and a trivialization $\mathcal{E}_x \cong \mathrm{G}$ of the fiber at x . Then the connection defines a canonical trivialization of \mathcal{E} on the formal completion of X at x . (Complex analytically, we obtain a trivialization on any simply connected neighborhood of x .) The section s then provides a map, the formal period map, from the formal completion of X at x to G/K . The reduction has relative position \mathbf{O} if the vector field ξ is tangent to the G -invariant distribution on G/K corresponding to \mathbf{O} .

2.1.7. Remark. — It is useful to observe that if the bundle \mathcal{E} has a global flat trivialization, then we can define a global period map $\mathrm{X} \rightarrow \mathrm{G}/\mathrm{K}$. According to Lemma 2.1.4, the relative position map ∇/\mathcal{E}_K is simply its differential.

2.1.8. Remark. — When the orbit \mathbf{O} is \mathbf{C}^\times -invariant, we can apply the definition of relative position to all vector fields simultaneously. Namely, we say that \mathcal{E}_K has relative position \mathbf{O} with respect to ∇ if the one-form ∇/\mathcal{E}_K takes values in $(\mathbf{O})_{\mathcal{E}_K} \otimes_{\mathbf{C}^\times} \Omega_X^1$.

2.1.9. Difference analog. — It is instructive to compare the above notion of relative position with the notion of relative position for equivariant bundles. Suppose a group A acts on X , and let \mathcal{E} be an A -equivariant G -torsor on X . Suppose furthermore that \mathcal{E} is equipped with a reduction \mathcal{E}_K to K . The group analogs of K -orbits in $\mathfrak{g}/\mathfrak{k}$ are K -orbits in G/K . These correspond bijectively to diagonal G -orbits on $G/K \times G/K$.

Let $a \in A$ and let $\mathbf{O} \subset G/K$ be a K -orbit. Then we can say \mathcal{E}_K has relative position \mathbf{O} with respect to a if $a^*\mathcal{E}_K \subset \mathbf{O}_{\mathcal{E}_K} \subset (G/K)_{\mathcal{E}_K}$. The notion of period map also carries over. Assume for simplicity that we may A -equivariantly trivialize the bundle. The relative position map then sends $X \times A \rightarrow G/K \times G/K$ via $(x, a) \mapsto (\mathcal{E}_K|_x, \mathcal{E}_K|_{a \cdot x}) \in G/K \times G/K$. Thus \mathcal{E}_K has relative position \mathbf{O} with respect to a if the period map sends $X \times \{a\}$ to the diagonal G -orbit corresponding to a .

This group-theoretic notion extends automatically to the case when A is an ind-group (or sheaf of groups)⁽¹⁾. It also allows one to define relative position for difference operators on G -torsors, as well as for G -torsors on a scheme over a finite field which are equivariant with respect to Frobenius. We will briefly return to this idea in § 7.3.12.

2.2. Harish-Chandra structures

We present here an elementary aspect of the theory of (\mathfrak{g}, K) (or Harish-Chandra) structures (see [BB]). Let G be an (ind-)group, with Lie algebra \mathfrak{g} , and $K \subset G$ a subgroup. A (\mathfrak{g}, K) -structure on a scheme M is a K -torsor \mathcal{P} over M , together with an action of \mathfrak{g} on \mathcal{P} . The restriction of the \mathfrak{g} -action on \mathcal{P} to the Lie subalgebra $\mathfrak{k} = \text{Lie } K \subset \mathfrak{g}$ is assumed to coincide with the action of the latter coming from the action of K on \mathcal{P} . Moreover, the action of \mathfrak{g} is assumed to be simply transitive, so that the natural map from \mathfrak{g} to the tangent space to \mathcal{P} at any point is an isomorphism. It follows then that the tangent bundle of M is identified with the \mathcal{P} -twist $(\mathfrak{g}/\mathfrak{k})_{\mathcal{P}}$ of $\mathfrak{g}/\mathfrak{k}$. The basic example of a (\mathfrak{g}, K) -scheme is G/K itself, with \mathfrak{g} acting *from the right* on the total space of the K -bundle $\mathcal{P} = G \rightarrow G/K$.

2.2.1. Lemma. — The G -torsor $\mathcal{P}_G = \mathcal{P} \times_K G$ induced from \mathcal{P} carries a canonical flat connection ∇ , such that the map $\nabla/\mathcal{P} : TM \cong (\mathfrak{g}/\mathfrak{k})_{\mathcal{P}}$ is the isomorphism induced by the \mathfrak{g} -action on \mathcal{P} .

2.2.2. Proof for $M = G/K$. — The G -torsor $\mathcal{P}_G = G \times_K G$ is canonically identified with $G/K \times G$, by the map $(g_1, {}_K g_2) \mapsto (g_1 K, g_1 g_2)$. To a point $g_1 K \in G/K$ we assign the

⁽¹⁾ Replacing groups by groupoids, we may take A to be the formal neighborhood of the diagonal and recover the case of flat connections discussed above.

K -reduction $(g_1, {}_K 1) \in G \times_K G$, which corresponds to the subset $(g_1 K, g_1 K) \subset G/K \times G$. Thus we obtain a global trivialization of \mathcal{P}_G , hence a flat connection. By construction, the period map $G/K \rightarrow G/K$ defined using this trivialization is the identity map, and the lemma follows.

2.2.3. General Case. — Let $(\mathfrak{g}, K)^\wedge$ denote the group ind-scheme generated by the group K and the formal group \mathfrak{g}^\wedge of \mathfrak{g} , in other words the formal completion of G along K . Then for any x in M , the completion of \mathcal{P} along the fiber \mathcal{P}_x is a $(\mathfrak{g}, K)^\wedge$ -torsor. The resulting principal bundle \mathcal{P}^\wedge on M carries a flat connection, since it is canonically trivialized over any local Artinian subscheme of M : the formal neighborhoods of infinitesimally nearby fibers of \mathcal{P} are the *same*. The induced G -torsor, which coincides with \mathcal{P}_G , inherits this flat connection ∇ as well as the K -reduction \mathcal{P} . By construction, the connection identifies the bundle $(\mathfrak{g}/\mathfrak{k})_{\mathcal{P}_G} = (\mathfrak{g}/\mathfrak{k})_{\mathcal{P}^\wedge}$ with the tangent bundle of M .

2.3. Double Quotients

Now let G be a group scheme, and H, K subgroup-schemes. Let us consider the double quotient stack $H \backslash G / K$. This means the following: to each scheme S we attach the groupoid $H \backslash G / K(S)$, whose objects are G -torsors on S together with reductions to K and to H . The morphisms are the isomorphisms of such triples. The definition of the functor $(H \backslash G / K)(S_2) \rightarrow (H \backslash G / K)(S_1)$ corresponding to a morphism $S_1 \rightarrow S_2$ is straightforward.

For example, when K and H are both equal to the identity subgroup, the objects of the groupoid $(1 \backslash G / 1)(S)$ are G -torsors endowed with two reductions to the identity, hence two sections. We may use the first section to trivialize the torsor, and the other section gives us a map from S to G , i.e., an S -point of G . Therefore $1 \backslash G / 1 = G$. For general K and $H = \{1\}$, we obtain the equivalence between reductions of the trivial G -torsor on S to K and maps from the base S to G/K . In general we have a surjective morphism from the scheme G/K to $H \backslash G / K$, realizing $H \backslash G / K$ as an algebraic stack.

The stack $H \backslash G / K$ carries a tautological G -torsor \mathcal{T} . Its fiber over an S -point of $H \backslash G / K$, thought of as a G -torsor \mathcal{P} on S , is identified with \mathcal{P} . Moreover \mathcal{T} comes equipped with tautological reductions \mathcal{T}_K and \mathcal{T}_H to K and H , respectively. Explicitly, $\mathcal{T} = G \times_H G / K = H \backslash G \times_K G$, $\mathcal{T}_H = G / K$, $\mathcal{T}_K = H \backslash G$.

2.3.1. Connections on Double Quotients. — When $H \subset G$ is the identity subgroup, the space G/K has an obvious (\mathfrak{g}, K) -structure (§ 2.2.1). Thus the tautological G -bundle \mathcal{T} carries a canonical flat connection, which has a “tautological” relative position with respect to K . For general H the construction of § 2.2.1 breaks down, since the action of \mathfrak{g} on the K -torsor $\mathcal{P} = H \backslash G$ is no longer simply transitive. Accordingly, there is no natural flat connection on \mathcal{P} . A well-know way to circumvent this problem is to

replace the flat vector bundles associated to our flat G -torsors by \mathcal{S} -modules, obtained by taking coinvariants by the stabilizers of the \mathfrak{g} -action. Our approach explained below is to replace an action of all vector fields on $H \backslash G$ by those coming from the action of an appropriate subgroup of G , and to construct connections not on $H \backslash G / K$ but on the subgroup itself. This leads, in Proposition 2.3.12, to an identification of $H \backslash G / K$ with a moduli stack of special connections on a subgroup.

2.3.2. Actions give connections. — Let A be a group-scheme, and M a scheme equipped with an A -action. Suppose \mathcal{T} is an A -equivariant G -torsor on M , so in particular we may lift the action of the Lie algebra \mathfrak{a} of A from M to \mathcal{T} . If the A action is not free, it does not follow that \mathcal{T} obtains a partial connection along the A -orbits. Namely, the action of the stabilizers in \mathfrak{a} on \mathcal{T} presents an obstruction for lifting vector fields consistently to the bundle.

However, for any $x \in M$, with A -orbit $\pi_x : A \rightarrow M$, the A -action naturally identifies the pullback bundle $\pi_x^* \mathcal{T}$ on A with the trivial bundle $A \times \mathcal{T}_x$. Therefore $\pi_x^* \mathcal{T}$ has a canonical flat connection (albeit isomorphic to a trivial connection).

2.3.3. Now we apply the construction of § 2.3.2 to $M = H \backslash G / K$ and \mathcal{T} . First we need to identify natural group actions on $H \backslash G / K$. Let A be an arbitrary subgroup of $N(K)$, the normalizer of K in G . Then the right action of A on $H \backslash G$ descends to $H \backslash G / K$. In fact, if $A_+ = A \cap N(K)$, then the quotient group A / A_+ acts on $H \backslash G / K$. Furthermore, since A acts on $G / K = \mathcal{T}_H$, we have the following obvious:

2.3.4. Lemma. — The A / A_+ -action on $H \backslash G / K$ lifts canonically to \mathcal{T} , preserving the reduction \mathcal{T}_H to H .

2.3.5. Therefore the bundle \mathcal{T} over $\mathcal{M} = H \backslash G / K$ is A / A_+ -equivariant. For any $x \in H \backslash G / K$, the construction of § 2.3.2 results in a G -torsor \mathcal{E}^x on A / A_+ with a flat connection ∇ (induced by a trivialization). The G -torsor \mathcal{E}^x also carries reductions $\mathcal{E}_H^x, \mathcal{E}_K^x$ to K, H . Since \mathcal{T}_H is preserved by the A / A_+ -action, \mathcal{E}_H^x is automatically flat with respect to ∇ .

The behavior of \mathcal{E}_K^x with respect to ∇ mirrors Lemma 2.2.1 — the connection ∇ is simply a part of the structure of Lemma 2.2.1 which descends to $H \backslash G / K$. Since \mathfrak{a} normalizes \mathfrak{k} , the action of K on the Lie algebra $\mathfrak{a} / \mathfrak{a}_+$ of A / A_+ is trivial. Thus for every $a \in \mathfrak{a}$ the K -orbit \mathbf{O}_a of $a \bmod \mathfrak{k}$ in $\mathfrak{g} / \mathfrak{k}$ is a point. This leads us to the following definition:

2.3.6. Definition. — Let \mathcal{E} be a G -torsor with a K -reduction \mathcal{E}_K on A / A_+ . Then \mathcal{E}_K has *tautological relative position* with respect to a connection ∇ if the image of the vector field ξ_a coming from the left action of a on A / A_+ under the map $\nabla / \mathcal{E}_K : T(A / A_+) \rightarrow (\mathfrak{g} / \mathfrak{k})_{\mathcal{E}_K}$ is in \mathbf{O}_{-a} .

2.3.7. — The flat connection on \mathcal{E}^x was obtained from an identification $\mathcal{E}^x \cong A/A_+ \times \mathcal{E}^x|_1$. The additional choice of an identification $\mathcal{E}^x|_1 \cong G$ trivializes the G -bundle $\mathcal{E}^x \cong A/A_+ \times G$. We now recall from 2.1.7 that this global flat trivialization allows us to define a global period map from A/A_+ to G/K , whose differential is the relative position map ∇/\mathcal{E}_K^x .

2.3.8. Lemma. — The K -reduction \mathcal{E}_K^x is in tautological relative position with ∇^x .

2.3.9. Proof. — We choose a trivialization $\mathcal{E}_H^x|_1 \cong H$, inducing $\mathcal{E}^x|_1 \cong G$ as above. The resulting trivialization of \mathcal{E}^x preserves the H -reduction \mathcal{E}_H^x . This trivialization of \mathcal{E}_H^x gives rise to a lift of the A/A_+ -orbit of x on $H \backslash G/K$ to an A/A_+ -orbit on G/K . The period map $A/A_+ \rightarrow G/K$ induced by the trivialization is precisely this orbit map. It follows that the relative position of ∇ is given by the right action of A/A_+ on G/K , and hence is tautological.

2.3.10. Difference version. — The bundle \mathcal{E}^x is A/A_+ -equivariant by construction. Therefore it is natural to replace the infinitesimal relative position above by its group analog, § 2.1.9. For $a \in A/A_+$, the K -double coset $KaK = K1Ka$ is a single point. Thus Definition 2.3.6 has an obvious version, with a^{-1} replacing $-a$. The proof of Lemma 2.3.8 carries over as well.

2.3.11. Denote by \mathcal{M}_A^∇ the stack classifying quadruples $(\mathcal{E}, \tau, \mathcal{E}_H, \mathcal{E}_K)$, where \mathcal{E} is a G -torsor on A/A_+ , τ is a trivialization of \mathcal{E} , i.e. an identification of \mathcal{E} with $A/A_+ \times \mathcal{E}_1$, where \mathcal{E}_1 is the fiber of \mathcal{E} at $1 \in A/A_+$ (this trivialization induces a flat connection ∇ on \mathcal{E}), \mathcal{E}_H is a flat H -reduction, and \mathcal{E}_K is a K -reduction in tautological relative position with ∇ . When A/A_+ is not connected, we will always automatically replace the infinitesimal formulation by its group (difference) version, as above. When A/A_+ is connected, the two are equivalent.

The statement of Lemma 2.3.8 holds over any base S , and hence we obtain a natural morphism of stacks $\phi : H \backslash G/K \rightarrow \mathcal{M}_A^\nabla$.

2.3.12. Proposition. — The morphism $\phi : H \backslash G/K \rightarrow \mathcal{M}_A^\nabla$ is an isomorphism of stacks.

2.3.13. Proof. — There is an obvious forgetful morphism $\psi : \mathcal{M}_A^\nabla \rightarrow H \backslash G/K$, sending $\mathcal{E} \in \mathcal{M}_A^\nabla$ to the fiber $\mathcal{E}|_1$ at the identity of A/A_+ , considered as a G -torsor with reductions to K and H . It is clear that $\psi \circ \phi = \text{Id}$. It remains to show that $\phi \circ \psi = \text{Id}$.

Given $(\mathcal{E}, \tau, \mathcal{E}_H, \mathcal{E}_K) \in \mathcal{M}_A^\nabla(S)$, where S is an arbitrary base, we obtain a map $\pi : A/A_+ \times S \rightarrow H \backslash G/K$ classifying the triple $(\mathcal{E}, \mathcal{E}_H, \mathcal{E}_K)$. Locally on $S' \rightarrow S$ (an

$fppf$ covering), we may trivialize the H -torsor $\mathcal{E}_H|_1$, and thus the G -torsor $\mathcal{E}|_1$. Then the map τ provides a trivialization of \mathcal{E}_H and \mathcal{E} , and hence a lift of π to map $\tilde{\pi} : A/A_+ \times S' \rightarrow G/K$. But this map is precisely the period map, as explained in Remark 2.1.7, and therefore its differential is the relative position map ∇/\mathcal{E}_K .

Since we know that the relative position of \mathcal{E}_K is tautological, it follows that the differential of $\tilde{\pi}$ coincides with that of the right A/A_+ -action on G/K . Hence $\tilde{\pi}$ (whence π) is $\mathfrak{a}/\mathfrak{a}_+$ -equivariant (for A/A_+ connected, or A/A_+ -equivariant in general). Therefore $\pi(A/A_+ \times S')$ equals the A/A_+ -orbit in $H \backslash G/K$ of $\pi(1 \times S')$. This shows that $\phi \circ \psi = \text{Id}$ and proves the proposition.

2.3.14. Remark. — We note that, due to its difference formulation, the proposition is applicable in a broader context where we allow G and A to be ind-groups. In applications below, A/A_+ will be an ind-group, while K will be a group scheme.

2.4. Alternative Formulation

We present a different viewpoint on the above constructions, motivated by the theory of shtukas in characteristic p , and more directly by that of Krichever sheaves developed by Laumon [Lau2] (see also [BS]). The rough idea is that in order to obtain a characteristic zero analog of constructions involving Frobenius, one should consider not schemes S but differential schemes (S, ∂) , where ∂ is a distinguished vector field on S ⁽²⁾.

Given a differential scheme (S, ∂) we have the notion of a differential G -torsor $(\mathcal{E}, \partial_{\mathcal{E}})$ on (S, ∂) , which is a G -torsor \mathcal{E} on S equipped with an action $\partial_{\mathcal{E}}$ of ∂ . There is also a notion of relative position for differential torsors, following Definition 2.1.5: we may require a reduction \mathcal{E}_H of \mathcal{E} to $H \subset G$ to be in relative position $[-p]$ with respect to $\partial_{\mathcal{E}}$ (where $p \in \mathfrak{a}/\mathfrak{a}_+$, $\mathfrak{a} = \text{Lie}(\mathbf{N}(H))$).

Let $p \in \mathfrak{a}/\mathfrak{a}_+$ act on $H \backslash G/K$ as before. We thus consider the pair $(H \backslash G/K, p)$ as a differential stack.

2.4.1. Proposition. — The pair $(H \backslash G/K, p)$ represents the functor from differential schemes to groupoids which assigns to (S, ∂) the category of quadruples $(\mathcal{E}, \partial_{\mathcal{E}}, \mathcal{E}_K, \mathcal{E}_H)$, where $(\mathcal{E}, \partial_{\mathcal{E}})$ is a differential G -torsor on (S, ∂) , \mathcal{E}_K is a reduction of \mathcal{E} to K preserved by $\partial_{\mathcal{E}}$, and \mathcal{E}_H is a reduction to H in relative position $[-p]$ with respect to $\partial_{\mathcal{E}}$ (morphisms being isomorphisms of such objects).

2.4.2. Proof. — The proof parallels that of Proposition 2.3.12. Let $(\mathcal{E}, \partial_{\mathcal{E}}, \mathcal{E}_K, \mathcal{E}_H)$ be as above. Since $H \backslash G/K$ classifies triples $(\mathcal{E}, \mathcal{E}_K, \mathcal{E}_H)$ we obtain a map $S \rightarrow H \backslash G/K$. The relative position condition implies that this map gives rise to

⁽²⁾ To obtain the full parallel of Proposition 2.3.12 one simply replaces differential schemes by schemes with an A -action.

differential morphism $(S, \partial) \rightarrow (H \backslash G/K, \rho)$ classifying $(\mathcal{E}, \partial_{\mathcal{E}}, \mathcal{E}_K, \mathcal{E}_H)$ as required. Conversely, given a differential morphism $(S, \partial) \rightarrow (H \backslash G/K, \rho)$, we may pull back the tautological G -torsor \mathcal{T} with reductions to K and H , and Lemma 2.3.8 guarantees that the resulting quadruple $(\mathcal{E}, \partial_{\mathcal{E}}, \mathcal{E}_K, \mathcal{E}_H)$ has the desired properties.

3. The Abelian Story

In this section we present the classical construction of Krichever of algebro-geometric solutions to soliton equations, following the approach of Drinfeld [Dr] and Mumford [Mum] (see also Rothstein [Ro1]). We will see that the Krichever construction can be viewed as a special case of the correspondence between bundles and flat connections established in the previous section. This is intended to make the comparison with our generalization in the following sections more transparent.

3.1. GL_n -opers and Differential Operators

3.1.1. Definition. — A GL_n -oper on a smooth curve Y is a rank n vector bundle \mathcal{E} , equipped with a flag

$$0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E},$$

and a connection ∇ , satisfying

- $\nabla(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes \Omega^1$.
- The induced maps $\mathcal{E}_i/\mathcal{E}_{i-1} \rightarrow (\mathcal{E}_{i+1}/\mathcal{E}_i) \otimes \Omega^1$ are isomorphisms for all i .

3.1.2. In local coordinates a GL_n -oper has the form

$$\partial_t + \begin{pmatrix} * & * & * & \cdots & * \\ + & * & * & \cdots & * \\ 0 & + & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & + & * \end{pmatrix},$$

where the $*$ are arbitrary and the $+$ are nonzero. The oper condition is a strict form of Griffiths transversality.

Recall that giving an n -th order differential operator L in one variable

$$\partial_t^n - q_1 \partial_t^{n-1} - q_2 \partial_t^{n-2} - \cdots - q_n$$

is equivalent to giving a system of n first-order equations which can be written in terms of the first-order matrix operator

$$\partial_t - \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

If the $q_i \in \mathbf{C}[[t]]$, then this is a GL_n -oper on the formal disc $\widehat{D} = \mathrm{Spf} \mathbf{C}[[t]]$. Conversely it is not hard to see that any oper may be locally brought into the above form. Thus GL_n -opers on the formal disc are equivalent to n -th order differential operators. (A similar statement holds on global curves if we twist by the appropriate line bundles.) We thus have

3.1.3. Lemma. — GL_n -opers on the formal disc \widehat{D} are in one-to-one correspondence with n -th order differential operators with principal symbol 1.

3.2. The Krichever Construction

Let X be a smooth, connected, projective curve and $\infty \in X$ a fixed base point. Denote by D the “disc” around $\infty \in X$, i.e., $\mathrm{Spec} \mathcal{O}$, where \mathcal{O} is the completed local ring at ∞ ⁽³⁾. If we choose a formal coordinate z^{-1} on D (so that z has a simple pole at ∞), we may identify \mathcal{O} with $\mathbf{C}[[z^{-1}]]$. Let D^\times denote the punctured disc at ∞ , i.e., $\mathrm{Spec} \mathcal{H}$, where \mathcal{H} is the field of fractions of \mathcal{O} . Choosing a formal coordinate z^{-1} identifies \mathcal{H} with $\mathbf{C}((z^{-1}))$. However, we note that all of our constructions will be independent of the choice of formal coordinates.

The field \mathcal{H} has a natural filtration, by orders of poles at ∞ : $f \in (\mathcal{H})_{\geq m}$ if $fz^m \in \mathcal{O}$ for any local coordinate z^{-1} on D . Thus $\mathcal{O} = \mathcal{H}_{\geq 0}$. While the gradation by order of poles depends on the choice of coordinate z , the filtration is clearly independent of this choice. Let \mathcal{H}^\times denote the group functor of invertible Laurent series: by definition, the set of \mathbf{R} -points of \mathcal{H}^\times is $(\mathbf{R} \widehat{\otimes} \mathcal{H})^\times \cong \mathbf{R}((z^{-1}))^\times$. Note that \mathcal{H}^\times is not representable by a scheme, but is a group ind-scheme. The sub-functor \mathcal{O}^\times is defined as follows: the set of \mathbf{R} -points of \mathcal{O}^\times is $(\mathbf{R} \widehat{\otimes} \mathcal{O})^\times \cong \mathbf{R}[[z^{-1}]]^\times$. This functor is representable by a group-scheme of infinite type, with Lie algebra \mathcal{O} .

The quotient ind-group $\mathcal{H}^\times / \mathcal{O}^\times$ is isomorphic to a product of \mathbf{Z} and a formal group. The group of \mathbf{C} -points of $\mathcal{H}^\times / \mathcal{O}^\times$ is naturally identified with \mathbf{Z} . But if \mathbf{R} is a ring with nilpotents, then the group of \mathbf{R} -points of $\mathcal{H}^\times / \mathcal{O}^\times$ is much larger: it equals

⁽³⁾ Note the difference between the disc $D = \mathrm{Spec} \mathcal{O}$, which is a scheme, and the formal disc $\widehat{D} = \mathrm{Spf} \mathcal{O}$, which is a formal scheme obtained by completing X at ∞ . While we can consider a punctured disc D^\times , there is no punctured formal disc.

the product of \mathbf{Z} with the group of all expressions of the form

$$r_{-n}t^{-n} + \cdots + r_{-1}t^{-1} + 1$$

where the r_i are nilpotent. In other words, $\mathcal{H}^\times/\mathcal{O}^\times$ is isomorphic to the constant group scheme \mathbf{Z} times the universal Witt formal group $\widehat{\mathbf{W}}$ (see [CC]), which is associated with the Lie algebra \mathcal{H}/\mathcal{O} .

3.2.1. Let Pic_X denote the Picard variety of X , i.e., the moduli scheme of line bundles on X together with a trivialization of the fiber at a fixed point $0 \in X$. Now consider the moduli scheme $\widetilde{\text{Pic}}_X$ of line bundles $\mathcal{L} \in \text{Pic}_X$ on X , together with a trivialization ϕ of \mathcal{L} over D . The group \mathcal{O}^\times acts naturally on $\widetilde{\text{Pic}}_X$ by changing trivializations and the quotient $\widetilde{\text{Pic}}_X/\mathcal{O}^\times$ is isomorphic to Pic_X .

Moreover, the \mathcal{O}^\times -action on $\widetilde{\text{Pic}}_X$ can be extended to an action of \mathcal{H}^\times . Informally speaking, given a pair $(\mathcal{L}, \phi) \in \widetilde{\text{Pic}}_X$, and an element $k \in \mathcal{H}^\times$, we define a new line bundle \mathcal{L}' by gluing $\mathcal{L}|_{X \setminus \infty}$ and \mathcal{O}_D over D^\times via $k\phi$; then the bundle \mathcal{L}' comes with a natural trivialization ϕ' over D . In other words, we multiply the transition function of \mathcal{L} on D^\times by k (see [LS] for a discussion of formal gluing of bundles).

Since \mathcal{H}^\times commutes with \mathcal{O}^\times , we obtain an action of \mathcal{H}^\times , and in fact of $\mathcal{H}^\times/\mathcal{O}^\times$, on Pic_X . This action is formally transitive: \mathcal{H}/\mathcal{O} surjects onto the tangent space $H^1(X, \mathcal{O}_X)$ to Pic_X at any point. This may be easily seen by identifying $\mathcal{H}/\mathcal{O} \cong H^0(X, i_*\mathcal{O}_{X \setminus \infty}/\mathcal{O}_X)$, where $i: X \setminus \infty \hookrightarrow X$, and studying the obvious long exact sequence in cohomology, noting that $H^1(X, i_*\mathcal{O}_{X \setminus \infty}) = 0$ since $X \setminus \infty$ is affine. It follows that we have a surjection from the connected component of $\mathcal{H}^\times/\mathcal{O}^\times$ onto the formal group $\widetilde{\text{Pic}}_X$ of Pic_X (while the full $\mathcal{H}^\times/\mathcal{O}^\times$ action changes degrees of bundles as well).

There is a tautological line bundle \mathcal{P} on $X \times \text{Pic}_X$ whose fiber at $x \times \mathcal{L}$ is the fiber of \mathcal{L} at x . The pushforward of $\mathcal{P}|_{(X \setminus \infty) \times \text{Pic}_X}$ to Pic_X is a quasi-coherent sheaf $\mathcal{P}_- = \mathcal{P}(X \setminus \infty)$ on Pic_X (its fiber over \mathcal{L} is the vector space of sections of \mathcal{L} over $X \setminus \infty$).

3.2.2. Proposition. — The action of $\mathcal{H}^\times/\mathcal{O}^\times$ on Pic_X naturally lifts to \mathcal{P}_- .

3.2.3. Proof. — This follows immediately from the definition of the \mathcal{H}^\times action on $\widetilde{\text{Pic}}_X$: in changing the transition function from D to $X \setminus \infty$ we do not affect the bundle on $X \setminus \infty$. In other words, for $\mathcal{L} \in \widetilde{\text{Pic}}_X(\mathbf{R})$, there is a canonical identification between $\mathcal{L}|_{X \setminus \infty}$ and $(k \cdot \mathcal{L})|_{X \setminus \infty}$, for any $k \in \mathcal{H}^\times(\mathbf{R})$ and \mathbf{R} any Artinian ring. This identification is \mathcal{O}^\times -equivariant, hence descends to Pic_X .

3.2.4. There is a distinguished line $(\mathcal{H}/\mathcal{O})_{\geq -1}$ in the Lie algebra \mathcal{H}/\mathcal{O} , consisting of Laurent series with first order pole modulo regular ones. In a local coordinate z^{-1} on \mathbf{D} , this is the line $\mathbf{C}z$. Thus we have a distinguished vector field ∂_1 on the Jacobian. The resulting line in the tangent space to the Jacobian Pic_X^0 at any point \mathcal{L} is naturally identified with the tangent line to the Abel-Jacobi map based at ∞ . By Proposition 3.2.2, the vector field ∂_1 naturally lifts to the sheaf \mathcal{P}_- and provides the latter with a partial connection along this distinguished direction. Given $\mathcal{L} \in \text{Pic}_X$, we may restrict \mathcal{P}_- to the formal disc $\widehat{\mathbf{D}}_t = \exp(t\partial_1) \cdot \mathcal{L}$ generated by ∂_1 , obtaining a flat vector bundle.

The sheaf \mathcal{P}_- carries a natural increasing filtration, by subsheaves of sections of \mathcal{P} with increasing order of pole at ∞ . These subsheaves are coherent, but not locally free in general. On the locus of bundles with vanishing H^0 and H^1 , namely the complement of the theta-divisor $\Theta \subset \text{Pic}_X^{g-1}$, these sheaves are vector bundles whose rank is the order of pole.

The sheaf \mathcal{P}_- carries one additional structure – namely, an action of the ring $\mathcal{O}(X \setminus \infty)$ of functions away from ∞ . This action is compatible with the filtrations on $\mathcal{O}(X \setminus \infty) \subset \mathcal{H}$ and \mathcal{P}_- : for $f \in \mathcal{O}(X \setminus \infty)$ with n -th order pole, the sheaf $\mathcal{P}_-/f \cdot \mathcal{P}_-$ is a rank n vector bundle over the locus $\text{Pic}_X^{g-1} \setminus \Theta$.

3.2.5. Proposition. — Let $\mathcal{L} \in \text{Pic}_X^{g-1} \setminus \Theta$, and $f \in \mathcal{O}(X \setminus \infty)$ with precisely n -th order pole at ∞ . Then the rank n bundle $\mathcal{P}_-/f \cdot \mathcal{P}_-$ restricted to the formal disc $\widehat{\mathbf{D}}_t$, with its natural filtration and connection, is a GL_n -oper.

3.2.6. Equivalently, to every \mathcal{L} and f we assign an n -th order operator on $\widehat{\mathbf{D}}_t$. This extends to a homomorphism $\mathbf{K} : \mathcal{O}(X \setminus \infty) \rightarrow \mathcal{D}_t$, such that $f \cdot \psi = \mathbf{K}(f) \cdot \psi$, where ψ (the Baker-Akhiezer function) is a section of \mathcal{P}_- with first order pole (cf. [Dr]).

3.2.7. Example. — Let $X = \mathbf{P}^1$, with z a coordinate on \mathbf{A}^1 with first order pole at ∞ . We construct a formal one-parameter deformation of the trivial line bundle \mathcal{O}_X by the action of $-z \in \mathcal{H}/\mathcal{O}$. Analytically, this means we are multiplying the transition function at ∞ by e^t , where t is a parameter on \mathbf{D} . The resulting line bundle on $\mathbf{P}^1 \times \widehat{\mathbf{D}}_t$ has a connection in the $\widehat{\mathbf{D}}_t$ direction (that is, an action of $\frac{\partial}{\partial t}$). The connection does not affect the trivialization of \mathcal{O}_X on $\mathbf{P}^1 \setminus \infty$ (in which coordinate it is written as $\frac{\partial}{\partial t}$). Using the transition function to pass to a trivialization on the punctured disc \mathbf{D}^\times around ∞ , the connection becomes $\frac{\partial}{\partial t} - z$ and the constant section 1 on $\mathbf{P}^1 \setminus \infty$ is written as $\psi(z, t) = e^{zt}$, which has an essential pole at ∞ (but is well defined when

t is a formal parameter). The Krichever homomorphism \mathbf{K} induced by $ze^{zt} = \frac{\partial}{\partial t} e^{zt}$ is simply the Fourier transform, sending $\mathbf{C}[z]$ to $\mathbf{C}[\frac{\partial}{\partial t}]$.

3.2.8. Suppose now that \mathbf{X} is endowed with a degree n map ϕ to \mathbf{P}^1 , such that $\phi^{-1}(\infty) = \infty$ (so that ϕ is completely branched over ∞). Giving such a ϕ is the same as specifying a function $f \in \mathcal{O}(\mathbf{X} \setminus \infty)$ with n -th order pole at ∞ . Applying the above construction to f we attach an n -th order differential operator L with principal symbol 1 (equivalently, a GL_n -oper, see Lemma 3.1.3) on the t -disc to every line bundle $\mathcal{L} \in \mathrm{Pic}_{\mathbf{X}} \setminus \Theta$. The key fact is the following

3.2.9. *Theorem* [Kr]. — The action of $\mathcal{H}^\times / \mathcal{O}^\times$ on $\mathrm{Pic}_{\mathbf{X}}$ corresponds to the flows of the n -th KdV hierarchy on the space of all GL_n -opers on $\widehat{\mathbf{D}}_t$.

3.2.10. *Remark.* — The main result of this paper is an extension of the above construction of commuting flows on opers on the formal disc from the case of line bundles on \mathbf{X} to the case of principal G -bundles on \mathbf{X} , where G is a semisimple algebraic group. The oper connection and the flows will come from an action of an ind-group A/A_+ generalizing $\mathcal{H}^\times / \mathcal{O}^\times$, the filtrations will come from a refinement of the order-of-pole filtration, and the action of $\mathcal{O}(\mathbf{X} \setminus \infty)$ will be replaced by the data of a reduction to the ind-group $G(\mathbf{X} \setminus \infty)$.

3.3. Other Perspectives

3.3.1. *The Fourier-Mukai Transform.* — The Krichever construction may be described as an application of the Fourier-Mukai transform, as was discovered by Rothstein [Ro2] (see also [N1, N2]), thus clarifying the meaning of the Krichever homomorphism $a \cdot \psi = \mathbf{K}(a) \cdot \psi$.

The Fourier-Mukai transform is the equivalence between the derived categories of \mathcal{O} -modules on an abelian variety A and its dual A^\vee , obtained by convolution with the universal line bundle on the product. Laumon [Lau1] generalized this transform, establishing in particular an equivalence between the derived category of \mathcal{D} -modules on A and the derived category of modules over a sheaf \mathcal{O}^{\natural} of commutative \mathcal{O} -algebras ($\mathcal{O}^{\natural} = \mathcal{O}(A^{\natural})$, where A^{\natural} is the moduli of line bundles on A equipped with a flat connection).

Now let $A = \mathrm{Jac}_{\mathbf{X}}$, so that the dual variety $A^\vee = \mathrm{Jac}_{\mathbf{X}}$ as well. Consider the Abel-Jacobi map $a_\infty : \mathbf{X} \rightarrow \mathrm{Jac}_{\mathbf{X}}$ based at ∞ . Let $\mathcal{O}_{\mathbf{X}}(*\infty)$ denote the sheaf of holomorphic functions on \mathbf{X} with arbitrary poles at ∞ allowed. Rothstein [Ro2] proves that the pushforward of $i_* \mathcal{O}_{\mathbf{X} \setminus x}$ to $\mathrm{Jac}_{\mathbf{X}}$ under the Abel-Jacobi map has a natural \mathcal{O}^{\natural} -module structure. Therefore the result of applying the Fourier-Mukai transform to $(a_\infty)_* \mathcal{O}_{\mathbf{X}}(*\infty)$

is a \mathcal{D} -module on Jac_X . But this transformed sheaf is easily seen to be precisely the sheaf \mathcal{P}_- on the Jacobian. Thus we obtain a \mathcal{D} -module structure on \mathcal{P}_- . This structure is consistent with our constructions above, in the sense that the action on \mathcal{P}_- of the subalgebra $\mathbf{C}\partial_1 \subset \mathcal{H}/\mathcal{O}$ comes from its embedding into \mathcal{D} .

3.3.2. Formal Jacobians. — We wish to comment on the geometric significance of the ind-group $\mathcal{H}^\times/\mathcal{O}^\times$ acting on Pic_X , following [CC] (see [AMP] where the ideas of [CC] are explained and developed in the context of conformal field theory). This group represents the moduli functor of line bundles on the disc, trivialized away from the basepoint. Thus $\mathcal{H}^\times/\mathcal{O}^\times$ may be considered as a substitute for the Picard variety of the disc. As we mentioned above, it is isomorphic to the constant group scheme \mathbf{Z} times the universal Witt formal group $\widehat{\mathbf{W}}$, thus identifying the latter as the Jacobian of the disc. It carries a formal Abel-Jacobi map, whose tangent line is $(\mathcal{H}/\mathcal{O})_{-1}$. There are formal analogues of many of the usual properties of the Jacobian, including the Fourier-Mukai transform (following a general construction of Beilinson).

3.3.3. Concluding Remarks. — The action of $\mathcal{H}^\times/\mathcal{O}^\times$ on the Picard scheme of a curve identifies the formal neighborhood of any $\mathcal{L} \in \text{Pic}_X$ with a double quotient of the group ind-scheme \mathcal{H}^\times . It is in this fashion that the Krichever construction relates to the general ideas of § 2. However, since this only captures a formal piece of the Picard variety, it is hard to characterize the connections coming from arbitrary line bundles all at once⁽⁴⁾. The solution adopted in the theory of Krichever sheaves ([Lau2, BS]), paralleling the theory of elliptic sheaves, is to retain the entire curve \mathbf{X} , and to consider line bundles \mathcal{L} on \mathbf{X} times a differential scheme (S, ∂) , with the ∂ action lifting to \mathcal{L} . One then finds that the scheme $\text{Pic}_X^{g-1} \setminus \Theta$ with its ∂_1 -action classifies Krichever sheaves (of rank 1) for \mathbf{X} (compare § 2.4 — the non-abelian version of this statement will be discussed in § 7.2.5, § 8.1.4). In this work, we will concentrate on the moduli of G -bundles for G semisimple, which do have a simple global double quotient description. Once we introduce “abelianized” versions of these moduli, we obtain interesting flows and a construction of differential data extending the above picture for line bundles.

4. Loop Groups and Moduli Spaces

In order to develop an analog of the Krichever construction for G -bundles on curves, we wish to apply the general construction of § 2 in the case when G is the (formal) loop group LG , H its subgroup LG_X of loops that extend “outside” on an algebraic curve, and K is A_+ , an abelian subgroup of LG_+ (loops that extend “inside”). This is the subject of the rest of this paper.

⁽⁴⁾ One may substitute this formal picture by an adelic one, realizing the entire Picard as a double quotient for the group of idèles of \mathbf{X} , though idèle bundles with connection seem rather daunting.

In this section we introduce the loop groups and review the relation between their quotient spaces and moduli of bundles on curves. We also write an explicit form for the flat connections in an important special case.

4.1. Loop Groups

In the rest of this paper, unless noted otherwise, G will denote a connected semisimple algebraic group over \mathbf{C} .

Recall the setting of § 3.2. Let LG be the group ind-scheme $G(\mathcal{H})$, whose \mathbf{R} -points are the $\mathbf{R}((z^{-1}))$ -points of G . We refer to LG as the loop group. The subgroup $LG_+ \subset LG$ is defined to be the group scheme (of infinite type) $G(\mathcal{O})$. The Lie algebra of LG is the loop algebra $\mathfrak{Lg} = \mathfrak{g}(\mathcal{H})$, with positive half $\mathfrak{Lg}_+ = \mathfrak{g}(\mathcal{O})$. These algebras may be identified, after choosing a coordinate, with $\mathfrak{g}((z^{-1}))$ and $\mathfrak{g}[[z^{-1}]]$ respectively. The loop algebra carries a natural filtration, generalizing the filtration on $\mathcal{H} = \mathfrak{g}_1(\mathcal{H})$:

4.1.1. Definition. — The homogeneous filtration on the loop algebra is defined by

$$\mathfrak{Lg}_{\geq l} = \{f \in \mathfrak{Lg} \mid f z^{-l} \in \mathfrak{Lg}_+\}.$$

The induced filtration on the loop group will similarly be denoted by $LG_{\geq l}$. Both filtrations are independent of the choice of z .

4.1.2. Define $LG_-^X = LG_{\leq 0} \subset LG$ to be those loops which extend holomorphically to maps $X \setminus \infty \rightarrow G$. We reserve the notation LG_- for the case when $X = \mathbf{P}^1$. For any projective X , $LG_-^X \cap LG_+ \cong G$, as the only global loops are constants. In the case $X = \mathbf{P}^1$, this leads to a direct sum decomposition on the level of Lie algebras, $\mathfrak{Lg} \cong \mathfrak{Lg}_- \oplus \mathfrak{Lg}_{\geq 1}$. (This is the infinitesimal form of the Birkhoff decomposition, [PS].)

We now introduce infinite Grassmannians and interpret them as moduli spaces of bundles. For a detailed treatment of this material, we refer the reader to [BL, LS, Tel].

An important fact about principal G -bundles on algebraic curves is that if G is semisimple, then any G -bundle on an affine curve over \mathbf{C} is trivial [Ha]. It follows that (in our previous notations) a G -bundle on X may be trivialized on $X \setminus \infty$ and on D , and is thus determined by a transition function on D^\times , which is an element of the loop group. This provides a description of the set of isomorphism classes of G -bundles on X as a double quotient of the loop group. However, to obtain a similar statement for moduli *stacks* (that is, to recover the algebraic structure behind this set) one must appeal to a theorem of Drinfeld and Simpson [DSi], which gives a version of the above trivialization statement for families.

4.1.3. Definition. — Let $\mathcal{L}\mathcal{G}$ be the stack that classifies the G -torsors on X (for G semisimple) equipped with trivializations on $X \setminus \infty$ and on D . More precisely,

given a scheme S , $\mathcal{L}\mathcal{G}(S)$ is a groupoid whose objects are G -torsors on $S \times X$ with a trivialization on $S \times (X \setminus \infty)$ and $S \times D$, and morphisms are isomorphisms between such objects.

The infinite Grassmannian $\mathcal{G}r^X$ of X is the moduli stack that classifies G -torsors on X , trivialized on D . More precisely, given a scheme S , $\mathcal{G}r^X(S)$ is a groupoid whose objects are G -torsors on $S \times X$ with a trivialization on $S \times D$, and morphisms are isomorphisms between such objects.

The following remarkable description of the moduli stack of G -bundles is due to Beauville-Laszlo and Drinfeld-Simpson [BL, DSi] (see [Tel, Sor] for more detailed discussions).

4.1.4. Uniformization Theorem [BL, DSi].

- (1) The stack $\mathcal{L}\mathcal{G}$ is representable by the ind-scheme LG .
- (2) For any scheme S and any G -torsor \mathcal{P} on $S \times X$, the restriction of \mathcal{P} to $S \times (X \setminus \infty)$ becomes trivial after an étale base change $S' \rightarrow S$.
- (3) The moduli stack \mathcal{M}_G of G -torsors on X is canonically isomorphic to the double quotient stack $LG_-^X \backslash LG/LG_+$. It is smooth and of finite type.

4.1.5. Proposition. — $\mathcal{G}r^X$ is canonically isomorphic to $LG_-^X \backslash LG$, and is representable by a scheme of infinite type.

4.1.6. Proof. — The fact that $\mathcal{G}r^X$ is canonically isomorphic to $LG_-^X \backslash LG$ follows immediately from parts (1) and (2) of Theorem 4.1.4. Indeed, part (2) of Theorem 4.1.4 shows that the canonical forgetful morphism $p: \mathcal{L}\mathcal{G} \rightarrow \mathcal{G}r^X$ is surjective. Part (1) shows that $\mathcal{L}\mathcal{G} \simeq LG$. The group LG_-^X acts simply transitively on trivializations of a G -torsor on $X \setminus \infty$, which are the fibers of p . Hence p gives us an isomorphism $LG_-^X \backslash LG \simeq \mathcal{G}r^X$.

It remains to prove that $\mathcal{G}r^X$ is a scheme. The following proof was communicated to us by C. Teleman. Let $LG_{\geq n} \subset LG_+$ ($n > 0$) be the congruence subgroup with Lie algebra $\mathfrak{lg}_{\geq n}$ (see § 4.1.1), consisting of loops regular at ∞ and agreeing with the identity $1 \in LG$ there to order n . Thus the double quotient stack $LG_-^X \backslash LG/LG_{\geq n}$ is the stack of G -torsors on X equipped with a trivialization on an n -th order neighborhood of ∞ . For any G -torsor $\mathcal{P} \in LG_-^X \backslash LG/LG_+$ we can find an $n > 0$ so that a choice of level n structure on the bundle fixes all of its automorphisms. More precisely, there is a fine local moduli scheme for bundles near \mathcal{P} with level n structure at ∞ (see [Tel], Construction 3.12). It follows that for every $\mathcal{E} \in LG_-^X \backslash LG$ there is an LG_+ -invariant Zariski neighborhood U and an $N > 0$ such that $U/LG_{\geq n}$ is an affine scheme for $n > N$. Thus U represents the projective limit of $U/LG_{\geq n}$ in the category of affine schemes, and hence is an affine scheme. Therefore every $\mathcal{E} \in \mathcal{G}r^X$ has a Zariski neighborhood which is an affine scheme (of infinite type), so that $\mathcal{G}r^X$ itself is a scheme of infinite type.

4.1.7. Warning. — It is important to note that the “thick” or “in” Grassmannian $\mathcal{G}r^X = LG_-^X \setminus LG$ is *not* the loop Grassmannian considered, e.g., in [Gin, MV, LS], which is the ind-scheme LG/LG_+ that classifies the G -torsors on X trivialized *outside* of ∞ (and is independent of X). In particular, $\mathcal{G}r^X$ is an ordinary scheme (of infinite type) which does depend on the curve X . It is however closely related to the Sato Grassmannian and its sub-Grassmannians studied in [SW]. In algebraic geometry the “in” and “out” Grassmannians are very different, while in the analytic context of [SW] this distinction is obscured (in genus 0) since Fourier series on S^1 can be infinite in both directions.

4.1.8. The Grassmannian comes equipped with several universal bundles. As a homogeneous space $LG_-^X \setminus LG$, it comes with a tautological LG -bundle $\mathcal{T}(D^\times)$, as in § 2.3, with a reduction $\mathcal{T}(X \setminus \infty)$ to LG_-^X and a trivialization. From the moduli space description, there is a tautological G -torsor \mathcal{T} , on $\mathcal{G}r^X \times X$ whose fiber over $\mathcal{E} \times x$ is the fiber of \mathcal{E} at x . The bundle $\mathcal{T}(D^\times)$ is recovered as the sections of \mathcal{T} over D^\times . More generally the sections of \mathcal{T} over an affine subscheme $X' \subset X$ form a $G(X') = \text{Mor}(X', G)$ torsor. In particular, for every point $x \in X$ there is a G -torsor $\mathcal{T}(x)$ on $\mathcal{G}r^X$, whose fiber at a point $\mathcal{E} \in \mathcal{G}r^X$ is the fiber \mathcal{E}_x of \mathcal{E} at x . Since $\mathcal{G}r^X$ parameterizes bundles which are trivialized on D , the bundles $\mathcal{T}(X')$ for $X' \subset D$ are canonically trivialized.

4.1.9. Proposition. — Let $U \subset X \setminus \infty$ be a subscheme. Then the LG -action on $\mathcal{G}r^X$ lifts to the tautological bundle $\mathcal{T}(U)$.

4.1.10. Proof. — The total space of the LG_-^X -bundle $\mathcal{T}(X \setminus \infty)$ is naturally identified with LG , and hence it is clearly LG -equivariant. For general $U \subset X \setminus \infty$, the bundle $\mathcal{T}(U)$ is associated to $\mathcal{T}(X \setminus \infty)$ under the restriction $\mathcal{T}(X \setminus \infty) \rightarrow \mathcal{T}(U)$. In other words, $\mathcal{T}(U) = G(U) \times_{LG_-^X} LG$, and hence $T(U)$ is also LG -equivariant (cf. Lemma 2.3.4).

Geometrically, the lifting property can be interpreted as saying that when we change a bundle by deforming the transition function near ∞ , fibers away from ∞ are unchanged.

4.1.11. It follows from Proposition 4.1.9 and Lemma 2.3.2 that we may construct connections on various subgroups of LG by considering their action on $\mathcal{G}r^X$ lifted to the tautological bundles $\mathcal{T}(U)$. In order to obtain interesting connections, however, we will need to pick out interesting subgroups of LG and nontrivial structures on $\mathcal{T}(U)$ they preserve. The desire to obtain commuting families of flows on the resulting spaces of connections singles out *Heisenberg subgroups* of LG , and we will take up this idea in § 5.

4.2. The Big Cell

Consider the action on $\mathcal{G}r^X$ of the subgroup $LG_+ \subset LG$ of loops that extend to D . Acting on a pair $(\mathcal{E}, \phi) \in \mathcal{G}r^X$, an element $g \in LG_+$ does not change the G -torsor \mathcal{E} , but changes the trivialization ϕ of $\mathcal{E}|_D$ to ϕg^{-1} .

Note that $\mathcal{G}r^X = LG_- \backslash LG$ has a distinguished point corresponding to the identity coset. From the point of view of the moduli description of $\mathcal{G}r^X$, this is the pair (\mathcal{E}_0, ϕ_0) , where \mathcal{E}_0 is a trivial G -torsor on X , and ϕ_0 is its trivialization on D , which extends to a global trivialization on the whole X .

Let $\mathcal{G}r^\circ \subset \mathcal{G}r^X$ be the LG_+ -orbit of (\mathcal{E}_0, ϕ_0) . This is a scheme of infinite type that classifies bundles on X , trivialized on D , which admit a global trivialization. Consider the bundle $\mathcal{T}(X)$ of global sections of the tautological G -torsor \mathcal{T} over $\mathcal{G}r^\circ \times X$ along X . Since X is projective, the only global sections of a trivial bundle are the constant sections, so that $\mathcal{T}(X)$ is a G -torsor over $\mathcal{G}r^\circ$. Furthermore, for any $x_1, x_2 \in X$, there are canonical isomorphisms $\mathcal{T}(x_1) \cong \mathcal{T}(X) \cong \mathcal{T}(x_2)$ obtained from restricting global sections to the different fibers. These isomorphisms enable us to transfer extra structures, such as decompositions or connections, from one fiber to another.

4.2.1. Unfortunately, most interesting group actions do not preserve the subscheme $\mathcal{G}r^\circ \subset \mathcal{G}r^X$, so our main construction cannot be applied there. However, if $\mathcal{G}r^\circ$ were open, we could restrict the action of any Lie subalgebra of $L\mathfrak{g}$, and any formal subgroup of LG , to $\mathcal{G}r^\circ$. The orbit $\mathcal{G}r^\circ$ is open when $H^1(X, \mathfrak{g}) = 0$, which is satisfied when $X = \mathbf{P}^1$ is the projective line. Therefore from now on we reserve the notation $\mathcal{G}r^\circ$ for the case of \mathbf{P}^1 and call $\mathcal{G}r^\circ$ the *big cell*.

The stabilizer of the LG_+ -action at $(\mathcal{E}, \phi) \in \mathcal{G}r^\circ$ consists of elements of LG_+ which extend to all of \mathbf{P}^1 as automorphisms of \mathcal{E} , namely the global sections of the adjoint group scheme $\mathcal{E} \times_G \text{Ad}G$ (where G acts on itself by conjugation). In the realization $\mathcal{G}r^{\mathbf{P}^1} = LG_- \backslash LG$, this stabilizer is the intersection $LG_- \cap LG_+ \cong G$. Thus we obtain:

4.2.2. Lemma. — The big cell $\mathcal{G}r^\circ$ is canonically isomorphic to $G \backslash LG_+$. Furthermore let $LG_{>0} \subset LG_+$ be the congruence subgroup, consisting of loops which take the value $1 \in G$ at ∞ . Then we have a canonical factorization $LG_+ = G \cdot LG_{>0}$, and therefore $\mathcal{G}r^\circ$ is isomorphic to $LG_{>0}$. Thus $\mathcal{G}r^\circ$ may be identified with a pro-unipotent group, and hence it is isomorphic to a projective limit of affine spaces.

4.2.3. Recall that the total space of the bundle $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ over $\mathcal{G}r^{\mathbf{P}^1}$ is naturally identified with LG . The restriction of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ to $\mathcal{G}r^\circ$ is then identified with an open part of LG that consists of elements K admitting the factorization $K = K_- K_+$,

with $K_- \in \mathrm{LG}_-$ and $K_+ \in \mathrm{LG}_{>0}$. This factorization is unique. We will similarly denote by $k = k_- + k_+$ the direct sum decomposition of

$$\mathrm{Lg} = \mathrm{Lg}_- \oplus \mathrm{Lg}_{\geq 1} \cong \mathfrak{g}[z] \oplus z^{-1}\mathfrak{g}[[z^{-1}]]$$

into negative and positive halves.

It follows that $\mathcal{T}(\mathbf{P}^1 \setminus \infty)|_{\mathcal{G}r^\circ}$ is canonically trivialized: the fiber over $K_+ \in \mathrm{LG}_{>0} \simeq \mathcal{G}r^\circ$ is identified with LG_- by sending $K \in \mathcal{T}(\mathbf{P}^1 \setminus \infty)|_{K_+}$ to K_- .

4.2.4. We are in the setting of Lemma 2.3.2, where $M = \mathcal{G}r^\circ$, $\mathcal{T} = \mathcal{T}(\mathbf{P}^1 \setminus \infty)$. For simplicity, assume that $\mathfrak{a} \cong \mathbf{C}\rho$ is a one-dimensional Lie subalgebra of Lg and choose as A its formal group $\widehat{A}_\rho = \{e^{t\rho}\}$. The group \widehat{A}_ρ acts on $\mathcal{G}r^\circ$. Hence we obtain for each $\mathcal{E} \in \mathcal{G}r^\circ$ a connection on the LG_- -bundle $\pi_{\mathcal{E}}^*(\mathcal{T}(\mathbf{P}^1 \setminus \infty))$ over \widehat{A}_ρ (here $\pi_{\mathcal{E}} : \widehat{A}_\rho \rightarrow \mathcal{G}r^\circ$ is the \widehat{A}_ρ -orbit of \mathcal{E}). The above trivialization of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ induces a trivialization of $\pi_{\mathcal{E}}^*(\mathcal{T}(\mathbf{P}^1 \setminus \infty))$, and allows us to write down an explicit formula for this connection.

4.2.5. Lemma. — In the trivialization of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ induced by the factorization of loops, the connection operator on the LG_- -bundle $\pi_{\mathcal{E}}^*(\mathcal{T}(\mathbf{P}^1 \setminus \infty))$ takes the form

$$\nabla = \partial_t + (K_+(t)\rho K_+(t)^{-1})_-,$$

where $K_-(t)K_+(t)$ is the factorization of $K_+e^{-t\rho}$, and $K_+ = K_+(0)$ is the representative of $\mathcal{E} \in \mathcal{G}r^\circ$ in $\mathrm{LG}_{>0}$.

4.2.6. Proof. — The total space of the bundle $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ is an open part LG° of LG that consists of elements admitting factorization $K = K_-K_+$. The group A acts on it as follows:

$$e^{t\rho} : K \mapsto Ke^{-t\rho}.$$

Consider an element $K_+ \in \mathrm{LG}_{>0} \simeq \mathcal{G}r^\circ$. The fiber of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ over K_+ consists of all $K' \in \mathrm{LG}^\circ$, which can be represented in the form $K' = K_-K_+$. By construction of Lemma 2.3.2, the flat sections of $\pi_{K_+}^*(\mathcal{T}(\mathbf{P}^1 \setminus \infty))$ are precisely the pull-backs of the A -orbits $K'(t)$ of such K' in LG° .

Under the trivialization of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ introduced in § 4.2.1, the pull-back of $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ to A is identified with the trivial LG_- -bundle. The A -orbit of K_+ in $\mathcal{T}(\mathbf{P}^1 \setminus \infty) = \mathrm{LG}^\circ$ looks as follows: $K(t) = K_+e^{-t\rho}$. Hence the corresponding section of the trivial LG_- -bundle over A is $K_-(t)$, where we write $K(t) = K_-(t)K_+(t)$. This is a flat section with respect to our connection. Therefore the connection operator reads

$$\nabla = K_-(t)\partial_t K_-(t)^{-1} = \partial_t - K_-(t)^{-1}K'_-(t).$$

Now we find:

$$\mathbf{K}'_-(t)\mathbf{K}_+(t) + \mathbf{K}_-(t)\mathbf{K}'_+(t) = -\mathbf{K}(t)\rho,$$

and so

$$\mathbf{K}_-(t)^{-1}\mathbf{K}'_-(t) + \mathbf{K}'_+(t)\mathbf{K}_+(t)^{-1} = -\mathbf{K}_+(t)\rho\mathbf{K}_+(t)^{-1}.$$

This gives us the formula

$$\mathbf{K}_-(t)^{-1}\mathbf{K}'_-(t) = -(\mathbf{K}_+(t)\rho\mathbf{K}_+(t)^{-1})_-,$$

and the lemma follows.

4.2.7. Remark. — In particular, we see that if $A \subset \mathrm{LG}_+$ then we obtain a trivial connection operator ∂_t (i.e., the connection preserves our trivialization). From this point of view the LG_+ -action on $\mathcal{G}r^X$ is not interesting. The action of LG_- , however, and in particular of Heisenberg subgroups of LG_- , is the subject of our interest, since they can be identified with the KdV flows. We will return to the above calculation in § 8, and use it to derive the zero curvature representation of soliton equations.

5. Heisenbergs and Spectral Curves

In this section we describe the geometry of Cartan subgroups of loop groups, also known as Heisenberg subgroups. The action of these subgroups on the moduli spaces from § 4 will produce interesting integrable systems of KdV type. We also discuss the theory of spectral curves, introduce filtrations on the loop algebra associated with a Heisenberg subalgebra, and consider examples.

5.1. Basic Properties

We first recall some facts about Cartan subgroups of G , in a form convenient for generalization. Next we introduce Heisenberg subgroups and their spectral curves. The latter are used to explain (following Kazhdan and Lusztig [KL]) the theorem of Kac and Peterson [KP] classifying Heisenberg subgroups up to conjugacy.

Let H be a Cartan subgroup of G , $N(H)$ its normalizer, and $W = N(H)/H$ the Weyl group. The variety of all Cartan subgroups of G (hence of all Cartan subalgebras of \mathfrak{g}) is naturally identified with $G/N(H)$. Equivalently, $G/N(H)$ parameterizes $N(H)$ -reductions of the trivial G -torsor: the $N(H)$ -torsor corresponding to $H' \in G/N(H)$ is the torsor $\mathrm{Isom}_G(H', H)$ of conjugacies between H' and H . The choice of an isomorphism of groups $[\rho] : H \rightarrow H'$ reduces this $N(H)$ -torsor to an H -torsor (conjugacies inducing the given isomorphism). This gives a point of the variety G/H , which is a W -torsor over $G/N(H)$. This W -torsor is usually identified with the set of Borel subgroups $B' \subset G$ containing H' .

5.1.1. Definition. — A *Heisenberg subgroup* of LG is a subgroup obtained by restriction of scalars from a Cartan subgroup of $G(\mathcal{H})$. A *Heisenberg subalgebra* of $L\mathfrak{g}$ is the Lie algebra of a Heisenberg subgroup.

5.1.2. Remarks.

- (1) According to this definition, the Heisenberg subgroups are abelian. The terminology is explained by the fact that the pull-back of a Heisenberg subgroup to the Kac-Moody central extension of LG is a Heisenberg group.
- (2) Recall that a Cartan subgroup A of $G(\mathcal{H})$ is by definition a subgroup of $G(\mathcal{H})$, which becomes isomorphic to the maximal torus (i.e., a product of multiplicative groups of maximal dimension) over the algebraic closure $\overline{\mathcal{H}}$ of \mathcal{H} . Since A is an algebraic subgroup, it becomes isomorphic to a maximal torus over a finite extension of \mathcal{H} .
- (3) Heisenberg subalgebras are the maximal commutative subalgebras of $L\mathfrak{g}$ consisting of semisimple elements.
- (4) A Heisenberg subgroup $A \subset LG$ of the loop group is uniquely determined by a classifying map $C_A : D^\times \rightarrow G/N(H)$ (i.e. a family of Cartans of G).

5.1.3. The simplest example of a Heisenberg subgroup of LG is the *homogeneous Heisenberg LH*, consisting of loops $D^\times \rightarrow H$ into the constant Cartan $H \subset G$. It is given by the constant classifying map $D^\times \rightarrow [H] \in G/N(H)$. A Heisenberg subgroup is said to be of homogeneous type if it is LG -conjugate to LH . In other words, after conjugacy the classifying map $C_A : D^\times \rightarrow G/N(H)$ maps D^\times to a constant $H \subset G/N(H)$.

Since the field \mathcal{H} of Laurent series is not algebraically closed (equivalently, since the punctured disc is not simply-connected), the Cartan subgroups of LG are not all conjugate. Intuitively, this happens because they may experience monodromy around the puncture, and these monodromies are given by automorphisms of the Cartan subgroup of G , in other words, by the action of the Weyl group W of G . The monodromy may best be described as a class in the Galois cohomology group $H^1(\text{Gal}(\overline{\mathcal{H}}/\mathcal{H}), W)$, as in [KL].

5.1.4. Definition. — Consider a Heisenberg subgroup $A \subset LG$, given by its classifying map $C_A : D^\times \rightarrow G/N(H)$. The *spectral curve* $D^\times[A] \rightarrow D^\times$ is defined to be the pull-back under C_A of the W -cover $G/H \rightarrow G/N(H)$. The monodromy of this Galois cover is a well-defined conjugacy class $[z]$ in W , which is called the type of A .

5.1.5. If two Heisenbergs A and A' are LG -conjugate, then their spectral curves are automatically isomorphic. Now the spectral curve C_A of A may be described as the W -torsor associated to the $N(H)$ -torsor on D^\times of all local conjugacies of A to LH . Denote by \underline{A} the sheaf of groups (in the étale topology) on D^\times defined by A . Consider

the pullback \underline{A}_C of \underline{A} to C , so that the spectral curve of A_C has a tautological section. Since every \mathbf{H} -torsor on C_A (or D^\times) is trivial (which follows from the vanishing of $H^1(\text{Gal}(\overline{\mathcal{H}}/\mathcal{H}), \mathbf{G}_m)$, see [KL]), it follows that we can lift this tautological section to a conjugacy of A_C and LH_C :

5.1.6. Proposition. — The pullback \underline{A}_C of $\underline{A} \subset \underline{G}$ to its own spectral curve $C = D^\times[A]$ is conjugate to the homogeneous Heisenberg \underline{H}_C on C .

5.1.7. Corollary [KP]. — The Cartan subgroups of the loop group LG are classified, up to conjugacy, by the conjugacy classes in the Weyl group of G .

5.1.8. Remark. — The spectral curve $D^\times[A]$ is usually disconnected. If we pick locally an isomorphism $A \rightarrow \text{LH}$ (in other words a sheet of the spectral curve) we obtain a reduction of the W -torsor $D^\times[A]$ to the cyclic subgroup $\mathbf{Z}/n\mathbf{Z}$ of W generated by the monodromy, corresponding to picking a component \tilde{C} of the curve $D^\times[A]$. In terms of a coordinate z on D^\times , \tilde{C} is isomorphic to the Galois cover $\zeta^n = z$. Thus the loop group $\text{LG}_{\tilde{C}}$ on \tilde{C} , namely the sections of the constant group scheme G over \tilde{C} , is isomorphic to $G((z^{-\frac{1}{n}}))$. Thus Lemma 5.1.6 may be paraphrased as saying that if we allow ourselves to take n -th roots of z , we may conjugate A to LH . The restriction from $D^\times[A]$ to \tilde{C} is inessential — it simply allows us to think of sections of $G \times \tilde{C}$ over \tilde{C} as a loop group and not a product of loop groups.

Sometimes, when speaking about spectral curves, we will restrict ourselves to a component \tilde{C} of C . This should be clear from the context.

5.2. The Principal Heisenberg

In this section we will discuss the most prominent Heisenberg subgroup, the *principal* Heisenberg. The important features of Heisenberg subgroups can be seen clearly in this case. To make contact with the material of § 4, we wish to view LG as attached to the disc at a point ∞ on a curve X . Although all local results below can be stated for an arbitrary curve X , we will assume, for concreteness, that $X = \mathbf{P}^1$ and z is a global coordinate on \mathbf{P}^1 with simple pole at ∞ .

Thus $\text{Lg} = \mathfrak{g}((z^{-1}))$ is the (formal) loop Lie algebra at ∞ with Lie group $\text{LG} = G(\mathbf{C}((z^{-1})))$. The positive and negative parts $\text{Lg}_+ = \mathfrak{g}[[z^{-1}]]$ and $\text{Lg}_- = \mathfrak{g}[z]$ consist of loops that extend to the disc D at ∞ and to $\mathbf{P}^1 \setminus \infty$ respectively.

5.2.1. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Cartan decomposition of \mathfrak{g} . Here \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and \mathfrak{n}_+ , \mathfrak{n}_- are the upper and lower nilpotent subalgebra. Let $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ be the (upper) Borel subalgebra of \mathfrak{g} . Recall that \mathfrak{g} has generators h_i, e_i, f_i ($i = 1, \dots, \ell = \text{rank } \mathfrak{g}$), where $h_i \in \mathfrak{h}, e_i \in \mathfrak{n}_+, f_i \in \mathfrak{n}_-$. Denote by e_θ (resp., f_θ) a non-zero element of \mathfrak{n}_+ (resp., \mathfrak{n}_-) of weight (minus) the maximal root θ .

Recall that \mathbf{Lg} has Kac-Moody generators $h_i, i = 1, \dots, \ell; e_i, f_i, i = 0, \dots, \ell$, where $f_i = f_i \otimes 1, e_i = f_i \otimes 1, i = 1, \dots, \ell; f_0 = e_0 \otimes z, e_0 = f_0 \otimes z^{-1}$.

Introduce the ‘‘cyclic element’’ of \mathbf{Lg}

$$p_{-1} = \sum_{i=0}^{\ell} f_i = f_1 + \dots + f_{\ell} + e_0 \otimes z.$$

In the case of \mathfrak{sl}_n , with conventional choices, we have

$$(5.2.1) \quad p_{-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & z \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

This is a regular semisimple element of \mathbf{Lg} , so that its centralizer in \mathbf{LG} is a Heisenberg subgroup A of \mathbf{LG} which is called the principal Heisenberg. The Lie algebra \mathfrak{a} of A is the centralizer of p_{-1} in \mathbf{Lg} . It contains a unique element of the form

$$p_1 = c_1 \cdot e_1 + \dots + c_{\ell} \cdot e_{\ell} + f_0 \otimes z^{-1}$$

(with $c_i \neq 0$ for all $i = 1, \dots, \ell$). In the case of \mathfrak{sl}_n , we have

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ z^{-1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that $p_1 \in \mathbf{Lg}_+$ is regular at ∞ , and is also a regular semisimple element of \mathbf{Lg} .

Define the principal gradation on \mathbf{Lg} by setting $\deg e_i = -\deg f_i = 1, \deg h_i = 0$. The principal Heisenberg Lie algebra \mathfrak{a} is homogeneous with respect to this gradation, and has a basis p_i with i (modulo the Coxeter number) an exponent of \mathfrak{g} (see [Kac1]). Except in the D_n case, all the homogeneous components $\mathfrak{a}^i \subset \mathfrak{a}$ are one-dimensional, and the (-1) -component $\mathfrak{a}_{-1} = \mathbf{C}p_{-1}$ is always so. In the \mathfrak{sl}_n case, we may take $p_{-i} = p_{-1}^i$, which is in \mathfrak{sl}_n if i is not divisible by n .

5.2.2. The monodromy. — For $z \in \mathbf{P}^1 \setminus \{0, \infty\}$ (in particular on the punctured disc near ∞), $p_1(z)$ (or $p_{-1}(z)$) is a regular semisimple element of \mathfrak{g} , and hence defines a unique Cartan subalgebra $\mathfrak{a}(z)$. There are two important features of the principal family $\mathfrak{a}(z)$. The first is that as z undergoes a loop around ∞ , the algebra $\mathfrak{a}(z)$ undergoes a monodromy, which is a well-defined conjugacy class in the Weyl group W of G . In fact, for the principal Heisenberg this is the conjugacy class of *Coxeter elements*, which

are the products of simple reflections in W taken in an arbitrary order (see [Kos, Kac1] and § 5.1.4).

In the case of \mathfrak{sl}_n this monodromy has the following explicit description. For every $z \in \mathbf{C}^\times = \mathbf{P}^1 \setminus \{0, \infty\}$, the fiber of the trivial bundle $\mathbf{P}^1 \times \mathbf{C}^n$ over z decomposes under the Cartan $\mathfrak{a}(z)$ into a direct sum of n lines. This defines an n -fold branched cover of \mathbf{P}^1 , whose fiber over z is given by the set of eigenvalues of $p_1(z)$ on \mathbf{C}^n . Recall that we also have a spectral curve $D^\times[A]$ as in the general setting, which is a principal $W = S_n$ -cover of D^\times . The n -fold cover above is the bundle associated to $D^\times[A]$ under the permutation representation of S_n .

Since the Coxeter number (the order of a Coxeter element) of \mathfrak{sl}_n equals n , the monodromy statement above translates into the statement that this n -fold cover is fully branched over ∞ , i.e., is described by the equation $\zeta^{-n} = z^{-1}$. (Note that $p_1^n = z^{-1} \cdot \text{Id}$, so that p_1 itself plays the role of ζ^{-1} .) The Coxeter class for S_n consists of n -cycles, and by labeling the cover in different ways we obtain the different n -cycles as monodromies. It is also worth noting that the n -fold cover is isomorphic to a component of the reducible curve $D^\times[A]$ – the S_n -bundle may be reduced to a cyclic subgroup generated by an n -cycle.

For general G , the spectral curve $D^\times[A]$ of the principal Heisenberg is a union of the fully branched cyclic covers of D^\times of order the Coxeter number of G .

5.2.3. Degeneration at ∞ . — The second important feature of $\mathfrak{a}(z)$ is the way it degenerates at $z = \infty$, as a subspace of \mathfrak{g} , to an $l = \text{rk } \mathfrak{g}$ -dimensional abelian subalgebra \mathfrak{a}_∞ . This limit is the centralizer of the regular nilpotent element $\bar{p}_1 = p_1(\infty)$ of \mathfrak{g} . Specifically, in the case of \mathfrak{sl}_n ,

$$(5.2.2) \quad \bar{p}_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The centralizer \mathfrak{a}_∞ of \bar{p}_1 consists solely of upper triangular matrices. Geometrically, we see that the n -sheeted spectral cover associated with \mathfrak{a} can be completed to the n -fold cover of the disc defined by $\zeta^{-n} = z^{-1}$, completely branched at infinity. The elements $p_i \in \mathfrak{a}$ are now identified with the powers ζ^{-i} of the coordinate on the spectral cover. Thus in particular the principal gradation on \mathfrak{a} agrees with the gradation in powers of the coordinate upstairs. The upper triangular matrices are those that preserve the canonical filtration induced on the n -dimensional vector space $\mathbf{C}[\zeta^{-1}]/z^{-1}\mathbf{C}[\zeta^{-1}]$. Thus for $z \neq \infty$ we obtain a decomposition of the trivial vector bundle into lines, while at $z = \infty$ we retain the structure of a flag.

In general there is a unique Borel subalgebra in the fiber $\text{Lg}|_\infty \cong \mathfrak{g}$ which contains \mathfrak{a}_∞ , namely \mathfrak{b}_+ . Consider the principal filtration, whose i -th piece $\text{Lg}^{>i}$ consists

of elements of degree $\geq i$ in the principal gradation. It defines the above Borel subalgebra as $\mathfrak{b}_+ = \mathbf{L}\mathfrak{g}_- \cap \mathbf{L}\mathfrak{g}^{\geq 0}$. The resulting filtration on \mathfrak{g} is canonically determined by the choice of \mathfrak{b}_+ , and hence \mathfrak{a}_+ .

It is also useful to note that due to the structure of the principal filtration, there is a distinguished line $\mathfrak{a}^{-1} \subset \mathfrak{a}/\mathfrak{a}_+$ given by the intersection of \mathfrak{a} with $\mathbf{L}\mathfrak{g}_{\geq -1}$. Thus the element p_{-1} is determined intrinsically by the structure of \mathfrak{a} , up to a constant. This line is the analog of the line in \mathcal{H}/\mathcal{O} of elements with first order pole — it consists of “first order poles in the spectral coordinate ζ ”.

Finally, the full branching of the spectral curve translates algebraically into the statement that A is an anisotropic torus in LG : it does not contain any split torus (i.e., a product of copies of \mathcal{H}^\times) as a subgroup. It follows that the ind-group A/A_+ is in fact just the formal group associated to $\mathfrak{a}/\mathfrak{a}_+$.

5.3. Singularities of Heisenbergs

In this paper we are interested in the classification of Heisenberg subgroups of LG not only up to LG -conjugacy, but up to LG_+ -conjugacy. Thus we are interested in the “integral models” of Cartan subgroups in $G(\mathcal{H})$, that is abelian subgroups \underline{A}_+ of the trivial group scheme \underline{G} on the *unpunctured* disc D (as opposed to the group scheme \underline{A} over the punctured disc D^\times). The reason for that will become apparent in § 6, where we consider the moduli of pairs $(\mathcal{E}, \mathcal{E}^{\Lambda_+})$, where \mathcal{E} is a G -torsor on a curve X , and \mathcal{E}_Δ is a reduction of $\mathcal{E}|_D$ to such an \underline{A}_+ .

5.3.1. While Heisenberg subalgebras of $\mathbf{L}\mathfrak{g}$ are classified up to LG -conjugacy by conjugacy classes in the Weyl group, their determination up to LG_+ -conjugacy is in fact much more subtle (as was first explained to us by R. Donagi). This finer structure describes how families of Cartan subalgebras on D^\times degenerate at $\infty \in D$. It may also be described in terms of singularities of completed spectral curves. The integrable systems we construct in subsequent sections reflect this intricate behavior.

Let $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$ be a Heisenberg Lie algebra of homogeneous type. In other words we can find $g \in LG$ such that $g\mathfrak{a}g^{-1} = \mathbf{L}\mathfrak{h}$. Such a g is unique up to left multiplication by the normalizer $N(\mathbf{L}\mathfrak{h}_\mathbb{C}) \subset LG_\mathbb{C}$. Note that $N(\mathbf{L}\mathfrak{h}_\mathbb{C})/\mathbf{L}\mathfrak{H}_\mathbb{C} \cong W$ is the (finite) Weyl group W , since for the homogeneous Heisenberg $N(\mathbf{L}\mathfrak{h})/\mathbf{L}\mathfrak{H} = \mathbf{L}(N(\mathfrak{h})/\mathfrak{H})$ and loops into W are necessarily constant. It follows that the map $char : \mathfrak{a} \rightarrow \mathbf{L}\mathfrak{h}/W$ induced by g is uniquely defined. Let $\mathfrak{a}_+ = \mathfrak{a} \cap \mathbf{L}\mathfrak{g}_+$, and let $g(\mathfrak{a}_+) \subset \mathbf{L}\mathfrak{h}$ be the image of \mathfrak{a}_+ under g .

5.3.2. Lemma. — The image $g(\mathfrak{a}_+)$ satisfies $g(\mathfrak{a}_+) \subset \mathbf{L}\mathfrak{h}_+ \subset \mathbf{L}\mathfrak{h}$. Moreover, $\mathbf{L}\mathfrak{h}_{\geq N} \subset g(\mathfrak{a}_+) \subset \mathbf{L}\mathfrak{h}_+$ for $N \gg 0$, where $\mathbf{L}\mathfrak{h}_{\geq j} = \mathbf{L}\mathfrak{h} \cap \mathbf{L}\mathfrak{g}_{\geq j}$ is the homogeneous filtration.

5.3.3. Proof. — According to the Chevalley theorem, we have a “characteristic polynomial” map $\mathfrak{g} \rightarrow \text{Spec } \mathbf{C}[\mathfrak{g}]^G \cong \mathfrak{h}/W$. Over the field \mathcal{H} of Laurent series, we

obtain a map $char : \mathbf{Lg} \rightarrow \mathbf{L}(\mathfrak{h}/W)$ from the pointwise application of the characteristic polynomial. In particular, \mathfrak{a} is homogeneous when $char$ sends it to \mathbf{Lh}/W , loops which may be lifted to \mathfrak{h} . In fact, to conjugate \mathfrak{a} to \mathbf{Lh} is equivalent to choosing a lift $\mathfrak{a} \rightarrow \mathbf{Lh}$ of the characteristic map. Since the characteristic map is defined over $\mathcal{O} \subset \mathcal{H}$, it follows that for $a \in \mathfrak{a}_+$, $char(a) \in \mathbf{Lh}_+/W \subset \mathbf{Lh}/W$. Since the map g is a lift of the characteristic map, the first statement follows immediately.

Since g has a finite order of pole at ∞ , it follows that $g^{-1}(h) \in \mathbf{Lg}_+$ for $h \in \mathbf{Lh}_{\geq N}$ and $N \gg 0$ (as can be easily seen for example for \mathbf{SL}_n and hence in a faithful matrix representation). This proves the second statement.

5.3.4. Now let $\mathfrak{a} \subset \mathbf{Lg}$ be a general Heisenberg subalgebra. Recall that the pullback \mathfrak{a}_C of \mathfrak{a} to its own spectral curve $C = D^\times[\mathfrak{a}]$ is conjugate to the homogeneous Heisenberg \mathbf{Lh}_C . If $g_1, g_2 \in \mathbf{LG}_C$ satisfy $g_i^{-1} \mathbf{Lh}_C g_i = \mathfrak{a}_C$, then the conjugation by $g_1 g_2^{-1}$ preserves the homogeneous filtration on \mathbf{Lh}_C . Therefore the following definition makes sense.

5.3.5. Definition. — The *canonical filtration* $\{\mathfrak{a}^{\geq i}\}$ on \mathfrak{a} is the filtration induced on \mathfrak{a} from the homogeneous filtration on \mathbf{Lh}_C via the embedding $\mathfrak{a} \subset \mathfrak{a}_C$ and any conjugacy $\mathfrak{a}_C \cong \mathbf{Lh}_C$.

We say \mathfrak{a} is *smooth* if the subalgebra $\mathfrak{a}^+ := \mathfrak{a}^{\geq 0}$ and its subalgebra $\mathfrak{a}_+ = \mathfrak{a} \cap \mathbf{Lg}_+$ coincide.

5.3.6. Let $A_+ \subset A^+ \subset A$ be the ind-groups corresponding to the Lie algebras $\mathfrak{a}_+ \subset \mathfrak{a}^+ \subset \mathfrak{a}$. It follows that the group A^+ is isomorphic to $\mathbf{LH}_+^{[w]}$, where $\mathbf{LH}^{[w]}$ is a smooth Heisenberg of the same type. Hence in particular A^+ is actually a group scheme, and A/A^+ is the product of the formal group associated to $\mathbf{Lh}^{[w]}/\mathbf{Lh}_+^{[w]}$ by a lattice. The “difference” between A and $\mathbf{LH}^{[w]}$ is the finite-dimensional group scheme A^+/A_+ .

5.3.7. The Heisenberg algebras which have been studied in the literature [KP, dGHM, Fe] are graded with respect to an associated gradation on the loop algebra. These graded Heisenbergs all satisfy the smoothness condition $\mathfrak{a}_+ = \mathfrak{a}^+$ (as well as $\mathbf{LG}^+ \subset \mathbf{LG}_+$), and provide standard smooth representatives of all \mathbf{LG} -types.

We want to stress that there are plenty of \mathbf{LG}_+ -conjugacy classes of non-smooth Heisenbergs — there exist continuous families of those within given type $[w]$. The collection of \mathbf{LG}_+ -classes of Heisenbergs of type $[w]$ is naturally parameterized by the infinite-dimensional double quotient $N(A) \backslash \mathbf{LG} / \mathbf{LG}_+$, where A is any such subgroup (say, a graded representative) and $N(A)$ its normalizer. While the integrable systems that have been studied in the literature so far are associated to graded Heisenbergs only, in this paper we construct integrable systems attached to arbitrary Heisenberg

subalgebras possessing strongly regular elements (that is, for all Heisenbergs of type $[w]$ where $[w]$ varies through many, but not all, conjugacy classes in W).

5.3.8. Remark. — The picture of Heisenberg algebras that emerges closely parallels the structure of branched covers of D . First there is a topological invariant, the monodromy of the cover over D^\times , which is resolved by passing to an étale cover. For Heisenberg algebras this is the monodromy of the spectral curve and the passage from \mathfrak{a} to \mathfrak{a}_C . The next step is to consider the behavior at the marked point. The normalization of the completed spectral curve is a smooth curve isomorphic to D , so that we have a finite codimension embedding $\mathcal{O}_C \subset \mathcal{O} \cong \mathbf{C}[[z^{-1}]]$ of coordinate rings. This is the meaning of the embedding $\mathfrak{a}_+ \hookrightarrow \mathrm{L}\mathfrak{h}_+$ of Lemma 5.3.2. The filtration $\{\mathfrak{a}^{\geq i}\}$ is the filtration by order of pole on $D = \mathrm{Spec} \mathcal{O}$, transferred to the subring \mathcal{O}_C , and the smoothness condition $\mathfrak{a}^+ = \mathfrak{a}_+$ is the Lie algebra version of the normality condition on C (so the group A^+/A_+ “measures” the singularity).

5.4. Regular Centralizers

Heisenberg algebras are classified up to LG conjugacy by the associated spectral curves over D^\times . Their classification up to LG_+ conjugacy reflects the geometry of curves over D . While we do not have a general theory of completed spectral curves for arbitrary Heisenberg algebras, such a theory is available (thanks to [D, DM, DG]) for the large class of regular Heisenberg algebras.

Since $G/N(H)$ parameterizes the Cartan subalgebras of \mathfrak{g} , it embeds into the Grassmannian $G^{\ell}(\mathfrak{g})$ of ℓ -dimensional subspaces of \mathfrak{g} . Let $\widetilde{G/N(H)}$ be the variety of all abelian subgroups of G , which are centralizers of regular elements. It also embeds into $G^{\ell}(\mathfrak{g})$ and hence can be thought of as a partial compactification of $G/N(H)$.

5.4.1. Definition. — A Heisenberg subgroup A is regular if the classifying map $C_A : D^\times \rightarrow G/N(H)$ extends to a map $C_{A_+} : D \rightarrow \widetilde{G/N(H)}$.

5.4.2. Equivalently, A is regular if the fiber of A_+ at ∞ is a regular centralizer in G .

In order to extend the spectral curve $D^\times[A] \rightarrow D^\times$ over ∞ , we introduce the scheme $\widetilde{G/H}$ which classifies pairs (R, B) with $R \in \widetilde{G/N(H)}$ a regular centralizer, and B Borel subgroup of G containing R . There is an obvious morphism $\widetilde{G/H} \rightarrow \widetilde{G/N(H)}$, and the fiber over R may be identified with the fixed point scheme \mathcal{B}^R of R in the flag variety $\mathcal{B} = G/B$.

5.4.3. Definition. — The (completed) spectral curve $D[A]$ of a regular Heisenberg A is the pullback to D of the morphism $\widetilde{G/H} \rightarrow \widetilde{G/N(H)}$ under the classifying map C_{A_+} of A .

5.4.4. Proposition. — Let $A, A' \subset \text{LG}$ be regular Heisenbergs. Then A, A' are LG_+ -conjugate if and only if the spectral curves $D[A]$ and $D[A']$ are isomorphic.

5.4.5. Proof. — It is clear that if A, A' are LG_+ -conjugate, then $D[A] \simeq D[A']$. To prove the converse, we note that there is a one-to-one correspondence between the set of LG_+ -conjugacy classes of regular Heisenbergs with a fixed spectral curve C and the set of isomorphism classes of Higgs bundles over D with spectral cover C (see § 6). A theorem from [DG] states that the category of G -Higgs bundles on X with fixed spectral cover \tilde{X} carries a simply transitive action of the Picard category of torsors over a specific (abelian) group scheme \tilde{T} on X . In the case $X = D$, all G - and \tilde{T} -torsors are trivial, and so we obtain from that statement that there is a unique up to isomorphism Higgs bundle over D with spectral cover C . Therefore there is a unique LG_+ -conjugacy class of regular Heisenbergs with a fixed spectral curve C .

5.5. Examples

5.5.1. The principal Heisenberg is smooth, reflecting the smoothness of the n -fold branched cover defined by taking the n -th root of z . In fact, the principal gradation on \mathfrak{a} is a refinement of the canonical filtration. In the \mathfrak{sl}_2 case, the generators

$$p_i = \begin{pmatrix} 0 & z^{1-i} \\ z^{-i} & 0 \end{pmatrix} \in \mathfrak{a}_+, \quad i > 0$$

of \mathfrak{a}_+ are conjugate to elements

$$\begin{pmatrix} z^{-i+1/2} & 0 \\ 0 & -z^{-i+1/2} \end{pmatrix},$$

which lie in the positive part of the homogeneous Heisenberg subalgebra of $\mathfrak{sl}_2((z^{-1/2}))$.

The centralizer of

$$\begin{pmatrix} 0 & 1 \\ z^{-2} & 0 \end{pmatrix} \in \mathbf{L}\mathfrak{sl}_2,$$

is, as in the principal case, a regular nilpotent centralizer at ∞ . But this Heisenberg Lie algebra is LG -conjugate to the *homogeneous* Heisenberg.

It is easy to see that it is not smooth. Indeed, denote by

$$\tilde{p}_i = \begin{pmatrix} 0 & z^{2-i} \\ z^{-i} & 0 \end{pmatrix}, \quad i \in \mathbf{Z},$$

the generators of this Lie algebra \mathfrak{a} . The matrix

$$(5.5.1) \quad \begin{pmatrix} -\frac{1}{2} & -z \\ -\frac{1}{2}z^{-1} & 1 \end{pmatrix}$$

conjugates \tilde{p}_i to

$$\begin{pmatrix} z^{-i+1} & 0 \\ 0 & -z^{-i+1} \end{pmatrix}.$$

Hence we see that \mathfrak{a}_+ is generated by $\tilde{p}_i, i > 1$, while \mathfrak{a}^+ is generated by $\tilde{p}_i, i > 0$.

Because \mathfrak{a} is not smooth, it is not LG_+ -conjugate to $\text{L}\mathfrak{h}$, as one can see from the explicit formula (5.5.1) for one of the conjugating elements.

The spectral curve $\mathbf{D}^\times[A]$ is isomorphic to the trivial \mathbf{Z}_2 -cover of \mathbf{D}^\times (the same as the spectral curve of LH). But the completed spectral curve is singular: it has two irreducible components, with a simple node over ∞ .

The above Heisenberg is obtained from the pullback of the principal Heisenberg to its own spectral curve, which is automatically homogeneous. More generally, let \mathfrak{a} be a Heisenberg subalgebra with the limit $\mathfrak{a}_\infty \subset \text{Lg}_+/\text{Lg}_{>1} \cong \mathfrak{g}$. By considering the pullback of \mathfrak{a} to its spectral curve we obtain an example of a Heisenberg subalgebra that LG -conjugate to $\text{L}\mathfrak{h}$ but with a limit \mathfrak{a}_∞ at ∞ .

5.5.2. Since there are continuous families of non-isomorphic local singularities, it is easy to find continuous families of Heisenbergs none of which are LG_+ conjugate. For example, one can consider a Heisenberg of homogeneous type for SL_4 , whose 4-sheeted spectral curve is planar and isomorphic to four copies of \mathbf{D} joined at ∞ . The tangent lines to the four components define four points in the projectivized Zariski tangent space at ∞ . The cross ratio of the resulting four points in \mathbf{P}^1 is an invariant of the curve (thus of the associated Heisenberg) which may be varied continuously. Below we will assign integrable systems to LG_+ -conjugacy classes of Heisenberg subgroups of LG . It follows that there are continuous families of integrable systems obtained by our construction.

5.6. Filtrations

In this section we prove some technical results concerning filtrations, which will be useful when we consider generalizations of the notion of an oper in § 7 (in particular Proposition 7.3.7).

5.6.1. The homogeneous Heisenberg algebra $\text{L}\mathfrak{h} \subset \text{L}\mathfrak{g}$ has a strong compatibility property with the homogeneous filtration § 4.1.1. Denote by $\text{L}\mathfrak{h}_{\geq i}$ the i -th piece of the induced filtration, $\text{L}\mathfrak{h}_{\geq i} = \text{L}\mathfrak{h} \cap \text{L}\mathfrak{g}_{\geq i}$. In particular $\text{L}\mathfrak{h}_{>0} = \text{L}\mathfrak{h}_{\geq 1}$ is the Lie algebra of loops to \mathfrak{h} which vanish at ∞ . To an element $p_i \in \text{L}\mathfrak{h}_{\geq i}$ we associate its (principal) symbol

$$\bar{p}_i = p_i \bmod \text{L}\mathfrak{g}_{>i} \in \text{L}\mathfrak{g}_{\geq i}/\text{L}\mathfrak{g}_{>i} \cong \mathfrak{g}.$$

5.6.2. Definition. — We say that p_i is *strongly regular* if its symbol $\bar{p}_i \in \mathfrak{g}$ is a regular element, i.e., if the centralizer of \bar{p}_i in \mathfrak{g} is precisely \mathfrak{h} .

5.6.3. Lemma. — Suppose $p_i \in \mathbf{L}\mathfrak{h}_{\geq i}$ is strongly regular. Then:

- (1) p_i is regular.
- (2) $\text{Ker}(\text{ad } p_i) = \mathbf{L}\mathfrak{h}$ and $\mathbf{L}\mathfrak{g} \cong \mathbf{L}\mathfrak{h} \oplus \text{Im}(\text{ad } p_i)$.
- (3) The operator $\text{ad } p_i$ induces isomorphisms $\mathbf{L}\mathfrak{g}_{\geq k}/\mathbf{L}\mathfrak{h}_{\geq k} \rightarrow \mathbf{L}\mathfrak{g}_{\geq k+i}/\mathbf{L}\mathfrak{h}_{\geq k+i}$.

5.6.4. Proof. — We may pick a coordinate z on \mathbf{D} , thereby picking a gradation refining the homogeneous filtration. Then we may write $p_i = \bar{p}_i + \sum_{j>i} \bar{p}_j$. Suppose $a = \sum_{k=k_0} a_k \in \mathbf{L}\mathfrak{g}$ is the graded decomposition of an element satisfying $[p_i, a] = 0$. By equating each graded component of the commutator to zero, we find equations $\sum_{j=0}^n [\bar{p}_{i+j}, a_{k_0+n-j}] = 0$. By induction on n , and using the regularity of \bar{p}_i , we obtain that each $a_k \in \mathfrak{h}$ and hence $a \in \mathbf{L}\mathfrak{h}$, establishing part (1).

Now since all elements of $\mathbf{L}\mathfrak{h}$ are semisimple, p_i is regular semisimple, and we obtain part (2).

Since the filtration $\{\mathbf{L}\mathfrak{g}_{\geq k}\}$ is a Lie algebra filtration, and since by part (2) we have an $\text{ad } p_i$ -invariant decomposition of $\mathbf{L}\mathfrak{g}$, we obtain a well defined operator $\mathbf{L}\mathfrak{g}_{\geq k}/\mathbf{L}\mathfrak{h}_{\geq k} \rightarrow \mathbf{L}\mathfrak{g}_{\geq k+i}/\mathbf{L}\mathfrak{h}_{\geq k+i}$ as required, only depending on the symbol \bar{p}_i . The fact that it is an isomorphism is now an easy consequence of the regularity of the symbol, as may be checked using the z -gradation.

5.6.5. Definition. — Let $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$ be a general Heisenberg subalgebra. By a *filtration associated with $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$* we will understand a filtration on $\mathbf{L}\mathfrak{g}$ induced by the homogeneous filtration on $\mathbf{L}\mathfrak{g}_{\mathbb{C}}$ via the homomorphism $\text{Ad } g : \mathbf{L}\mathfrak{g} \rightarrow \mathbf{L}\mathfrak{g}_{\mathbb{C}}$, where g is an element of $\mathbf{L}\mathfrak{G}_{\mathbb{C}}$, such that $g\mathfrak{a}_{\mathbb{C}}g^{-1} = \mathbf{L}\mathfrak{h}_{\mathbb{C}}$, where $\mathfrak{a}_{\mathbb{C}}$ is the pullback to $\mathbf{L}\mathfrak{g}$ of \mathfrak{a} to its own spectral curve \mathbb{C} .

5.6.6. The restriction of the above filtration to $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$ is canonical, i.e., it does not depend on the choice of g . But the filtration on the whole $\mathbf{L}\mathfrak{g}$ does depend on the choice of g , because g is specified by the above condition only up to left $\mathbf{N}(\mathbf{L}\mathfrak{H}_{\mathbb{C}})$ multiplication. We do not know how to endow $\mathbf{L}\mathfrak{g}$ with a canonical filtration that restricts to the canonical filtration on \mathfrak{a} defined in § 5.3.5. However, any of these many filtrations (for varying g) have the following nice property.

5.6.7. Definition. — An element $p_i \in \mathfrak{a}$ is *strongly regular* if it corresponds to a strongly regular element in $\mathbf{L}\mathfrak{h}_{\mathbb{C}}$.

5.6.8. Lemma. — Let $\{\mathbf{L}\mathfrak{g}^{\geq k}\}$ be any filtration on the loop algebra associated with $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$ as above. If $p_i \in \mathfrak{a}^{\geq i}$ is strongly regular, then $\text{ad } p_i$ induces isomorphisms $\mathbf{L}\mathfrak{g}^{\geq k}/\mathfrak{a}^{\geq k} \rightarrow \mathbf{L}\mathfrak{g}^{\geq k+i}/\mathfrak{a}^{\geq k+i}$.

5.6.9. Proof. — Note first that the centralizer of p_i in \mathbf{Lg} is the intersection of its centralizer \mathfrak{a}_C in \mathbf{Lg}_C with \mathbf{Lg} , hence \mathfrak{a} , so that p_i is indeed a regular element of \mathbf{Lg} and $\mathbf{Lg} \cong \text{Ker}(\text{ad } p_i) \oplus \text{Im}(\text{ad } p_i)$. Now the statement of the lemma for \mathbf{Lg}_C and \mathfrak{a}_C follows immediately from the corresponding statement for \mathbf{Lh}_C .

Let us decompose \mathbf{Lg}_C into characters for the Galois group of $C \twoheadrightarrow D^\times$, producing a Lie algebra gradation. The subalgebra $\mathbf{Lg} \subset \mathbf{Lg}_C$ consists of the invariants of this action, the zeroth graded component. Since $p_i \in \mathbf{Lg}$, $\text{ad } p_i$ preserves the gradation on \mathbf{Lg}_C . It follows that the statement of the lemma must be true component by component in \mathbf{Lg}_C , and hence in \mathbf{Lg} itself, as desired.

5.6.10. Remark. — The most important classes of Heisenbergs, the homogeneous and principal, have canonical filtrations associated with them and contain many strongly regular elements. It is not true unfortunately that every Heisenberg algebra contains strongly regular elements. However this property depends only on the type of \mathfrak{a} , and not on its fine structure: if \mathfrak{a} is LG-conjugate to \mathfrak{a}' then \mathfrak{a} contains strongly regular elements precisely when \mathfrak{a}' does. Thus one may inquire for which conjugacy classes $[w]$ in the Weyl group Heisenbergs of type $[w]$ contain strongly regular elements. It suffices to answer this question for graded Heisenbergs, and in this setting it has been shown in [DF] (see also [Fe]), that the graded Heisenberg of type $[w]$ contains strongly regular elements precisely when $[w]$ is a regular conjugacy class of the Weyl group. Those have been previously classified by Springer. For instance, in the case of $\mathfrak{g} = \mathfrak{sl}_n$ these conjugacy classes correspond to partitions on n either into equal integers, or into equal integers plus 1 (see [FHM]).

The generalized Drinfeld-Sokolov construction of integrable systems described below is only applicable to Heisenbergs of these types. However, within each “topological” type of Heisenbergs, there usually exist continuous families of Heisenbergs, which are not LG₊-conjugate, and therefore continuous families of integrable hierarchies.

6. Abelianization

In this section we combine the contents of the previous two to produce a class of interesting moduli spaces, the abelianized Grassmannians $\mathcal{G}r_A^X$, parameterizing bundles with additional structure on the disc. This structure is a reduction to the positive part A_+ of a Heisenberg subgroup of LG. These moduli spaces carry actions of abelian (ind-) groups A/A_+ , deforming the abelian structure on the disc. Following the prescription of § 2, we use these actions to relate these moduli to moduli of special connections, which will turn out to be affine opers (see § 7).

We then identify the abelianized Grassmannians (when A is a *regular* Heisenberg) with moduli spaces of Higgs bundles. Higgs bundles have a natural interpretation in terms of line bundles on spectral curves, thus “abelianizing” our moduli spaces by comparing them to Picard varieties. In particular, we will identify the ind-group A/A_+

as a generalized Prym variety associated with the spectral curve attached to A . This allows us to interpret the action of A/A_+ on $\mathcal{G}r_A^X$ as a generalization of the Jacobian flows in the Krichever construction § 3. In § 8 we identify the action of A/A_+ on $\mathcal{G}r_A^X$ with a hierarchy of generalized KdV flows on the space of affine opers.

6.1. Abelianized Grassmannians

Let $A \subset LG$ be a Heisenberg subgroup, with Lie algebra \mathfrak{a} . Let $A_+ = A \cap LG_+$ be its positive half, with Lie algebra \mathfrak{a}_+ .

6.1.1. Definition. — The A -Grassmannian $\mathcal{G}r_A^X$ is the moduli stack of G -torsors on X , equipped with a reduction \mathcal{E}^{A_+} of the LG_+ -torsor $\mathcal{E}|_D$ to A_+ .

6.1.2. Lemma. — $\mathcal{G}r_A^X \cong \mathcal{G}r^X/A_+$.

6.1.3. Proof. — Since $\mathcal{G}r^X$ parameterizes G -torsors on X with an isomorphism $G|_D \cong \mathcal{E}|_D$, there is a natural surjection $\mathcal{G}r^X \rightarrow \mathcal{G}r_A^X$, and the group A_+ acts transitively along the fibers.

6.1.4. Remark. — The relevance of the double quotient space $LG_-^X \backslash LG/A_+$ and its adélic versions to the study of integrable systems has been pointed out in [EF1].

6.1.5. Since $\mathcal{G}r_A^X$ is the quotient of the scheme $\mathcal{G}r^X$ by the action of the group A_+ , it is an (Artin) algebraic stack. $\mathcal{G}r_A^X$ does not, however, possess a coarse moduli scheme. This is because the automorphisms of different bundles $\mathcal{E} \in \mathcal{G}r_A^X$ may be very different. These automorphisms are given by the sections of the adjoint group-scheme $\text{Ad } \mathcal{E}^{A_+}$ of the A_+ -torsor associated with \mathcal{E} , which extend to global sections of the group-scheme $\text{Ad}(\mathcal{E})$. Thus if \mathcal{E} is trivial as a G -torsor, $\text{Aut}(\mathcal{E}) \cong A_+ \cap G$, since $G = LG_-^X \cap LG_+$ are the global sections of the trivial G -torsor. For general \mathcal{E} the automorphisms will be isomorphic to the intersection of A_+ with some conjugate of LG_-^X which is not transversal to LG_+ .

However, recall that (in the case $X = \mathbf{P}^1$) $\mathcal{G}r^{\mathbf{P}^1}$ has a big cell $\mathcal{G}r^\circ$, which is canonically isomorphic to $G \backslash LG_+ \simeq LG_{>0}$. Define the big cell $\mathcal{G}r_A^\circ$ of $\mathcal{G}r_A^{\mathbf{P}^1}$ as the image of $\mathcal{G}r^\circ$ under the projection $\mathcal{G}r^{\mathbf{P}^1} \rightarrow \mathcal{G}r_A^{\mathbf{P}^1}$. While $\mathcal{G}r_A^{\mathbf{P}^1}$ is the moduli stack of triples $(\mathcal{V}, \mathcal{V}_-, \mathcal{V}^{A_+})$, where \mathcal{V} is an LG -torsor with LG_- -reduction \mathcal{V}_- and A_+ -reduction \mathcal{V}^{A_+} , $\mathcal{G}r_A^\circ$ is its open substack that classifies those triples for which the LG_- -reduction \mathcal{V}_- and the induced LG_+ -reduction $\mathcal{V}^{A_+} \times_{A_+} LG_+$ are in general position.

We want to show that $\mathcal{G}r_A^\circ$ possesses a coarse moduli space, Gr_A° . Hence we will be able to translate results concerning the stack $\mathcal{G}r_A^\circ$ into results concerning the scheme Gr_A° . We start with a preliminary result.

6.1.6. Lemma.

- (1) Let $A \subset LG$ be a Heisenberg subgroup, and $G \subset LG$ the subgroup of constants. Then either A is LG_+ -conjugate to a homogeneous Heisenberg LH (where H is a Cartan subgroup of G), in which case $A \cap G$ is a Cartan subgroup of G ; or $A \cap G$ is the center of G .
- (2) Let $A_0 = A \cap G$, $A'_+ = A \cap LG_{>0}$, for homogeneous A , and $A'_+ = \exp \mathfrak{a}_+$ for all other A . Then $A_+ = A_0 \cdot A'_+$.

6.1.7. Proof. — An element $a \in A \cap G$ is a constant section of the group scheme \underline{A} on D . Its value at the base point $\infty \in D$ is then a semi-simple element of G . Thus, the fiber A_∞ of \underline{A} is an abelian subgroup of G of dimension greater than or equal to ℓ , containing a semi-simple element a . It follows that either A_∞ is a Cartan subgroup, or a is a central element of G . This proves part (1). Part (2) follows immediately from part (1).

6.1.8. Lemma. — $\mathcal{G}r_A^\circ$ is isomorphic to the quotient of the affine scheme of infinite type $\text{Gr}_A^\circ = G \backslash LG_+ / A'_+$ by the trivial action of the group A_0 .

6.1.9. Proof. — Since $\mathcal{G}r^\circ \simeq G \backslash LG_+$, $\mathcal{G}r_A^\circ$ is isomorphic to $G \backslash LG_+ / A_+$. By Lemma 6.1.6 (2), it is isomorphic to the quotient of $G \backslash LG_+ / A'_+$ by the trivial action of A_0 . Since $gA'_+g^{-1} \cap G = \{1\}$ for any $g \in LG_+$ by 6.1.6 (1), the pro-unipotent group A'_+ acts freely on $G \backslash LG_+$. It is easy to find locally a transversal slice for this action. Therefore $G \backslash LG_+ / A'_+$ is a scheme.

6.1.10. Actions. — The stack $\mathcal{G}r_A^X = LG_-^X \backslash LG / A_+$ occupies an important intermediate position between the Grassmannian $\mathcal{G}r^X = LG_-^X \backslash LG$ and the moduli stack of bundles $\mathcal{M}_G^X = LG_-^X \backslash LG / LG_+$. Unlike the Picard group Pic_X (the moduli scheme of line bundles on X), \mathcal{M}_G^X does not have a group structure and carries no natural group actions. The Grassmannian does carry an action of the entire loop group LG . It is however too big, with “redundant” directions coming from the action of its subgroup LG_+ , which merely changes the trivialization on D .

The abelianized Grassmannian $\mathcal{G}r_A^X$ carries an interesting remnant of the LG action on $\mathcal{G}r^X$, which is similar to the action of the Jacobian on itself. Indeed, the quotient $\mathcal{G}r^X / A_+$ carries the right action of the normalizer of A_+ in LG . In particular, the group A acts on $\mathcal{G}r_A^X$. Of course, the right action of A_+ on $\mathcal{G}r_A^X$ is trivial, and so we are left with the action of the quotient ind-group A / A_+ , discussed in § 5.3.6. Recall that this group has three “parts” — the finite-dimensional group scheme A^+ / A_+ , a finite rank lattice, and the formal group of $\mathfrak{a} / \mathfrak{a}^+$.

6.1.11. Remark. — This action is the key property of $\mathcal{G}r_A^X$. It can also be used as a motivation for studying $\mathcal{G}r_A^X$. Indeed, Lemma 2.3.2 gives us a general procedure for

constructing flat connections from the actions of a group A on a homogeneous space. We take as the homogeneous space, the Grassmannian $\mathcal{G}r^X$. The most interesting case of actions to consider is that of a maximal abelian subgroup. That is why we look at the action of a Heisenberg subgroup A . But as we explained in Remark 4.2.7, the LG_+ -action on $\mathcal{G}r^X$ is not interesting. Therefore we mod out $\mathcal{G}r^X$ by $A_+ = A \cap LG_+$ and look at the residual action of A/A_+ .

6.2. The Principal Grassmannian

In this section we concentrate on the abelianized Grassmannian $\mathcal{G}r_A^X = LG_-^X \backslash LG/A_+$ in the case when $A \subset LG$ is the principal Heisenberg and $X = \mathbf{P}^1$. For $G = GL_n$ it may be described as the moduli stack of rank n vector bundles on \mathbf{P}^1 , identified over D with the pushforward of a line-bundle from the n -fold branched cover $\zeta^n = z$. For general G , $\mathcal{G}r_A^{\mathbf{P}^1}$ classifies G -torsors \mathcal{E} on \mathbf{P}^1 , with a local structure on D that can be described as follows: $\mathcal{E}|_{D^\times}$ is reduced to an abelian group subscheme of the constant group scheme \underline{G} , whose fibers are Cartan subgroups of G undergoing a Coxeter class monodromy around ∞ and degenerating at ∞ to a regular nilpotent centralizer.

6.2.1. The tautological LG -bundle $\mathcal{T}(D^\times)$ on $\mathcal{G}r_A^{\mathbf{P}^1}$ comes with reductions $\mathcal{T}(\mathbf{P}^1 \setminus \infty)$ and $\mathcal{T}^{\Lambda_+}(D) \subset \mathcal{T}(D)$ to LG_- and $A_+ \subset LG_+$, respectively. Since A_+ defines a unique Borel subgroup $B \subset G$ at ∞ , it follows that the fiber of any $\mathcal{E} \in \mathcal{G}r_A^{\mathbf{P}^1}$ at ∞ has a canonical flag. In particular, $\mathcal{T}(D^\times)$ has a canonical reduction to the Iwahori subgroup $LG^+ \subset LG_+$, whose sections take values in B at ∞ .

Recall that since A is smooth and “maximally twisted”, i.e. anisotropic, the ind-group A/A_+ is actually a formal group. Thus the formal group A/A_+ and its Lie algebra $\mathfrak{a}/\mathfrak{a}_+$ act on $\mathcal{G}r_A^{\mathbf{P}^1}$. We are particularly interested in the action of the element $p_{-1} \in \mathfrak{a}/\mathfrak{a}_+$ and of the formal one-dimensional additive group $\widehat{A}_{-1} = \{e^{p_{-1}}\}$ that it generates.

6.2.2. We are now in the setting of our general result, Proposition 2.3.12, describing the correspondence between the double quotients $H \backslash G/K$ equipped with an action of a group A and the moduli of certain flat bundles on A . Namely, we take $\mathcal{G}r_A^{\mathbf{P}^1} = LG_- \backslash LG/A_+$ as the double quotient and \widehat{A}_{-1} as the group A . The moduli stack of flat bundles on the other side of the correspondence classifies quadruples $(\mathcal{Z}, \nabla, \mathcal{Z}_-, \mathcal{Z}^{\Lambda_+})$, where \mathcal{Z} is an LG -torsor on \widehat{A}_{-1} with a flat connection ∇ , a flat LG_- -reduction, and an A_+ -reduction \mathcal{Z}^{Λ_+} -reduction in tautological relative position with respect to ∇ ⁽⁵⁾. We denote this moduli stack by $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1}$.

⁽⁵⁾ Since \widehat{A}_{-1} is isomorphic to the formal disc, ∇ automatically induces a trivialization of \mathcal{Z} and \mathcal{Z}_- .

On the other hand, we can consider the action of the whole group A/A_+ on $\mathcal{G}r_A^{\mathbf{P}^1}$. The corresponding moduli space $\mathcal{M}_A^{\mathbf{P}^1}$ classifies quadruples as above defined on all of A/A_+ rather than on \widehat{A}_{-1} . Then Proposition 2.3.12 gives us the following result.

6.2.3. Proposition. — $\mathcal{G}r_A^{\mathbf{P}^1}$ is canonically isomorphic to $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1}$, and to $\mathcal{M}_A^{\mathbf{P}^1}$.

6.3. General Case

Now let X be an arbitrary smooth curve, ∞ a point of X , and LG the loop group corresponding to the formal neighborhood of ∞ . Recall that LG has subgroups LG_- and LG_+ . Let A be an arbitrary Heisenberg subgroup of LG , $A_+ = A \cap LG_+$, and $\mathfrak{a}, \mathfrak{a}_+$ be the Lie algebras of A, A_+ , respectively. Each non-zero element $p \in \mathfrak{a}/\mathfrak{a}_+$ gives rise to a one-dimensional formal additive subgroup $\widehat{A}_p = \{e^{tp}\}$ of the group A/A_+ .

Consider the corresponding abelianized Grassmannian $\mathcal{G}r_A^X$. The group \widehat{A}_p acts on $\mathcal{G}r_A^X$ from the right, and we can again apply Proposition 2.3.12. Denote by $\mathcal{M}_{A,p}^X$ the moduli stack that classifies quadruples $(\mathcal{Z}, \nabla, \mathcal{Z}_-, \mathcal{Z}^{A_+})$, where \mathcal{Z} is an LG -torsor on \widehat{A}_{-1} with a flat connection ∇ , a flat LG_- -reduction \mathcal{Z}_- , and an A_+ -reduction \mathcal{Z}^{A_+} in tautological relative position with ∇ .

Similarly, \mathcal{M}_A^X will denote quadruples as above, but defined on all of A/A_+ rather than on \widehat{A}_p . Then we obtain the following generalization of Proposition 6.2.3.

6.3.1. Proposition. — $\mathcal{G}r_A^X$ is canonically isomorphic to $\mathcal{M}_{A,p}^X$ and to \mathcal{M}_A^X .

6.3.2. Differential Setting. — Rather than consider connections on \widehat{A}_p , we may follow the prescription of § 2.4 and introduce differential schemes. Thus \mathcal{Z} will now be a LG -torsor on a differential scheme (S, ∂) with a lifting $\partial_{\mathcal{Z}}$ of ∂ , and reductions to LG_- and A_+ which are respectively preserved and in relative position $[-p]$ with respect to $\partial_{\mathcal{Z}}$. Proposition 2.4.1 immediately implies

6.3.3. Proposition. — The pair $(\mathcal{G}r_A^X, p)$ represents the functor which assigns to a differential scheme (S, ∂) the groupoid of quadruples $(\mathcal{Z}, \partial_{\mathcal{Z}}, \mathcal{Z}_-, \mathcal{Z}^{A_+})$ as above.

6.4. Higgs Fields and Formal Jacobians

Recall that the Heisenberg algebra \mathfrak{a} is represented by a classifying map $C_A : D^\times \rightarrow G/N(\mathfrak{H})$. Since $G/N(\mathfrak{H})$ is embedded into the Grassmannian $\text{Gr}^\ell(\mathfrak{g})$, which is proper, we can extend this map to a map $D \rightarrow \text{Gr}^\ell(\mathfrak{g})$. The image of the base point of D under this map will be an abelian Lie subalgebra of G . Intuitively, a reduction \mathcal{E}^{A_+} of $\mathcal{E}|_D$ to A_+ is a reduction of the G -torsor \mathcal{E} on D^\times to a Cartan subgroup, which twists in a prescribed way around ∞ , and degenerates at ∞ to a reduction of \mathcal{E}_∞ to the limiting abelian subgroup of G .

6.4.1. Definition [DM].

- (1) A (regular) principal Higgs field on a G -torsor \mathcal{E} over a scheme Y is a sub-bundle $c \subset \text{ad}(\mathcal{E})$ of regular centralizers.
- (2) The spectral cover associated with a Higgs bundle (\mathcal{E}, c) on Y is the scheme $Y[c] \rightarrow Y$ parameterizing Borel subgroups of $\text{Ad } \mathcal{E}$ containing c . More precisely, $Y[c]$ is the fixed point scheme $(\mathcal{B})_{\mathcal{E}}^c$ of c on the relative flag manifold $(\mathcal{B})_{\mathcal{E}} = \mathcal{E} \times_G G/B$.

6.4.2. Remark. — The spectral cover $Y[c]$ may also be defined as follows. Let $\widetilde{G/N(H)}$ be the partial compactification of $G/N(H)$ parameterizing regular centralizers in G . Let $\widetilde{G/H} \rightarrow \widetilde{G/N(H)}$ be the scheme parameterizing pairs (r, B) where $r \in \widetilde{G/N(H)}$ is a regular centralizer and $B \subset G$ is a Borel subgroup containing c . Now trivialize the bundle \mathcal{E} locally on some flat covering $Y' \rightarrow Y$, so that c defines a local classifying map $Y' \rightarrow \widetilde{G/N(H)}$. The pullback to Y' of the morphism $\widetilde{G/H} \rightarrow \widetilde{G/N(H)}$ is independent of the trivialization up to isomorphism. Thus the resulting covers $Y'[c] \rightarrow Y'$ glue together to give a scheme $Y[c]$ over Y . (Without choosing trivializations, we obtain a classifying morphism $Y \rightarrow G \backslash \widetilde{G/N(H)}$, and $Y[c]$ is the pullback of the representable morphism $G \backslash \widetilde{G/H} \rightarrow G \backslash \widetilde{G/N(H)}$.)

6.4.3. Recall from § 5.1 the relation between Cartan subgroups of G and reductions: a reduction of the trivial G -torsor to a Cartan subgroup H (in other words, a point $s \in G/H$) is equivalent to the data of a subgroup of G conjugate to H , together with a distinguished group isomorphism $\phi : H' \cong H$. Equivalently, we may replace the isomorphism ϕ by the data of an identification between the set of Borels containing H' and those containing H . This is a consequence of the fact that $N(H)/H$ acts faithfully on the set \mathcal{B}^H of Borels containing H .

6.4.4. Proposition [DG]. — Let $R \subset G$ be a regular centralizer.

- (1) For any regular element $r \in R$, the fixed point scheme \mathcal{B}^R is identified with $\pi^{-1}(\text{char}(r))$, where $\pi : \mathfrak{h} \rightarrow \mathfrak{h}/W$ and $\text{char} : \mathfrak{g} \rightarrow \mathfrak{h}/W$ is the adjoint quotient map. In particular, \mathcal{B}^R carries a natural action of the Weyl group $W = N(H)/H$.
- (2) There is a morphism $R \rightarrow \text{Hom}_W(\mathcal{B}^R, H)$, which is an isomorphism for G simply connected. Thus $N(R)/R$ acts faithfully on \mathcal{B}^R .
- (3) The data of a reduction of G to R is equivalent to the data of a regular centralizer $R' \subset G$ and an identification of W -schemes $\mathcal{B}^{R'} \cong \mathcal{B}^R$.

6.4.5. Corollary. — Let $A \subset LG$ be a regular Heisenberg, with positive part $A_+ \subset A$. Then $\mathcal{G}r_A^X$ is naturally isomorphic to the moduli stack of G -torsors \mathcal{E} on X , equipped with a principal Higgs field c on $\mathcal{E}|_D$ and an isomorphism $D[c] \cong D[A_+]$ of spectral covers over D .

6.4.6. Proof. — The corollary is a version of Proposition 6.4.4 (3) over D . Recall from Definitions 5.4.3 and 6.4.1 that the spectral cover of a Heisenberg algebra or Higgs field is the global version of the fixed point schemes \mathcal{B}^R . A reduction of $\mathcal{E}|_D$ to A_+ is determined by a section of the associated scheme $(\underline{G}/\underline{A}_+)_{\mathcal{E}} = \mathcal{E} \times_{\underline{G}} \underline{G}/\underline{A}_+$ over D . Using Proposition 6.4.4, we can identify sections of $\underline{G}/\underline{A}_+$ with pairs (c, ϕ) , where $c \in \underline{G}/N(\underline{A}_+)$ is a subgroup of \underline{G} and $\phi : D[c] \rightarrow D[A_+]$ is an isomorphism of spectral covers. It follows that sections of $(\underline{G}/\underline{A}_+)_{\mathcal{E}}$ are identified with pairs (c, ϕ) , where $c \in (\underline{G}/N(\underline{A}_+))_{\mathcal{E}}$ is a principal Higgs field on $\mathcal{E}|_D$ and $\phi : D[c] \rightarrow D[A_+]$ is an isomorphism of spectral covers. This implies the Corollary.

6.4.7. Higgs Bundles and Line Bundles. — The most important feature of Higgs bundles on a scheme X is their relation to line bundles on the associated spectral cover. We offer a brief review of the theory as developed in [D,DM] and completed in [DG]. (See also [LM].)

Let us first suppose that $G = GL_n$, and that \mathcal{E} is a rank n vector bundle on X equipped with a subbundle $c \subset \text{ad } \mathcal{E}$ of Cartan subalgebras. Locally, the action of a regular section s of c on \mathcal{E} decomposes \mathcal{E} into a direct sum of n eigen-line bundles, parameterized by the n eigenvalues of s . In other words c defines an n -sheeted étale cover $Y = X[c]_n$ of X , whose points over $x \in X$ represent weights of c_x on \mathcal{E}_x . The cover Y may be recovered from the spectral cover $X[c]$ associated to the Higgs field c as follows: $\pi : Y \rightarrow X$ is the bundle $X[c] \times_{S_n} \{1, \dots, n\}$ associated to the principal S_n -bundle $X[c]$ under the permutation representation of S_n on the set $\{1, \dots, n\}$.

We have expressed $\mathcal{E} \cong \pi_* \mathcal{L}$, where $\pi : Y \rightarrow X$ and \mathcal{L} is a line bundle on Y . Conversely, any line bundle \mathcal{L} on Y pushes forward to a rank n vector bundle, equipped with a canonical semisimple Higgs field c (i.e. bundle of Cartans) preserving the decomposition $\mathcal{E}_x \cong \bigoplus_{\pi^{-1}(x)} \mathcal{L}_x$. In the case $G = SL_n$, we obtain a correspondence between semisimple Higgs bundles (\mathcal{E}, c) on X , and line bundles \mathcal{L} on Y equipped with an isomorphism $\det \pi_* \mathcal{L} \cong \mathcal{O}_X$.

Now suppose the Higgs field c is allowed to be an arbitrary bundle of regular centralizers in $\text{ad } \mathcal{E}$. It is still possible to define an n -sheeted spectral cover $Y = X[c]_n$, which will now be ramified over the locus where c is not a Cartan. Moreover \mathcal{E} will be identified with the pushforward of a line bundle on Y . Conversely, the pushforward of a line bundle from an n -sheeted cover produces a rank n vector bundle. The case when X is a curve, with local coordinate z near ∞ , and $X[c]$ is locally isomorphic to

the cover $\zeta^n = z$, was discussed in § 5.2: the stalk of $\pi_*\mathcal{L}$ at $z = \infty$ will be isomorphic to the stalk of the bundle of $(n-1)$ -jets of \mathcal{L} at ∞ . Thus instead of a direct sum decomposition, we see a flag on the stalk. The flag is defined by multiplication by the coordinate ζ , which acts as a regular nilpotent element on the stalk.

In [D, DM, DG], the case of arbitrary G is worked out in detail. The category of Higgs bundles with a given spectral cover (possibly ramified) is described precisely in terms of Prym varieties (that is, in terms of W -equivariant torus bundles on the spectral covers). We only describe the very first step in this work, to motivate our interpretation of A/A_+ as a Prym variety.

Suppose that G is an arbitrary reductive group and (\mathcal{E}, c) a regular semisimple Higgs bundle on X (so that c is a bundle of Cartans). The W -cover $X[c] \rightarrow X$ is then étale, and the pullback $\mathcal{E}_{X[c]}$ of \mathcal{E} to $X[c]$ carries a tautological reduction to a Borel containing c . Thus given an irreducible representation V of G , the associated bundle $V_{\mathcal{E}_{X[c]}}$ on $X[c]$ has a distinguished line bundle, defined as the highest weight space for the tautological Borel. Thus for every (semisimple) Higgs bundle on X , one obtains a W -equivariant homomorphism from the weight lattice Λ of G to the Picard variety of $X[c]$, in other words an element of the cameral Prym variety:

6.4.8. Definition [DM]. — The cameral Prym variety associated with the Higgs bundle (\mathcal{E}, c) on X is $\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(X[c]))$.

6.4.9. Recall from § 3.3.2 that the ind-group $\mathcal{H}^\times/\mathcal{O}^\times$ plays the role of the Picard of the disc. Definition 6.4.8 then suggests an analogous role for A/A_+ . Let us denote by $\mathcal{H}^\times[A]$ the invertible functions on $D^\times[A]$, considered as a group ind-scheme. Similarly $\mathcal{O}^\times[A]$ will be the group scheme of invertible functions on $D[A]$ and $\mathcal{H}^\times/\mathcal{O}^\times[A]$ the quotient group ind-scheme. The following lemma is an easy consequence of Proposition 6.4.4.

6.4.10. Lemma. — Let G be simply connected, and $A \subset \mathrm{LG}$ a regular Heisenberg subgroup. Then $A \cong \mathrm{Hom}_W(D^\times[A], \mathbf{H}) \cong \mathrm{Hom}_W(\Lambda, \mathcal{H}^\times[A])$, $A_+ \cong \mathrm{Hom}_W(\Lambda, \mathcal{O}^\times[A])$ and $A/A_+ \cong \mathcal{H}^\times/\mathcal{O}^\times[A]$.

6.4.11. Definition. — The Prym variety $\mathrm{Prym}(A)$ associated with a Heisenberg A is the ind-group A/A_+ .

6.4.12. Corollary.

- (1) For any Heisenberg subgroup A of homogeneous type, there is a map $\mathrm{Prym}(A) \rightarrow \mathrm{Prym}(\mathrm{LH})$, uniquely specified up to W .
- (2) For A regular, there is an injection $\mathrm{Prym}(A) \hookrightarrow \mathrm{Prym}(\tilde{A})$, where $\tilde{A} \subset \underline{G}_{D[A]}$ is the Heisenberg subgroup of homogeneous type obtained from pulling A back to $D[A]$.

6.4.13. *Proof.* — The first statement is a consequence of Lemma 5.3.2, which describes \mathfrak{a} in terms of Lh . The second follows from the inclusion $i : A \hookrightarrow \tilde{A}$ and the equality $i(A_+) = i(A) \cap \tilde{A}_+$.

6.4.14. The first part of the corollary is the analog, for an arbitrary Heisenberg subgroup, of the morphism between the Prym varieties corresponding to the pullback of line bundles from a cover $(D[A])$ to its normalization $(D[\text{LH}])$. The second is the analog of the morphism corresponding to pulling back line bundles from a curve (D) to its branched cover $(D[A])$.

6.5. Krichever Construction, Revisited

Let $G = \text{GL}_n$ and $A \subset \text{LGL}_n$ a regular Heisenberg, with n -sheeted spectral cover $D[A]_n$. Let Σ be a projective curve, $i : D[A]_n \hookrightarrow \Sigma$ an inclusion and $\pi : \Sigma \rightarrow X$ a morphism extending $D[A]_n \rightarrow D$. Let $\mathcal{H}^\times / \mathcal{O}^\times[\Sigma]$ denote the ind-group representing the quotient of $\mathcal{O}^\times(\pi^{-1}(D^\times))$ by $\mathcal{O}^\times(\pi^{-1}(D))$.

6.5.1. *Definition.* — The Σ -Grassmannian $\mathcal{G}r_A^X[\Sigma]$ is the moduli stack of GL_n Higgs bundles (\mathcal{E}, c) on X , equipped with an isomorphism $X[c]_n \rightarrow \Sigma$ of the n -sheeted spectral cover of c with Σ .

6.5.2. Proposition.

- (1) The Σ -Grassmannian $\mathcal{G}r_A^X[\Sigma]$ is a substack of $\mathcal{G}r_A^X$, which is preserved by the action of A/A_+ .
- (2) $\mathcal{G}r_A^X[\Sigma] \cong \text{Pic}_\Sigma$.
- (3) There is a natural isomorphism $A/A_+ \cong \mathcal{H}^\times / \mathcal{O}^\times[\Sigma]$, which identifies their actions on Pic_Σ .

6.5.3. *Proof.* — The isomorphism of Corollary 6.4.5 may be reformulated in the case $G = \text{GL}_n$, by replacing spectral covers $D[c]$ with n -sheeted spectral covers $D[c]_n$. It follows that there is a natural morphism $\mathcal{G}r_A^X[\Sigma] \rightarrow \mathcal{G}r_A^X$, obtained by restricting c to D and composing the identification $D[c]_n \rightarrow \pi^{-1}(D) \subset \Sigma$ and the inverse of the inclusion $i : D[A]_n \rightarrow \Sigma$. This morphism is a monomorphism, since $c|_D$ determines the extension c on X (when it exists) and $\pi : \Sigma \rightarrow X$ has no automorphisms preserving i .

The action of A/A_+ on $\mathcal{G}r_A^X$ affects neither the G -torsor $\mathcal{E}|_{X \setminus \infty}$, nor the reduction of $\mathcal{E}|_{D^\times}$ to A (since the action is deduced from an A action, which clearly has this property). It follows that the action preserves the substack $\mathcal{G}r_A^X[\Sigma]$, whose definition depends only on $\mathcal{E}|_{X \setminus \infty}$ and its reduction to A on D^\times . This completes the proof of part (1).

Part (2) of the proposition is the standard identification between the moduli stack of GL_n Higgs bundles with n -sheeted spectral cover isomorphic to Σ and that

of line bundles on Σ ([DG]). Applying this statement to D^\times , we obtain a canonical isomorphism between the automorphism group of a GL_n Higgs bundle on D^\times with spectral curve $D^\times[A]$ and the automorphism group of a (trivial) line bundle on $D^\times[A]$. This gives us an identification $A \cong \mathcal{H}^\times[\Sigma]$. The equivalent statement $A_+ \cong \mathcal{O}^\times[\Sigma]$ over D then implies the desired identification $A/A_+ \cong \mathcal{H}^\times/\mathcal{O}^\times[\Sigma]$. Moreover, it is clear that the actions of A/A_+ on $\mathcal{G}r_A^X[\Sigma]$ and of $\mathcal{H}^\times/\mathcal{O}^\times[\Sigma]$ on Pic_Σ coincide. This concludes the proof.

6.5.4. We may now replace GL_n by an arbitrary reductive group G . Let $A \subset LG$ be a regular Heisenberg, with spectral curve $D[A]$. Let Σ be a projective curve, $i : D[A] \hookrightarrow \Sigma$ an inclusion and $\pi : \Sigma \rightarrow X$ a morphism extending $D[A] \rightarrow D$.

6.5.5. Definition. — The Σ -Grassmannian $\mathcal{G}r_A^X[\Sigma]$ is the moduli stack of G -Higgs bundles (\mathcal{E}, c) on X , equipped with an isomorphism $X[c] \rightarrow \Sigma$ of the spectral cover of c with Σ .

6.5.6. Conclusion. — As in Proposition 6.5.2, the stack $\mathcal{G}r_A^X[\Sigma]$ is naturally a substack of $\mathcal{G}r_A^X$, and is preserved by A/A_+ . By the results of [D, DM, DG], $\mathcal{G}r_A^X[\Sigma]$ has a description in terms of line bundles on Σ . As we observed, A/A_+ serves as a generalized Prym variety for the spectral curve $D[A]$. Thus, the Σ -Grassmannian $\mathcal{G}r_A^X[\Sigma]$ together with the action of A/A_+ is a natural generalization of the setting of the Krichever construction to arbitrary semisimple groups G . The points of $\mathcal{G}r_A^X[\Sigma]$ correspond to an interesting class of global “algebro-geometric” solutions of generalized soliton hierarchies. On the other hand, the total abelianized Grassmannian $\mathcal{G}r_A^X$ captures all formal solutions of those hierarchies, as we will see in § 8.

6.5.7. Remark. — Cherednik [Ch1, Ch2, Ch3] also considers versions of the stacks $\mathcal{G}r_A^X[\Sigma]$ as parameterizing generalized algebro-geometric solutions to soliton equations. He uses the language of reductions to subgroups of the group scheme $G \times X$ over X , which are generically Cartans. As in Corollary 6.4.5, this is essentially equivalent to the notion of principal Higgs bundles.

7. Opers

In this section we study special connections, introduced by Drinfeld-Sokolov [DS] and Beilinson-Drinfeld [BD1] under the name opers, and their “spectral” generalizations, affine opers. Opers may be defined on any curve, and have an interpretation in terms of differential operators. Opers are also equivalent to a special class of affine opers. The Drinfeld-Sokolov gauge (Proposition 7.3.7) relates affine opers with the connections produced in § 6. Tying the loose ends together, we obtain in the

next section a broad generalization of the relation between line bundles on a curve and differential operators reviewed in § 3.

7.1. *Introducing Opers*

Let G be a reductive Lie group and B a Borel subgroup, with Lie algebras $\mathfrak{b} \subset \mathfrak{g}$. There is a distinguished B -orbit $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b}$, which is open in the subspace of vectors stabilized by the radical $N \subset B$. \mathbf{O} consists of vectors which have precisely their negative simple root components nonzero — that is, the B - (or H -)orbit of the sum of the Chevalley generators f_i in $\mathfrak{g}/\mathfrak{b}$. In the case of GL_n , \mathbf{O} can be identified with the set of matrices of the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ + & * & * & \cdots & * \\ 0 & + & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & + & * \end{pmatrix}$$

where the $+$ represent arbitrary nonzero entries.

Now recall the notion of relative position of a reduction of a bundle with respect to a connection given in § 2.1, and the notion of GL_n -oper from § 3.1.

7.1.1. Definition. — Let Y be a smooth curve. A G -oper on Y is a G -torsor \mathcal{E} on Y with a connection and a reduction \mathcal{E}_B to B , which has relative position \mathbf{O} with respect to ∇ .

In other words, the one-form ∇/\mathcal{E}_B takes values in $\mathbf{O}_{\mathcal{E}_B} \otimes \Omega^1 \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{E}_B} \otimes \Omega^1$ (the orbit \mathbf{O} is \mathbf{C}^\times -invariant).

The G -opers on Y form a stack denoted by $\mathcal{O}p_G(Y)$, or $\mathcal{O}p(Y)$ when there is no ambiguity. A \mathfrak{g} -oper is by definition an oper for the adjoint group of \mathfrak{g} .

7.1.2. Identifying GL_n -principal bundles equipped with B -reductions with rank n vector bundles equipped with flags, we see that this notion agrees with the GL_n -opers as introduced in § 3.1. Thus in the case of GL_n , opers are identified with n -th order differential operators with principal symbol 1. The SL_n -opers correspond (in the notation of § 3.1) to the case $q_1 = 0$, that is differential operators with vanishing subprincipal symbol. For the classical series B_n and C_n , we obtain differential operators which are either self- or skew-adjoint.

7.1.3. Lemma [BD1]. — If G is a group of adjoint type, then the moduli stack $\mathcal{O}p_G(Y)$ of G -opers on Y is an affine ind-scheme denoted by $Op_{\mathfrak{g}}(Y)$ (which is in fact a scheme for Y projective or $Y = \widehat{D}$). For general G , the stack $\mathcal{O}p_G(Y)$ is the quotient of the (ind-)scheme $\mathcal{O}p_{\mathfrak{g}}(Y)$ of \mathfrak{g} -opers by the trivial action of the center $Z(G)$.

7.1.4. Let $\widehat{\mathbf{D}}$ be the formal disc. In § 8 we will need an explicit description of the scheme $\mathbf{Op}_{\mathfrak{g}}(\widehat{\mathbf{D}}) = \mathbf{Op}(\widehat{\mathbf{D}})$.

Consider $(\mathcal{E}, \nabla, \mathcal{E}_{\mathbf{B}}) \in \mathbf{Op}_{\mathfrak{g}}(\widehat{\mathbf{D}})$. If we choose a trivialization of $\mathcal{E}_{\mathbf{B}}$, then the connection operator ∇ can be written as

$$(7.1.1) \quad \partial_t + \sum_{i=1}^{\ell} \phi_i(t) \cdot f_i + \mathbf{b}(t).$$

Here t is a coordinate on $\widehat{\mathbf{D}}$, $\phi_i(t)$ are invertible elements of $\mathbf{C}[[t]]$, and $\mathbf{b}(t) \in \mathfrak{b}_+[[t]]$.

Let $\widetilde{\mathbf{Op}}(\widehat{\mathbf{D}})$ be the affine space of operators of the form (7.1.1). The group $\mathbf{B}_+[[t]]$ acts on $\widetilde{\mathbf{Op}}(\widehat{\mathbf{D}})$ by gauge transformations corresponding to the changes of trivialization. We have: $\mathbf{Op}(\widehat{\mathbf{D}}) \simeq \widetilde{\mathbf{Op}}(\widehat{\mathbf{D}})/\mathbf{B}_+[[t]]$. This allows us to identify $\mathbf{Op}(\widehat{\mathbf{D}})$ with a projective limit of affine spaces.

Recall that $\bar{p}_{-1} = \sum_{i=1}^{\ell} f_i$. We have a direct sum decomposition $\mathfrak{b}_+ = \bigoplus_{i \geq 0} \mathfrak{b}_{+,i}$ with respect to the principal gradation. The operator $\text{ad} \bar{p}_{-1}$ acts from $\mathfrak{b}_{+,i+1}$ to $\mathfrak{b}_{+,i}$ injectively for all $i > 1$. Hence we can find for each $j > 0$ a vector subspace $\mathbf{V}_j \subset \mathfrak{b}_{+,j}$, such that $\mathfrak{b}_{+,j} = [\bar{p}_{-1}, \mathfrak{b}_{+,j+1}] \oplus \mathbf{V}_j$. Note that $\mathbf{V}_j \neq 0$ if and only if j is an exponent of \mathfrak{g} , and in that case $\dim \mathbf{V}_j$ is the multiplicity of the exponent j . In particular, $\mathbf{V}_0 = 0$. Let $\mathbf{V} = \bigoplus_{j \in \mathbf{E}} \mathbf{V}_j \subset \mathfrak{n}_+$, where \mathbf{E} is the set of exponents of \mathfrak{g} . We call such \mathbf{V} a transversal subspace.

7.1.5. Lemma [DS]. — The action of $\mathbf{B}_+[[t]]$ on $\widetilde{\mathbf{Op}}(\widehat{\mathbf{D}})$ is free. Moreover, each $\mathbf{B}_+[[t]]$ -orbit contains a unique representative of the form

$$(7.1.2) \quad \partial_t + \bar{p}_{-1} + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathbf{V}[[t]].$$

7.1.6. Corollary. — The ring of functions on $\mathbf{Op}(\widehat{\mathbf{D}})$ is isomorphic to the ring of functions on $\mathbf{V}[[t]]$ for any choice of transversal subspace $\mathbf{V} \subset \mathfrak{n}_+$.

7.2. Affine Opers

While opers provide a Lie-theoretic viewpoint on differential equations $\mathbf{L}\psi = 0$ (where $\mathbf{L} = \partial^n - \dots$), the affine opers incorporate the eigenvalue problem $\mathbf{L}\psi = z\psi$ for \mathbf{L} : by rewriting this as $(\mathbf{L} - z)\psi = 0$, we find we are dealing with a natural one-parameter deformation of the operator \mathbf{L} . Similarly, an oper for a general group \mathbf{G} comes with a natural one-parameter “spectral” deformation. As Drinfeld and Sokolov discovered, it beneficial to consider this entire family as a single connection, by replacing the group \mathbf{G} by its loop group in z .

Affine opers are thus the analogs of opers for loop groups. Let p_{-1} be as in § 5.2. Recall that p_{-1} is the “loop analog” of the principal nilpotent \bar{p}_{-1} of \mathfrak{g} . On a differential scheme (such as the line $\widehat{\mathbf{A}}_{-1}$) we may thus define an affine oper as a

LG-bundle with connection and reductions, where the distinguished vector field acts in relative position p_{-1} (see Definition 7.3.1). However, in contrast to the case of G , the LG^+ -orbit of $[p_{-1}] \in \mathrm{Lg}/\mathrm{Lg}^+$ is not \mathbf{C}^\times -invariant. Hence this definition needs to be modified in the setting of arbitrary curves, where there are no distinguished vector fields.

To this end we appeal to the symmetries of the projective line. Consider the multiplicative group \mathbf{C}^\times of automorphisms of \mathbf{P}^1 preserving the points 0 and ∞ . It acts naturally on $\mathrm{LG} \supset \mathrm{LG}^+$ and $\mathrm{Lg} \supset \mathrm{Lg}^+$ by rescaling the coordinate z . We can therefore form the semidirect products $\widetilde{\mathrm{LG}} \supset \widetilde{\mathrm{LG}}^+$ of LG and LG^+ with \mathbf{C}^\times . Denote by $\mathbf{O}_{\mathrm{aff}}$ the $\widetilde{\mathrm{LG}}^+$ -orbit of $[p_{-1}] \in \mathrm{Lg}/\mathrm{Lg}^+$. Thus orbit is the \mathbf{C}^\times -span of the LG^+ -orbit of $[p_{-1}]$. It consists of all elements of the form $\sum_{i=0}^{\ell} \lambda_i f_i$, where λ_i 's are arbitrary non-zero complex numbers.

7.2.1. Definition. — Let Y be a smooth curve. An *affine oper* on Y is a quadruple $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+)$, where \mathcal{V} is an $\widetilde{\mathrm{LG}}$ -torsor on Y with a connection ∇ , a flat reduction \mathcal{V}_- to $\widetilde{\mathrm{LG}}_-$ and a reduction \mathcal{V}^+ to $\widetilde{\mathrm{LG}}^+$ in relative position $\mathbf{O}_{\mathrm{aff}}$.

The affine oper on Y form a stack that is denoted by $\mathcal{A}\mathcal{O}(Y)$.

7.2.2. Definition. — A *generic affine oper* is an affine oper, whose $\widetilde{\mathrm{LG}}_-$ and $\widetilde{\mathrm{LG}}^+$ reductions are in general position (i.e., they correspond to the open $\widetilde{\mathrm{LG}}^+$ -orbit of $\widetilde{\mathrm{LG}}_- \setminus \widetilde{\mathrm{LG}}$). Let $\mathcal{A}\mathcal{O}^\circ(Y)$ be the open substack of generic affine oper in $\mathcal{A}\mathcal{O}(Y)$.

7.2.3. Geometric Reformulation. — An affine oper on Y may also be described as follows. Let \mathcal{L} be a principal \mathbf{C}^\times -bundle on Y (equivalently, a line bundle). The group \mathbf{C}^\times acts naturally on the projective line \mathbf{P}^1 (preserving two points, 0 and ∞) and on the loop group LG of maps from the punctured formal neighborhood of $\infty \in \mathbf{P}^1$ to G . Let $\mathbf{P}_{\mathcal{L}}^1 = \mathcal{L} \times_{\mathbf{C}^\times} \mathbf{P}^1$ and $(\mathrm{LG})_{\mathcal{L}} = \mathcal{L} \times_{\mathbf{C}^\times} \mathrm{LG}$ be the induced bundles on Y . Note that $\mathbf{P}_{\mathcal{L}}^1$ is equipped with two disjoint sections 0 and ∞ .

Let \mathcal{P} be a G -bundle on $\mathbf{P}_{\mathcal{L}}^1$, equipped with a flag along the section ∞ (i.e. reduction of the G -bundle $\mathcal{P}|_{\infty}$ to B). The sections of \mathcal{P} on the punctured formal neighborhood of the section ∞ give rise to a principal bundle \mathcal{V} on Y for the \mathcal{L} -twist $(\mathrm{LG})_{\mathcal{L}}$ of LG (clearly, the data of \mathcal{V} and \mathcal{L} are equivalent to those of an $\widetilde{\mathrm{LG}}$ -torsor $\widetilde{\mathcal{V}}$ on Y .) Moreover the sections \mathcal{V}_- over $\mathbf{P}_{\mathcal{L}}^1 \setminus \infty$ give rise to a reduction of \mathcal{V} to the twist $(\mathrm{LG}_-)_{\mathcal{L}}$, and sections near ∞ respecting the flag give rise to a reduction \mathcal{V}^+ to $(\mathrm{LG}^+)_{\mathcal{L}}$. An affine oper is given by a connection on \mathcal{V} respecting \mathcal{V}_- and having relative position $\mathbf{O}_{\mathrm{aff}}$ with respect to \mathcal{V}^+ . Conversely, given an affine oper we obtain a \mathbf{C}^\times -bundle \mathcal{L} on Y and a G -bundle \mathcal{P} on $\mathbf{P}_{\mathcal{L}}^1$ satisfying the above conditions.

An affine oper is *generic* if the bundle \mathcal{P} is trivial along the fibers of $\mathbf{P}_{\mathcal{Z}}^1 \rightarrow Y$. In this case we can identify the G -bundles $\mathcal{P}|_{\infty}$ and $\mathcal{P}|_0$ canonically. The former comes equipped with a flag, while the latter inherits a connection from the affine oper connection on \mathcal{Z}_- (via the evaluation at 0, $(LG_-)_{\mathcal{Z}} \rightarrow G$.) Thus, a generic affine oper gives rise to a G -bundle with a flag and a connection, which are the data required for a G -oper. Moreover, since the evaluation of p_{-1} at $z = 0$ gives us \bar{p}_{-1} , it is reasonable to expect that this gives rise to a morphism $\mathcal{A}\mathcal{O}^\circ(Y) \rightarrow \mathcal{O}p(Y)$:

7.2.4. Proposition. — There is a canonical isomorphism $\mathcal{A}\mathcal{O}^\circ(Y) \rightarrow \mathcal{O}p(Y)$ between the moduli of generic affine opers and the moduli of opers on any smooth curve Y .

7.2.5. Proof. — As indicated above, to an affine oper we assign a G -torsor $\mathcal{E} = \mathcal{Z}_0$ with connection ∇ , and a B -reduction \mathcal{E}_+ coming from the identification of \mathcal{Z}_0 with \mathcal{Z}_∞ . We must verify that the connection has relative position \mathbf{O} with respect to \mathcal{E}_+ . This is a consequence of the relation between the B -orbit $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b}$ and the \widetilde{LG}^+ -orbit $\mathbf{O}_{\text{aff}} \subset \mathbf{Lg}/\mathbf{Lg}^+$. Under the standard inclusion $\mathfrak{g} \hookrightarrow \mathbf{Lg}$, we have $\mathfrak{b} \hookrightarrow \mathbf{Lg}^+$. It follows from the explicit form of the orbits that $\mathbf{O}_{\text{aff}} \subset \mathbf{O} + (\mathbf{Lb})_{\text{out}}$, where $(\mathbf{Lb})_{\text{out}}$ consists of loops into \mathfrak{b} which extend to $\mathbf{P}^1 \setminus \infty$. Hence in any trivialization of \mathcal{E}_+ , evaluation at 0 sends ∇ to \mathbf{O} , so that $(\mathcal{E}, \mathcal{E}_+, \nabla)$ is indeed an oper.

We now construct the inverse morphism, by giving a coordinate-free version of the formula

$$p_{-1} = \bar{p}_{-1} + e_{\theta} \otimes z.$$

Given a G -oper $(\mathcal{E}, \mathcal{E}_+, \nabla)$, let \mathcal{E}^θ denote the line subbundle of the adjoint bundle $(\mathfrak{g})_{\mathcal{Z}}$ corresponding to the highest weight line $\mathbf{C}e_{\theta} \subset \mathfrak{g}$ with respect to the Borel reduction \mathcal{E}_+ . Let \mathcal{L} denote the \mathbf{C}^\times -bundle dual to the line bundle $\Omega_Y \otimes \mathcal{E}^\theta$ of one-forms valued in the highest weight line.

Consider the \mathbf{P}^1 -bundle $\mathbf{P}_{\mathcal{Z}}^1$ on Y as before, and the corresponding twisted loop groups $(\mathbf{LG})_{\mathcal{Z}} \supset (\mathbf{LG}^+)_{\mathcal{Z}}$ etc. Let \mathcal{P} denote the G -bundle on $\mathbf{P}_{\mathcal{Z}}^1$ which is the pullback of \mathcal{E} from Y , and $(\mathcal{Z}, \mathcal{Z}_-, \mathcal{Z}^+)$ the corresponding $((\mathbf{LG})_{\mathcal{Z}}, (\mathbf{LG}^-)_{\mathcal{Z}}, (\mathbf{LG}^+)_{\mathcal{Z}})$ -torsors, where the $(\mathbf{LG}^+)_{\mathcal{Z}}$ -reduction \mathcal{Z}^+ is defined so that the values of its sections at ∞ lie in \mathcal{E}_+ . Finally, \mathcal{Z} carries an induced connection $\tilde{\nabla}$. This connection needs to be corrected to make \mathcal{Z} into an affine oper, because while \mathcal{Z}^+ has relative position $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b} = \mathbf{Lg}_+/\mathbf{Lg}^+$ with respect to $\tilde{\nabla}$, we need relative position \mathbf{O}_{aff} . But by construction, there is a canonical section $e_{\theta} \otimes z$ of the line bundle $\mathcal{L} \otimes \Omega_Y \otimes \mathcal{E}^\theta$, which is a one-form on Y with values in $\text{ad}\mathcal{Z}_-$. The connection ∇ obtained by adding this to $\tilde{\nabla}$ still preserves \mathcal{Z}_- , and it clearly has the correct relative position.

The above constructions work over an arbitrary base and are functorial. Hence we obtain an isomorphism of stacks $\mathcal{AO}^\circ(\mathbf{Y}) \simeq \mathcal{Op}(\mathbf{Y})$.

Now Lemma 7.1.3 implies:

7.2.6. Corollary. — The stack $\mathcal{AO}^\circ(\mathbf{Y})$ is the quotient of a scheme $\mathbf{AO}^\circ(\mathbf{Y}) \simeq \mathcal{Op}(\mathbf{Y})$ by the trivial action of the center $\mathbf{Z}(\mathbf{G})$ of \mathbf{G} .

7.3. Drinfeld-Sokolov Gauge

We now turn to our main objects of study: the affineopers on the formal group $\widehat{\mathbf{A}}_{-1}$. Recall that $\widehat{\mathbf{A}}_{-1}$ is the formal additive group with Lie algebra $\mathbf{C}p_{-1}$, where p_{-1} is a fixed element of \mathbf{Lg} given by formula (5.2.1) (i.e., from now on we fix our choice of the generators f_i , $i = 0, \dots, \ell$). Therefore we have a canonical coordinate t on $\widehat{\mathbf{A}}_{-1}$, such that $\widehat{\mathbf{A}}_{-1} = \{e^{tp_{-1}}\}$, and a canonical vector field on $\widehat{\mathbf{A}}_{-1}$ corresponding to p_{-1} (also denoted by p_{-1}). Because of that, we can avoid twisting by the group \mathbf{C}^\times that was used in the general definition 7.2.1.

Recall the notion of tautological relative position from Definition 2.3. The following definition is equivalent to the definitions 7.2.1, 7.2.2 in the case when \mathbf{Y} is replaced by $\widehat{\mathbf{A}}_{-1}$.

7.3.1. Definition. — An affine oper on $\widehat{\mathbf{A}}_{-1}$ is a quadruple $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+)$, where \mathcal{V} is an LG-torsor on $\widehat{\mathbf{A}}_{-1}$ with a connection ∇ , a flat reduction \mathcal{V}_- to \mathbf{LG}_- and a reduction \mathcal{V}^+ to \mathbf{LG}^+ in tautological relative position with ∇ .

A generic affine oper on $\widehat{\mathbf{A}}_{-1}$ is a quadruple $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+)$ as above, such the reductions \mathcal{V}_- and \mathcal{V}^+ are in general position.

The affineopers on $\widehat{\mathbf{A}}_{-1}$ form a stack that is denoted by $\mathcal{AO}(\widehat{\mathbf{A}}_{-1})$, and generic affineopers form an open substack $\mathcal{AO}^\circ(\widehat{\mathbf{A}}_{-1})$.

7.3.2. Suppose we are given a quadruple $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^{\mathbf{A}_+}) \in \mathcal{M}_{p_{-1}}$, where \mathcal{V} is an LG-torsor on $\widehat{\mathbf{A}}_{-1}$ with a flat connection ∇ , a flat reduction \mathcal{V}_- to \mathbf{LG}_- and a reduction $\mathcal{V}^{\mathbf{A}_+}$ to \mathbf{A}_+ in tautological relative position with respect to ∇ . Then the induced reduction of \mathcal{V} to \mathbf{LG}^+ , $\mathcal{V}^+ = \mathcal{V}^{\mathbf{A}_+} \times_{\mathbf{A}_+} \mathbf{LG}^+$, is clearly in relative position $\mathbf{O}_{p_{-1}} \subset \mathbf{Lg}/\mathbf{Lg}^+$ with respect to ∇ . Hence $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+)$ is an affine oper on $\widehat{\mathbf{A}}_{-1}$. This can certainly be done over an arbitrary base. Thus, we obtain:

7.3.3. Lemma. — There is a natural morphism of stacks $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1} \rightarrow \mathcal{AO}(\widehat{\mathbf{A}}_{-1})$.

7.3.4. Drinfeld-Sokolov Gauge. — Recall that in Proposition 6.2.3 we established an isomorphism between the abelianized Grassmannian $\mathcal{G}_\Lambda^{\mathbf{P}^1}$, associated to the principal Heisenberg subalgebra $\mathfrak{a} \subset \mathbf{Lg}$, and $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1}$. Our goal now is to prove that the above

morphism $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1} \rightarrow \mathcal{A}\mathcal{O}(\widehat{\mathbf{A}}_{-1})$, and hence the composition $\mathcal{G}_A^{\mathbf{P}^1} \rightarrow \mathcal{A}\mathcal{O}(\widehat{\mathbf{A}}_{-1})$, are in fact isomorphisms. In order to do that we prove in this section a technical result, which shows the existence of a canonical gauge for all affine opers on $\widehat{\mathbf{A}}_{-1}$. This result will enable us to show that any affine oper has a canonical reduction to \mathbf{A}_+ , thus giving us an inverse morphism $\mathcal{A}\mathcal{O}(\widehat{\mathbf{A}}_{-1}) \rightarrow \mathcal{M}_{p_{-1}}^{\mathbf{P}^1}$. This gauge goes back to the original works on the inverse scattering method in soliton equations. It was first formulated in the language of connections by Drinfeld and Sokolov [DS].

We will state our result in a much more general situation of a strongly regular element p of an arbitrary Heisenberg subalgebra \mathfrak{a} of \mathbf{Lg} . (Recall Definition 5.6.7 of strongly regular element $p \in \mathfrak{a}^{\geq l}$.) This statement is a direct generalization of the Drinfeld-Sokolov lemma [DS] (see also [dGHM]), and our proof essentially follows their argument.

7.3.5. Let us fix notation: $\mathfrak{a} \subset \mathbf{Lg}$ is a Heisenberg algebra with the canonical filtration $\{\mathfrak{a}^{\geq j}\}$; $p \in \mathfrak{a}^{\geq l}$, $l > 0$; $\{\mathbf{Lg}^{\geq j}\}$ is a filtration compatible with \mathfrak{a} , in the sense of § 5.6.

Since $p \in \mathfrak{a}$, it is semisimple. If p is regular, then we have the decomposition $\mathbf{Lg} \cong \text{Ker}(\text{ad } p) \oplus \text{Im}(\text{ad } p)$. Furthermore, if p is strongly regular, then Lemma 5.6.8 gives us the following

7.3.6. Lemma. — $\mathbf{Lg}^{\geq j} \equiv \mathfrak{a}^{\geq j} \oplus \text{ad } p \cdot \mathbf{Lg}^{\geq j-l}$ for every j .

7.3.7. Proposition. — Let $p \in \mathfrak{a}^{\geq l}$, $l < 0$, be a strongly regular element, and $\nabla_t = \partial_t + p + q$, with $q(t) \in \mathbf{Lg}^+$, be a connection on the trivial \mathbf{LG} -bundle on the formal disc $\widehat{\mathbf{D}}_t$. Then there is a gauge transformation $\mathbf{M} \in \mathbf{LG}^{>0}$, such that

$$(7.3.1) \quad \mathbf{M}^{-1}(\partial_t + p + q)\mathbf{M} = \partial_t + p + p_+(t),$$

with $p_+(t) \in \mathfrak{a}^+$. Furthermore, this equation determines \mathbf{M} uniquely up to right multiplication by \mathbf{A}^+ .

7.3.8. Proof. — Let us solve (7.3.1) for some $\mathbf{M} \in \mathbf{LG}^{>0}$ and $p_+ \in \mathfrak{a}^+$ (to simplify notation, we will write ∂ for ∂_t and ∇ for ∇_t). Since $\mathbf{LG}^{>0}$ is a pronipotent group, every $\mathbf{M} \in \mathbf{LG}^{>0}$ may be written as $\exp m$ for some $m \in \mathbf{Lg}^{>0}$. We construct m and p_+ by induction on the filtration as $m = \sum_{i=l}^{\infty} m_i$ and $p_+ = \sum_{i=1}^{\infty} p_i$ with $m_i, p_i \in \mathbf{Lg}^{\geq i}$. Write

$$\mathbf{M}^{-1}\nabla\mathbf{M} = \sum_{k \geq 0} \frac{1}{k!} \text{ad}^k m \cdot \nabla.$$

At the first step, $\nabla \equiv \partial + p \pmod{\mathbf{Lg}^+}$ and so $m \equiv 0 \pmod{\mathbf{Lg}^{\geq -l}}$ indeed works modulo \mathbf{Lg}^+ .

For the inductive step, suppose that $k \geq -l$, and $M_k = \exp(\sum_{i=-l}^k m_i)$ satisfies

$$M_k^{-1} \nabla M_k \equiv \partial + p + \sum_{i=1}^{k-l-1} p_i \pmod{\mathbf{Lg}^{\geq k+l}}.$$

Denote $\delta_k = M_k^{-1} \nabla M_k - (\partial + p + \sum_{i=1}^{k+l-1} p_i)$, so that $\delta_k \in \mathbf{Lg}^{\geq k+l}$. By Lemma 7.3.6, we may decompose $\delta_k = p_{k+l} + [p, m_k]$ for some $p_{k+l} \in \mathfrak{a}^{\geq k+l}$ and $m_k \in \mathbf{Lg}^{\geq k}$. If we set $M_{k+1} = \exp(\sum_{i=-l}^{k+1} m_i)$, then

$$M_{k+1}^{-1} \nabla M_{k+1} \equiv \partial + p + \sum_{i=1}^{k+l} p_i \pmod{\mathbf{Lg}^{\geq k+l+1}},$$

completing the inductive step.

Note that at each step, M_k is unique up to the right multiplication by an element of A^+ . This completes the proof.

7.3.9. Proposition 7.3.7 implies the following statement for the principal Heisenberg, whose proof we postpone to Proposition 8.4.4, where the case of general Heisenbergs is taken up.

7.3.10. Corollary. — Let $(\mathcal{Y}, \nabla, \mathcal{Y}_-, \mathcal{Y}^+)$ be an affine oper on \widehat{A}_{-1} . Then \mathcal{Y} has a unique reduction $\mathcal{Y}^{A^+} \subset \mathcal{Y}^+$ to A_+ , such that

- (1) the induced A -torsor $\mathcal{Y}^A \subset \mathcal{Y}$ is flat;
- (2) \mathcal{Y}^{A^+} has the tautological relative position with respect to ∇ .

7.3.11. Remark. — The above corollary may be applied to the variants of affineopers defined on arbitrary differential schemes. Thus we obtain a morphism from affineopers on a differential scheme to the bundles considered in Proposition 6.3.3, and thus to \mathcal{G}_A^X .

7.3.12. Remark: The Extended Oper Family. — The direct analog of the notion of an oper for the loop group would not require a flat reduction to \mathbf{LG}_- . The resulting objects relate to pseudo-differential operators. We only note here that the Drinfeld-Sokolov lemma applies to them as well.

We may also replace the curve \mathbf{P}^1 in the above definition by an arbitrary smooth curve X (in particular, replacing the \mathbf{LG}_- -reduction by an \mathbf{LG}_-^X -reduction). The resulting X -affineopers are analogous to elliptic sheaves for X . Moreover we have a notion of “generic” X -affine oper, akin to the genericity conditions on line bundles in the Krichever construction. Choosing a basepoint $0 \in X \setminus \infty$ we obtain a map from generic affineopers for any X toopers on the disc. Other generalizations of affineopers will be discussed in §8.4.

We briefly mention some other relatives of opers. The difference opers and Frobenius opers arise from replacing differential operators by difference operators and polynomials in the Frobenius, respectively. They may be expected to be useful in the theory of the q -KdV hierarchies and elliptic sheaves, respectively. Recall from § 2.1 that there is a notion of relative position for equivariant bundles with reductions. Let (Y, l) be a pair consisting of a scheme Y and an automorphism l of Y . Let χ be an LG^+ -orbit in $\mathrm{LG}/\mathrm{LG}^+$. Recall that these orbits are in one-to-one correspondence with the affine Weyl group. In applications one usually takes χ to be a Coxeter element.

7.3.13. Definition. — A difference oper on (Y, l) is a quadruple $(\mathcal{V}, l_{\mathcal{V}}, \mathcal{V}_-, \mathcal{V}^+)$, where \mathcal{V} is an LG -torsor on Y , $l_{\mathcal{V}}$ a lifting of l to \mathcal{V} , \mathcal{V}_- an LG_- -reduction of \mathcal{V} preserved by $l_{\mathcal{V}}$, and \mathcal{V}^+ an LG^+ -reduction of \mathcal{V} in relative position χ with respect to $l_{\mathcal{V}}$.

7.3.14. If Y is a scheme defined over a finite field, we take l to be the Frobenius of Y . The resulting objects, the Frobenius opers, are generalizations for semisimple groups G of the notion of elliptic sheaves (see e.g. [BS]). As was the case for their analogs, Krichever sheaves from § 3.3.3, the definition of elliptic sheaves for X as sheaves on $X \times Y$ is simplified in the semisimple case: by the Drinfeld-Simpson theorem [DSi] we may replace G -torsors on $X \times Y$ by LG -torsors on Y , with reductions as above.

8. Integrable Systems

In this section we combine the results of sections § 6 and § 7 to obtain isomorphisms between abelianized Grassmannians and the moduli of affine opers. Under this isomorphism, the action of the group A/A_+ gives rise to a collection of infinitely many commuting flows on the space of affine opers. They form a generalized Drinfeld-Sokolov hierarchy. We first work out in detail the case of the principal Heisenberg subalgebra with \mathbf{P}^1 as the underlying curve X . After that we generalize the construction to the case of an arbitrary Heisenberg subalgebra.

8.1. The Principal Case: KdV

Let us recall the notation: ∞ is a point on \mathbf{P}^1 , LG is the loop group associated to the disc at ∞ , and Lg its Lie algebra. In this subsection we denote by \mathfrak{a} the principal Heisenberg subalgebra of Lg , and by p_{-1} a generator of the one-dimensional space $\mathfrak{a}_{\geq -1}/\mathfrak{a}_+$. Finally, \widehat{A}_{-1} is the formal additive subgroup of A/A_+ generated by p_{-1} .

8.1.1. Theorem.

- (1) There is a canonical isomorphism between the abelianized Grassmannian $\mathcal{G}_A^{\mathbf{P}^1}$ and the moduli stack $\mathcal{AO}(\widehat{A}_{-1})$ of affine opers on \widehat{A}_{-1} .

- (2) The isomorphism of part (1) identifies the big cell $\mathcal{G}r_A^\circ$ of $\mathcal{G}r_A^{\mathbf{P}^1}$ and the moduli stack $\mathcal{A}\mathcal{O}^\circ(\widehat{A}_{-1})$ of generic affineopers on \widehat{A}_{-1} .
- (3) For each point $0 \in \mathbf{P}^1 \setminus \infty$, there is a canonical isomorphism between $\mathcal{G}r_A^\circ$ and the moduli stack $\mathcal{O}p(\widehat{A}_{-1})$ of G-opers on \widehat{A}_{-1} .

8.1.2. Proof. — The construction of Corollary 7.3.10 gives us a morphism $\mathcal{A}\mathcal{O}(\widehat{A}_{-1}) \rightarrow \mathcal{M}_{p-1}^{\mathbf{P}^1}$, which sends $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+) \in \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$ to $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+) \in \mathcal{M}_{p-1}^{\mathbf{P}^1}$. On the other hand, in Lemma 7.3.3 we constructed a morphism $\mathcal{M}_{p-1}^{\mathbf{P}^1} \rightarrow \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$. It is clear from the construction that these morphisms are inverse to each other. Hence we obtain an isomorphism $\mathcal{M}_{p-1}^{\mathbf{P}^1} \simeq \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$. But $\mathcal{G}r_A^{\mathbf{P}^1}$ is isomorphic to $\mathcal{M}_{p-1}^{\mathbf{P}^1}$ by Proposition 6.2.3. Therefore $\mathcal{G}r_A^{\mathbf{P}^1} \simeq \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$. This proves part (1).

By definition, the big cell $\mathcal{G}r_A^\circ$ of $\mathcal{G}r_A^{\mathbf{P}^1}$ classifies LG-torsors \mathcal{V} with reductions \mathcal{V}_- to LG_- and \mathcal{V}^+ to A_+ , such that \mathcal{V}_- and the induced LG^+ -reduction $\mathcal{V}^+ \times_{A_+} \text{LG}^+$ are in general position (see § 6.1.5). On the other hand, $\mathcal{A}\mathcal{O}^\circ(\widehat{A}_{-1})$ classifies the quadruples $(\mathcal{V}, \nabla, \mathcal{V}_-, \mathcal{V}^+)$, such that the reductions \mathcal{V}_- and \mathcal{V}^+ are in general position. Hence under the above isomorphism $\mathcal{G}r_A^\circ$ is mapped to $\mathcal{A}\mathcal{O}^\circ(\widehat{A}_{-1})$ and we obtain part (2).

Finally, part (3) follows from part (2) and Proposition 7.2.3.

8.1.3. Remark. — Let us once again spell out the definition of the morphisms $\mathcal{G}r_A^{\mathbf{P}^1} \rightarrow \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$ and $\mathcal{A}\mathcal{O}(\widehat{A}_{-1}) \rightarrow \mathcal{G}r_A^{\mathbf{P}^1}$.

Let $(\mathcal{V}, \mathcal{V}_-, \mathcal{V}^+)$ be a point of $\mathcal{G}r_A^{\mathbf{P}^1}$. Here \mathcal{V} is an LG-torsor, and $\mathcal{V}_-, \mathcal{V}^+$ are its reductions to LG_-, A_+ , respectively. Let \mathcal{T} be the universal bundle over $\mathcal{G}r_A^{\mathbf{P}^1}$, whose fiber at $(\mathcal{V}, \mathcal{V}_-, \mathcal{V}^+)$ is \mathcal{V} ; it has canonical reductions \mathcal{T}_- and \mathcal{T}^+ . Denote by $\pi: \widehat{A}_{-1} \rightarrow \mathcal{G}r_A^{\mathbf{P}^1}$ the map corresponding to the action of \widehat{A}_{-1} on $(\mathcal{V}, \mathcal{V}_-, \mathcal{V}^+)$. Note that the action of \widehat{A}_{-1} lifts to \mathcal{T} . Now as in Lemma 2.3.8, pulling back \mathcal{T} to \widehat{A}_{-1} by π , we obtain an LG-bundle \mathcal{V}_t on \widehat{A}_{-1} with a flat connection ∇ and the reductions $\mathcal{V}_{-,t}$ and \mathcal{V}_t^+ to LG_- and A_+ . Moreover, by Lemma 2.3.8, $\mathcal{V}_{-,t}$ is preserved by ∇ , while \mathcal{V}_t^+ is in tautological relative position with respect to ∇ . Denote by \mathcal{V}_t^\dagger the induced LG^+ -reduction $\mathcal{V}_t^+ \times_{A_+} \text{LG}^+$. Then $(\mathcal{V}_t, \nabla, \mathcal{V}_{-,t}, \mathcal{V}_t^\dagger)$ is the affine oper corresponding to $(\mathcal{V}, \mathcal{V}_-, \mathcal{V}^+) \in \mathcal{G}r_A^{\mathbf{P}^1}$.

The inverse map is constructed as follows. Given an affine oper $(\mathcal{V}_t, \nabla, \mathcal{V}_{-,t}, \mathcal{V}_t^\dagger)$, we obtain a canonical A_+ -reduction \mathcal{V}_t^+ of \mathcal{V}_t using Corollary 7.3.10. Then we look

at the fiber \mathcal{Z}'_0 of \mathcal{Z}'_i at $0 \in \widehat{A}_{-1}$. It comes with the reductions $\mathcal{Z}'_{-,0}$ and $\mathcal{Z}'_0^{\text{A}_+}$ to LG_- and A_+ , respectively. Hence we attach to our affine oper the point $(\mathcal{Z}'_0, \mathcal{Z}'_{-,0}, \mathcal{Z}'_0^{\text{A}_+})$ of $\mathcal{G}_\Lambda^{\mathbf{P}^1}$.

The above construction is almost tautological. However, there is one place where in addition to the general functorial correspondence of Proposition 2.3.12 we really have to use the specifics of our situation: namely, when we switch between reductions to A_+ and LG^+ . It is easy to pass from an A_+ -reduction to an LG^+ -reduction by using induction, and this allows us to pass from the unwieldy moduli space $\mathcal{M}_{p_{-1}}^{\mathbf{P}^1}$ to the much nicer moduli of affine opers. But a priori one can not go back from an LG^+ -reduction to an A_+ -reduction. For this we need to rely on the rather technical Drinfeld-Sokolov gauge (see Corollary 7.3.10). As we will see below, this construction works if we replace p_{-1} (the generator of the formal group \widehat{A}_{-1}) by any strongly regular element p of a general Heisenberg subalgebra \mathfrak{a} .

8.1.4. Geometric interpretation. — The morphism $\mathcal{G}_\Lambda^{\mathbf{P}^1} \rightarrow \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$ has a simple geometric interpretation, which is close in spirit to the Krichever construction as in [Mum] (see also § 7.2.5, § 3.3.3). To simplify the picture, we explain it using analytic rather than algebro-geometric language.

Let \mathcal{E} be a G -bundle on \mathbf{P}^1 , equipped with a reduction to A_+ on D . We wish to deform \mathcal{E} by the action of p_{-1} , as we did in the trivial abelian case § 3.2.7. Thus we construct a G -bundle $\widetilde{\mathcal{E}}$ on $\mathbf{P}^1 \times \widehat{A}_{-1}$ by multiplying the transition function of \mathcal{E} on D^\times by $e^{-\#}$. This change in the transition function does not affect \mathcal{E} away from ∞ , so that we can canonically identify sections $\mathcal{E}(\mathbf{P}^1 \setminus \infty)$ at time $t = 0$ with the sections $\mathcal{E}_t(\mathbf{P}^1 \setminus \infty)$ of its deformation at any time $t \in \widehat{A}_{-1}$. In other words, we have a canonical (flat) partial connection ∇ over \widehat{A}_{-1} on $\widetilde{\mathcal{E}}|_{(\mathbf{P}^1 \setminus \infty) \times \widehat{A}_{-1}}$ in the direction of \widehat{A}_{-1} .

Thus the restriction $\widetilde{\mathcal{E}}_0$ of $\widetilde{\mathcal{E}}$ to $0 \times \widehat{A}_{-1}$, where $0 \in \mathbf{P}^1 \setminus \infty$, is a G -bundle on \widehat{A}_{-1} with a flat connection. On the other hand, the restriction \widetilde{E}_∞ of $\widetilde{\mathcal{E}}$ to $\infty \times \widehat{A}_{-1}$ is a G -torsor on \widehat{A}_{-1} with a B -reduction induced by the A_+ -reduction of $\mathcal{E}|_D$. The point is that when \mathcal{E} is a trivial bundle on \mathbf{P}^1 , we may identify $\widetilde{\mathcal{E}}_0$ with \widetilde{E}_∞ . Hence we obtain a G -bundle on \widehat{A}_{-1} with a flag and a flat connection. Explicit calculation (see below) shows that it is a G -oper. Thus, we attach to \mathcal{E} a G -oper on \widehat{A}_{-1} .

On the other hand, we can consider the LG -bundle on \widehat{A}_{-1} corresponding to taking the sections of \widetilde{E} over D^\times . It carries a reduction to A_+ , and hence the induced reduction to LG^+ . But this reduction is not preserved by the connection ∇ . The particular form of p_{-1} shows that in fact we obtain an affine oper.

Conversely, an affine oper gives rise to a period map $\widehat{A}_{-1} \rightarrow \text{LG}_- \setminus \text{LG} / \text{LG}^+$, which by the (Griffiths) transversality of the connection is tangent to a certain completely non-integrable distribution (compare [Mum]). The Drinfeld-Sokolov gauge

picks out a canonical lifting of this period map to $\mathrm{LG}_- \backslash \mathrm{LG} / \mathbf{A}_+$ which is tangent to the vector field p_{-1} , and from this data we recover our original \mathcal{E} .

8.1.5. According to Lemma 6.1.6, $\mathcal{G}r_{\mathbf{A}}^\circ$ is the quotient of the scheme $\mathrm{Gr}_{\mathbf{A}}^\circ = \mathrm{G} \backslash \mathrm{LG}_+ / \mathbf{A}'_+$ by the trivial action of $\mathbf{A}_0 = \mathbf{Z}(\mathrm{G})$. On the other hand, $\mathcal{A}\mathcal{O}^\circ(\widehat{\mathbf{A}}_{-1})$ (resp., $\mathcal{O}p(\widehat{\mathbf{A}}_{-1})$) is the quotient of a scheme $\mathrm{AO}^\circ(\widehat{\mathbf{A}}_{-1})$ (resp., $\mathrm{Op}(\widehat{\mathbf{A}}_{-1}) = \mathrm{Op}_{\mathfrak{g}}(\widehat{\mathbf{A}}_{-1})$) by the trivial action of $\mathbf{Z}(\mathrm{G})$, see Lemma 7.2.6 and Lemma 7.1.3. Hence we obtain:

8.1.6. Corollary.

- (1) There is a canonical isomorphism of schemes $\mathrm{Gr}_{\mathbf{A}}^\circ \simeq \mathrm{AO}^\circ(\widehat{\mathbf{A}}_{-1})$.
- (2) For each point $0 \in \mathbf{P}^1 \setminus \infty$, there is a canonical isomorphism $\mathrm{Gr}_{\mathbf{A}}^\circ \simeq \mathrm{Op}(\widehat{\mathbf{A}}_{-1})$.

8.1.7. The Universal Oper. — Let \mathcal{T} be the tautological G -bundle over $\mathrm{Op}(\widehat{\mathbf{A}}_{-1}) \times \widehat{\mathbf{A}}_{-1}$, whose restriction to $(\mathcal{E}, \nabla, \mathcal{E}_{\mathbf{B}}) \times \widehat{\mathbf{A}}_{-1}$ is \mathcal{E} . The two additional structures of an oper translate into a \mathbf{B} -reduction $\mathcal{T}_{\mathbf{B}}$ of \mathcal{T} and a partial connection $\nabla_{\mathcal{T}}$ along $\widehat{\mathbf{A}}_{-1}$. This is the “universal oper” on $\mathrm{Op}(\widehat{\mathbf{A}}_{-1}) \times \widehat{\mathbf{A}}_{-1}$.

On the other hand, consider the homogeneous space $\mathrm{LG}_+ / \mathbf{A}'_+$, which is a G -bundle over $\mathrm{Gr}_{\mathbf{A}}^\circ = \mathrm{G} \backslash \mathrm{LG}_+ / \mathbf{A}'_+$. The subgroup $\mathrm{LG}^+ \subset \mathrm{LG}_+$, which is canonically associated to \mathbf{A}_+ gives us a \mathbf{B} -reduction $\mathrm{LG}^+ / \mathbf{A}'_+$ of LG_+ . Now fix a point $0 \in \mathbf{P}^1$ (i.e., choose a generator z of $\mathbf{C}[\mathbf{P}^1 \setminus \infty]$ up to a scalar multiple) and the corresponding subgroup $\mathrm{LG}_{<0}$ of LG . Then we can view LG_+ as an open part of $\mathrm{LG}_{<0} \backslash \mathrm{LG}$. Hence we have an action of p_{-1} from the right on $\mathrm{LG}_+ / \mathbf{A}'_+$. Define a partial connection $\nabla_{p_{-1}}$ on $(\mathrm{LG}_+ / \mathbf{A}'_+) \times \widehat{\mathbf{A}}_{-1}$ along $\widehat{\mathbf{A}}_{-1}$ by the formula $\partial_t + p_{-1}$. Corollary 8.1.6 (2) can be interpreted as follows:

8.1.8. Corollary. — The universal oper $(\mathcal{T}, \nabla_{\mathcal{T}}, \mathcal{T}_{\mathbf{B}})$ on $\mathrm{Op}(\widehat{\mathbf{A}}_{-1}) \times \widehat{\mathbf{A}}_{-1}$ is canonically isomorphic to the triple $(\mathrm{LG}_+ / \mathbf{A}'_+, \partial_t + p_{-1}, \mathrm{LG}^+ / \mathbf{A}'_+)$ on $(\mathrm{G} \backslash \mathrm{LG}_+ / \mathbf{A}'_+) \times \widehat{\mathbf{A}}_{-1}$.

8.1.9. Differential Polynomials. — According to Lemma 7.1.5, $\mathrm{Op}(\widehat{\mathbf{A}}_{-1})$ is isomorphic to the pro-vector space $\mathbf{V}[[t]]$, where \mathbf{V} is a subspace of \mathfrak{n}_+ satisfying $\mathfrak{n}_+ = \mathbf{V} \oplus \mathrm{Im} \mathrm{ad} p_{-1}$. Let us fix such a subspace \mathbf{V} , and a homogeneous basis of \mathbf{V} with respect to the principal gradation. Then we can identify $\mathbf{V}[[t]]$, and hence $\mathrm{Op}(\widehat{\mathbf{A}}_{-1})$, with the pro-vector space of ℓ -tuples $(v_i(t))_{i=1, \dots, \ell}$ of formal Taylor power series. For instance, in the case $\mathfrak{g} = \mathfrak{sl}_2$ we identify $\mathrm{Op}(\widehat{\mathbf{A}}_{-1})$ with the space of operators of the form

$$(8.1.1) \quad \partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}.$$

Let $v_i^{(n)}$ be the linear functional on $\mathbf{V}[[t]]$, whose value at $(v_i(t))_{i=1, \dots, \ell}$ is $\partial_t^n v_i(t)|_{t=0}$. We can now identify the ring of regular functions $\mathbf{C}[\mathrm{Op}(\widehat{\mathbf{A}}_{-1})]$ on $\mathrm{Op}(\widehat{\mathbf{A}}_{-1})$ with the

ring of differential polynomials $\mathbf{C}[v_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$. Furthermore, the natural action of ∂_t on $\mathbf{C}[\text{Op}(\widehat{\mathbf{A}}_{-1})]$ is given by $\partial_t \cdot v_i^{(n)} = v_i^{(n+1)}$. We will use the notation v_i for $v_i^{(0)}$.

8.1.10. Now Corollary 8.1.6 gives us a new proof of the following result, which is equivalent to Theorem 4.1 from [F] (to make the connection with [F] clear, note that $\text{Gr}_\Lambda^\circ \simeq \mathbf{N}_+ \backslash \text{LG}^{>0} / \mathbf{A}'_+$).

8.1.11. Theorem. — The ring of functions $\mathbf{C}[\text{Gr}_\Lambda^\circ]$ on Gr_Λ° is isomorphic to the ring of differential polynomials $\mathbf{C}[v_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$. Under this isomorphism, the right infinitesimal action of p_{-1} on Gr_Λ° is given by $p_{-1} \cdot v_i^{(n)} = v_i^{(n+1)}$.

8.1.12. KdV hierarchy. — The infinite-dimensional formal group \mathbf{A}/\mathbf{A}_+ and its Lie algebra $\mathfrak{a}/\mathfrak{a}_+$ act from the right on Gr_Λ° . Hence by Theorem 8.1.11 we obtain an action of $\mathfrak{a}/\mathfrak{a}_+$ on the space $\text{Op}(\widehat{\mathbf{A}}_{-1})$ of \mathbf{G} -opers on $\widehat{\mathbf{A}}_{-1}$. The action of the element p_{-1} of $\mathfrak{a}/\mathfrak{a}_+$ coincides with the flow generated by ∂_t . Other elements of $\mathfrak{a}/\mathfrak{a}_+$ act on $\text{Op}(\widehat{\mathbf{A}}_{-1})$ by vector fields commuting with ∂_t . It is known that $\mathfrak{a}/\mathfrak{a}_+$ has a basis p_i , $i \in -\mathbf{I}$, where \mathbf{I} is the set of all positive integers equal to the exponents of \mathfrak{g} modulo the Coxeter number. The degree of p_i equals i with respect to the principal gradation of Lg (see [Kac1]).

Given an element $p_{-m} \in \mathfrak{a}/\mathfrak{a}_+$, $m \in \mathbf{I}$, let \widetilde{p}_{-m} be the corresponding derivation of $\mathbf{C}[v_j^{(n)}]_{j=1, \dots, \ell; n \geq 0}$. We know that each \widetilde{p}_{-m} commutes with $\widetilde{p}_{-1} = \partial_t$, and hence is an *evolutionary* derivation. Because of the Leibnitz rule, the action of an evolutionary derivation on $\mathbf{C}[v_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$ is uniquely determined by its action on v_i , $i = 1, \dots, \ell$.

We know that $\widetilde{p}_{-m} \cdot v_i$ is a differential polynomial in v_i 's. The system of partial differential equations

$$(8.1.2) \quad \partial_{t_m} v_i = \widetilde{p}_{-m} \cdot v_i, \quad i = 1, \dots, \ell,$$

(considered as equations on the functions $v_i(t)$, $i = 1, \dots, \ell$, belonging to some reasonable class of functions), is called the m -th equation of the *generalized KdV hierarchy corresponding to \mathfrak{g}* , and the time t_m is called the m -th time of the hierarchy. The totality of these equations as m runs over \mathbf{I} is called the generalized KdV hierarchy corresponding to \mathfrak{g} .

For instance, for $\mathfrak{g} = \mathfrak{sl}_2$, we obtain the KdV hierarchy. In this case, \mathbf{I} consists of all positive odd integers. We already know the action of p_{-1} : it is given by ∂_t . In particular, the equations (8.1.2) read in this case: $\partial_{t_1} v = \partial_t v$ (here we use v for v_1), which means that $t_1 = t$ (this is true for all \mathfrak{g}). The next element is p_{-3} . The corresponding derivation \widetilde{p}_{-3} has been computed explicitly in [F]. The resulting equation is

$$(8.1.3) \quad \partial_{t_3} v = \frac{3}{2} v \partial_t v - \frac{1}{4} \partial_t^3 v,$$

which is the KdV equation (up to a slight redefinition of variables).

8.2. Zero Curvature Representation

It is well-known that the KdV equations can be written in the zero curvature, or Zakharov-Shabat, form (see [DS]). The zero curvature formalism is one of the standard methods to write down equations of completely integrable systems, and it is convenient for explicit description of the associated Hamiltonian structures. This form of the equations arises very naturally in our approach as the equations expressing the flatness of the connection on the formal group A/A_+ . Recall that we have *identified* Gr_A^o with the moduli space of flat connections on the entire formal group A/A_+ (see Proposition 6.2.3 for the precise statement). The flatness condition, written in explicit coordinates, takes the familiar form of zero curvature equations as we will now demonstrate.

8.2.1. The isomorphism of Proposition 6.2.3 assigns to each point $K_+ \in G \backslash LG_+ / A'_+ = Gr_A^o$ a quadruple $(\mathcal{Z}, \nabla, \mathcal{Z}_-, \mathcal{Z}^{A_+})$, where \mathcal{Z} is an LG -torsor on A/A_+ with a flat connection ∇ , a flat LG_- -reduction \mathcal{Z}_- , and an A_+ -reduction in tautological relative position with ∇ . We want to trivialize \mathcal{Z} and calculate the contraction ∇_p of ∇ with the vector field on A/A_+ coming from the left action of $p \in \mathfrak{a}/\mathfrak{a}_+$.

It is convenient to trivialize \mathcal{Z} in such a way that the LG_- -reduction \mathcal{Z}_- and the induced LG^+ -reduction $\mathcal{Z}^{A_+} \times_{A_+} LG^+$ are preserved. Since we started with a point on the big cell Gr_A^o , these reductions are in general position. Therefore such a trivialization is unique up to the gauge action of the group $LG_- \cap LG^+ = B_+$. Let us first choose one such trivialization using the factorization of loops as in § 4.2. The computation of the connection operator ∇_p in this trivialization is the content of Lemma 4.2.5, and we obtain:

$$\nabla_p = \partial_t + (K_+(t)pK_+(t)^{-1})_-, \quad p \in \mathfrak{a}/\mathfrak{a}_+.$$

But the connection ∇ is flat by our construction. Therefore $[\nabla_p, \nabla_{p'}] = \nabla_{[p, p']} = 0$ for all $p, p' \in \mathfrak{a}/\mathfrak{a}_+$. This gives us the zero curvature equations

$$(8.2.1) \quad \left[\partial_{t_m} + (K_+(t)p_{-m}K_+(t)^{-1})_-, \partial_{t_n} + (K_+(t)p_{-n}K_+(t)^{-1})_- \right] = 0,$$

where $m, n \in \mathbb{I}$.

In the special case $n = 1$ we obtain the equation

$$(8.2.2) \quad \left[\partial_{t_m} + (K_+(t)p_{-m}K_+(t)^{-1})_-, \partial_t + (K_+(t)p_{-1}K_+(t)^{-1})_- \right] = 0.$$

But the component of our connection in the direction of p_{-1} is an affine oper:

$$(8.2.3) \quad \partial_t + (K_+(t)p_{-1}K_+(t)^{-1})_- = \partial_t + p_{-1} + \mathbf{b}(t),$$

where $\mathbf{b}(t) \in \mathfrak{b}_+[[t]]$. Denote $L_m = (K_+(t)p_{-m}K_+(t)^{-1})_-$. It is easy to see that any solution for L_m as an element of $\mathfrak{g}[[t]]$ is a differential polynomial in the matrix

elements $\mathbf{b}_\alpha(t)$, $\alpha \in \Delta_+$ of $\mathbf{b}(t)$. Hence formula (8.3.8) expresses $\partial_{t_m} \mathbf{b}_\alpha(t)$ as a differential polynomial in $\mathbf{b}_\alpha(t)$'s.

However, we should remember that the formulas for the connection operators that we obtained refer to a particular trivialization of \mathcal{Z} , which was unique up to the gauge action of $B_+[[t]]$. We may require the trivialization to preserve not only \mathcal{Z}_- and \mathcal{Z}^+ , but also the form (8.2.3) – in other words to have symbol p_{-1} – thus further reducing the gauge freedom to $N_+[[t]]$. The equations (8.2.2) are invariant under this action, and we should consider them as equations not on the space of connections of the form (8.2.3), but on the space of affine opers, which is its quotient by the free action of $N_+[[t]]$. Under this action, we can bring (8.2.3) to the form

$$\partial_t + p_{-1} + \mathbf{v}(t), \quad \mathbf{v}(t) \in V[[t]],$$

where V is the transversal subspace to $\text{ad} \bar{p}_{-1}$ in \mathfrak{n}_+ (see Lemma 7.1.5). If we choose a basis $\{v_i\}_{i=1, \dots, \ell}$, we can interpret the equation (8.2.2) as an equation expressing $\partial_{t_m} v_i(t)$ as a differential polynomial in v_j 's. Thus we obtain the *zero curvature representation* of the m -th generalized KdV equation (8.1.2). This form of the generalized KdV equations was first introduced in [DS].

8.3. Flag Manifolds and the m KdV Hierarchy

8.3.1. Recall from § 4.1.1 that for the homogeneous filtration of the loop group, we were able to define one piece $\text{LG}_- = \text{LG}_{\leq 0}$ of an opposite filtration, just using the global curve \mathbf{P}^1 . In order to find a similar partial splitting of the principal filtration, let us fix a Borel subgroup $B_- \subset G$ which is transverse to the Borel subgroup B defined by A_+ at ∞ (in other words, $[B_-] \in G/B$ lies in the open B -orbit).

Let us fix a point $0 \in \mathbf{P}^1 \setminus \infty$. Define $\text{LG}^- = \text{LG}^{\leq 0} \subset \text{LG}$ as the subgroup that consists of loops x which extend to all of $\mathbf{P}^1 \setminus \infty$ (that is, $x \in \text{LG}_-$) whose value at 0 lies in B_- . Thus LG^- is a “lower Iwahori subgroup” of LG . (Replacing B_- by a parabolic, one obtains “parahori subgroups” of LG .)

8.3.2. Definition. — The *affine flag manifold* $\mathcal{F}l$ is the scheme of infinite type representing the moduli functor of G -torsors on \mathbf{P}^1 , equipped with a trivialization on D and a reduction to B_- at 0 .

The following proposition is proved in the same way as Proposition 4.1.5.

8.3.3. Proposition. — $\mathcal{F}l \cong \text{LG}^- \backslash \text{LG}$.

8.3.4. The analog of abelianization of \mathcal{G}_r^X for the affine flag manifold is the stack $\mathcal{F}l_A$ classifying G -torsors \mathcal{Z} on \mathbf{P}^1 , equipped with a reduction \mathcal{E}^{A_+} of the LG_+ -torsor $\mathcal{E}|_D$ to A_+ , and a reduction to B_- at 0 .

The group $\text{LG}^{>0}$ acts on $\mathcal{F}l$ from the right. The infinitesimal decomposition $\text{Lg} \cong \text{Lg}^- \oplus \text{Lg}^{>0}$ implies that the orbit of the identity coset is open, and since

$\mathrm{LG}^- \cap \mathrm{LG}^{>0} = 1$, the orbit is in fact isomorphic to $\mathrm{LG}^{>0}$. This orbit is denoted by $\mathcal{F}l$. There is an obvious map $\mathcal{F}l \rightarrow \mathcal{G}r^{\mathbf{P}^1}$, forgetting the flag at 0. The fibers of this map are isomorphic to the flag manifold G/B of G , and $\mathcal{F}l^\circ$ is the inverse image of the big cell $\mathcal{G}r^\circ$ under this map. The restriction $\mathcal{F}l^\circ \rightarrow \mathcal{G}r^\circ$ is an N-bundle, whose fibers are identified with the big cell of G/B .

The image of $\mathcal{F}l$ in $\mathcal{F}l_\Lambda$ is an open substack of $\mathcal{F}l_\Lambda$, which we denote by $\mathcal{F}l_\Lambda^\circ$. In the same way as in the case of abelianized Grassmannians, one shows that $\mathcal{F}l_\Lambda^\circ$ is the quotient of a scheme of infinite type $\mathrm{Fl}_\Lambda^\circ = \mathrm{LG}^{>0}/A_+'$ by the trivial action of $Z(G)$. The natural morphism $\mathrm{Fl}_\Lambda^\circ \rightarrow \mathrm{Gr}_\Lambda^\circ$ is again an N-bundle.

The following are generalizations of the notions of opers and affine opers for the flag manifold. Note that though one can define affine Miura opers on an arbitrary curve Y as in Definition 7.2.1, we only need them when Y is \widehat{A}_{-1} . Therefore the definition below will suffice for our current purposes.

8.3.5. Definition. — A *Miura G-oper* on a curve Y is a quadruple $(\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}'_B)$, where \mathcal{E} is a G -torsor on Y with a connection ∇ and two reductions: $\mathcal{E}_B, \mathcal{E}'_B$, to the Borel subgroup B of G , which are in generic position. The reduction \mathcal{E}'_B is preserved by ∇ , while \mathcal{E}_B is in relative position \mathbf{O} with respect to ∇ .

An *affine Miura oper* is a quadruple $(\mathcal{V}, \nabla, \mathcal{V}^-, \mathcal{V}^+)$, where \mathcal{V} is an LG -torsor on \widehat{A}_{-1} with a connection ∇ , a flat reduction \mathcal{V}^- to LG^- and a reduction \mathcal{V}^+ to LG^+ in tautological relative position with ∇ .

A *generic affine Miura oper* on \widehat{A}_{-1} is a quadruple $(\mathcal{V}, \nabla, \mathcal{V}^-, \mathcal{V}^+)$ as above, such the reductions \mathcal{V}^- and \mathcal{V}^+ are in generic position.

The Miura G -opers on \widehat{A}_{-1} form a stack $\mathcal{M}\mathcal{O}p(\widehat{A}_{-1})$. The affine opers on \widehat{A}_{-1} form a stack that is denoted by $\mathcal{A}\mathcal{M}\mathcal{O}(\widehat{A}_{-1})$, and generic affine opers form its open substack $\mathcal{A}\mathcal{M}\mathcal{O}^\circ(\widehat{A}_{-1})$.

We have obvious surjective morphisms $\mathcal{M}\mathcal{O}p(\widehat{A}_{-1}) \rightarrow \mathcal{O}p(\widehat{A}_{-1})$ and $\mathcal{A}\mathcal{M}\mathcal{O}(\widehat{A}_{-1}) \rightarrow \mathcal{A}\mathcal{O}(\widehat{A}_{-1})$.

The following lemma is proved in the same way as Proposition 7.2.3.

8.3.6. Lemma. — There is a canonical isomorphism $\mathcal{A}\mathcal{M}\mathcal{O}^\circ(\widehat{A}_{-1}) \simeq \mathcal{M}\mathcal{O}p(\widehat{A}_{-1})$, which makes the following diagram commutative.

$$(8.3.1) \quad \begin{array}{ccc} \mathcal{A}\mathcal{M}\mathcal{O}^\circ(\widehat{A}_{-1}) & \xrightarrow{\sim} & \mathcal{M}\mathcal{O}p(\widehat{A}_{-1}) \\ \downarrow & & \downarrow \\ \mathcal{A}\mathcal{O}^\circ(\widehat{A}_{-1}) & \xrightarrow{\sim} & \mathcal{O}p(\widehat{A}_{-1}) \end{array}$$

Here the isomorphism $\mathcal{A}\mathcal{O}^\circ(\widehat{A}_{-1}) \simeq \mathcal{O}p(\widehat{A}_{-1})$ from Lemma 7.2.3 corresponds to the point 0 used in the definition of the subgroup LG^- .

8.3.7. The stack $\mathcal{A}\mathcal{M}\mathcal{O}^\circ(\widehat{\mathbb{A}}_{-1})$ (resp., $\mathcal{M}\mathcal{O}p(\widehat{\mathbb{A}}_{-1})$) is the quotient of an affine scheme of infinite type $\text{AMO}^\circ(\widehat{\mathbb{A}}_{-1})$ (resp., $\text{MO}p(\widehat{\mathbb{A}}_{-1})$) by the trivial action of $Z(\mathbf{G})$.

The following statement is proved in the same way as Corollary 8.1.6.

8.3.8. Theorem. — $F\ell_{\mathbb{A}}^p \simeq \text{AMO}_{-1}^\circ$ and $F\ell_{\mathbb{A}}^p \simeq \text{MO}p(\widehat{\mathbb{A}}_{-1})$.

8.3.9. Let us describe explicitly the ring of functions on $\text{MO}p(\widehat{\mathbb{A}}_{-1})$. Let $(\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}'_B)$ be a Miura oper on $\widehat{\mathbb{A}}_{-1}$. Since the B-reductions \mathcal{E}_B and \mathcal{E}'_B are in generic position, they produce a unique compatible reduction \mathcal{E}_H of \mathcal{E} to H . Let us trivialize \mathcal{E}_H . The operator of connection ∇ reads relative to this trivialization as follows:

$$\partial_t + \sum_{i=1}^{\ell} \phi_i(t) \cdot f_i + \mathbf{h}(t), \quad \mathbf{h}(t) \in \mathfrak{h}[[t]].$$

By changing trivialization of \mathcal{E}_H (i.e., applying a gauge transformation from $H[[t]]$), it can be brought to the form

$$(8.3.2) \quad \partial_t + \bar{p}_{-1} + \mathbf{h}(t), \quad \mathbf{h}(t) \in \mathfrak{h}[[t]].$$

Denote $u_i(t) = \alpha_i(\mathbf{h}(t))$. We can now identify $\text{AO}p(\widehat{\mathbb{A}}_{-1})$ with the pro-vector space of ℓ -tuples $(u_i(t))_{i=1, \dots, \ell}$ of formal Taylor series. For instance, in the case $\mathfrak{g} = \mathfrak{sl}_2$, we identify $\text{AO}p(\widehat{\mathbb{A}}_{-1})$ with the space of operators of the form

$$\partial_t + \begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 1 & -\frac{1}{2}u(t) \end{pmatrix}.$$

Let $u_i^{(n)}$, $i = 1, \dots, \ell; n \geq 0$, be the function on $\text{AO}p(\widehat{\mathbb{A}}_{-1})$, whose value at $(u_i(t))_{1, \dots, \ell}$ equals $\partial_t^n u_i(t)|_{t=0}$. The ring of functions $\mathbf{C}[\text{AO}p(\widehat{\mathbb{A}}_{-1})]$ on $\text{AO}p(\widehat{\mathbb{A}}_{-1})$ is isomorphic to the polynomial ring $\mathbf{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$, on which ∂_t acts by $\partial_t \cdot u_i^{(n)} = u_i^{(n+1)}$. Thus we obtain a new proof of Proposition 4 from [FF2] (see also [F]).

8.3.10 Theorem. — The ring of functions $\mathbf{C}[F\ell_{\mathbb{A}}^p]$ on $F\ell_{\mathbb{A}}^p = \text{LG}^{>0}/A_+$ is isomorphic to the ring of differential polynomials $\mathbf{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$, on which the action of p_{-1} is given by $p_{-1} \cdot u_i^{(n)} = u_i^{(n+1)}$.

8.3.11. mKdV Hierarchy. — The infinite-dimensional abelian Lie algebra $\mathfrak{a}/\mathfrak{a}_+$ acts from the right on $F\ell_{\mathbb{A}}^p$. Hence we obtain an infinite set of commuting flows on $\text{AO}p(\widehat{\mathbb{A}}_{-1})$, and an infinite set of commuting evolutionary derivations of $\mathbf{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$.

Denote by \bar{p}_{-m} the derivation corresponding to $p_{-m} \in \mathfrak{a}/\mathfrak{a}_+$, $m \in \mathbf{I}$. In particular, we have: $\bar{p}_{-1} = \partial_t$. The equation

$$(8.3.3) \quad \partial_{t_m} u_i = \bar{p}_{-m} \cdot u_i, \quad i = 1, \dots, \ell,$$

is called the m -th equation of the *generalized modified KdV hierarchy* (or mKdV hierarchy) associated to \mathfrak{g} .

8.3.12. Miura transformation. — We have the following commutative diagram of differential rings

$$(8.3.4) \quad \begin{array}{ccc} (\mathbf{C}[Gr_{\Lambda}^{\circ}], p_{-1}) & \xrightarrow{\sim} & (\mathbf{C}[v_i^{(n)}], \partial_t) \\ \downarrow & & \downarrow \\ (\mathbf{C}[Fl_{\Lambda}^{\circ}], p_{-1}) & \xrightarrow{\sim} & (\mathbf{C}[u_i^{(n)}], \partial_t) \end{array}$$

where the vertical arrows are embeddings and the horizontal arrows are isomorphisms. Furthermore, the above diagram is compatible with the action of $\mathfrak{a}/\mathfrak{a}_+$ on all four rings.

The embedding $\mathbf{C}[v_i^{(n)}]_{i=1, \dots, \ell; n \geq 0} \rightarrow \mathbf{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$ is called the *Miura transformation*. The corresponding map of spectra $\mathrm{AO}p(\widehat{\mathbf{A}}_{-1}) \rightarrow \mathrm{Op}(\widehat{\mathbf{A}}_{-1})$ is simply the forgetting of the flat \mathbf{B} -reduction $\mathcal{E}'_{\mathbf{B}}$.

Explicitly, given a Miura oper (8.3.2), we view it as an element of $\widetilde{\mathrm{Op}}(\widehat{\mathbf{A}}_{-1})$ (§ 7.1.4) and take its projection onto $\mathrm{Op}(\widehat{\mathbf{A}}_{-1})$, i.e. apply a gauge transformation by an appropriate element of $\mathbf{N}[[t]]$ to bring it to the form (7.1.2). For instance, in the case $\mathfrak{g} = \mathfrak{sl}_2$ we have the following transformation:

$$\begin{pmatrix} 1 & -\frac{u}{2} \\ 0 & 1 \end{pmatrix} \left(\partial_t + \begin{pmatrix} \frac{u}{2} & 0 \\ 1 & -\frac{u}{2} \end{pmatrix} \right) \begin{pmatrix} 1 & -\frac{u}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \partial_t + \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}.$$

Therefore the Miura transformation is in this case

$$(8.3.5) \quad v \rightarrow \frac{1}{4}u^2 + \frac{1}{2}\partial_t u.$$

8.3.13. Zero curvature representation. — We start out with an analog of Lemma 4.2.5 in the context of the flag manifold.

Recall that the big cell $\mathcal{F}^{\ell} \subset \mathcal{F}l$ is isomorphic to the group $\mathrm{LG}^{>0}$. Consider the LG^{-} -bundle $\mathcal{T}: \mathrm{LG} \rightarrow \mathcal{F}l = \mathrm{LG}^{-} \backslash \mathrm{LG}$. The fiber of \mathcal{T} over $\mathbf{K}^+ \in \mathrm{LG}^{>0} = \mathcal{F}^{\ell}$ consists of all elements \mathbf{K} of LG that can be written in the form $\mathbf{K} = \mathbf{K}^- \mathbf{K}^+$, for some $\mathbf{K}^- \in \mathrm{LG}^{-}$. Thus the restriction of \mathcal{T} to \mathcal{F}^{ℓ} is canonically trivialized.

We are now in the setting of Lemma 2.3.2, where $M = \mathcal{F}^\ell$, and $A = \widehat{A}_p \subset A/A_+$, where $p \in \mathfrak{a}/\mathfrak{a}_+$. The group \widehat{A}_p acts on \mathcal{F}^ℓ from the right. Hence we obtain for each $K^+ \in \text{LG}^{>0} = \mathcal{F}^\ell$ a connection on the LG^- -bundle $\pi_{K^+}^*(\mathcal{S})$ over \widehat{A}_p (here $\pi_{K^+} : \widehat{A}_p \rightarrow \mathcal{F}^\ell$ is the \widehat{A}_p -orbit of K^+). The above trivialization of \mathcal{S} induces a trivialization of $\pi_{K^+}^*(\mathcal{S})$, and allows us to write down an explicit formula for this connection in the same way as in Lemma 4.2.5.

8.3.14. Lemma. — In the above trivialization of \mathcal{S} the connection operator on $\pi_{K^+}^*(\mathcal{S})$ equals

$$(8.3.6) \quad \partial_t + (\mathbf{K}^+(t)p\mathbf{K}^+(t)^{-1})^-,$$

where $\mathbf{K}^-(t)\mathbf{K}^+(t)$ is the factorization of $\mathbf{K}^+e^{-\psi}$.

8.3.15. The isomorphism of Proposition 6.2.3, reformulated for $F\ell_A^p$, assigns to each point $K^+ \in F\ell_A^p$ a quadruple $(\mathcal{Z}, \nabla, \mathcal{Z}^-, \mathcal{Z}^+)$, where \mathcal{Z} is an LG -torsor on A/A_+ with a flat connection ∇ , a flat LG^- -reduction \mathcal{Z}^- , and an LG^+ -reduction in tautological relative position with ∇ . We want to calculate the contractions $\nabla_p, p \in \mathfrak{a}/\mathfrak{a}_+$, of ∇ in the trivialization of \mathcal{Z} , which preserves both \mathcal{Z}^- and \mathcal{Z}^+ , and has symbol p . Such a trivialization is unique, because $\text{LG}^- \cap \text{LG}^{>0} = \text{Id}$. From Lemma 8.3.14, we obtain the following formula:

$$\nabla_p = \partial_t + (\mathbf{K}^+(t)p\mathbf{K}^+(t)^{-1})^-.$$

The flatness of the connection ∇ leads to the zero curvature equations

$$(8.3.7) \quad \left[\partial_{t_m} + (\mathbf{K}^+(t)p_{-m}\mathbf{K}^+(t)^{-1})^-, \partial_{t_n} + (\mathbf{K}^+(t)p_{-m}\mathbf{K}^+(t)^{-1})^- \right] = 0.$$

The p_{-1} component of the connection is a modified affine oper

$$\partial_t + (\mathbf{K}^+(t)p_{-1}\mathbf{K}^+(t)^{-1})^- = \partial_t + p_{-1} + \mathbf{h}(t),$$

where $\mathbf{h}(t) \in \mathfrak{h}[[t]]$ (cf. § 8.3.9). Therefore in the special case $n = 1$ we obtain the equation

$$(8.3.8) \quad \left[\partial_{t_m} + (\mathbf{K}^+(t)p_{-m}\mathbf{K}^+(t)^{-1})^-, \partial_t + p_{-1} + \mathbf{h}(t) \right] = 0.$$

This is the *zero curvature representation* of the m -th mKdV equation (8.3.3). Any solution for $(\mathbf{K}^+(t)p_{-m}\mathbf{K}^+(t)^{-1})^-$ as an element of $\mathfrak{g}[[t]]$ is a differential polynomial in $u_i(t) = \alpha_i(\mathbf{h}(t))$, i.e., $L_m \in \mathbf{C}[u_i^{(n)}] \otimes \mathfrak{g}$. Hence formula (8.3.8) expresses $\partial_{t_m} u_i$ as a differential polynomial in u_j 's (see [EF1, F] for more detail).

Note that in contrast to the KdV equations, we do not have any residual gauge freedom in equations (8.3.8), because the trivialization of \mathcal{Z} compatible with \mathcal{Z}^- and \mathcal{Z}^+ is unique.

8.3.16. Example. — Let us derive the mKdV equation, which is (8.3.8) in the case $\mathfrak{g} = \mathfrak{sl}_2$, $m = 3$, following [F].

We have:

$$(8.3.9) \quad \partial_t + p_{-1} + \mathbf{h} = \partial_t + \begin{pmatrix} \frac{1}{2}u & z \\ 1 & -\frac{1}{2}u \end{pmatrix}.$$

Now we have to compute $(\mathbf{K}^+(t)p_{-3}\mathbf{K}^+(t)^{-1})^-$. This can be done recursively using the equation

$$[p_{-1} + \mathbf{h}, (\mathbf{K}^+(t)p_{-3}\mathbf{K}^+(t)^{-1})^-] = 0$$

(see [F]). It gives:

$$(\mathbf{K}^+(t)p_{-3}\mathbf{K}^+(t)^{-1})^- = \begin{pmatrix} \frac{1}{2}uz^2 - \left(\frac{1}{16}u^3 - \frac{1}{8}\partial_t^2 u\right) & z^2 + \left(-\frac{1}{8}u^2 + \frac{1}{4}\partial_t u\right)z \\ z - \left(\frac{1}{8}u^2 + \frac{1}{4}\partial_t u\right) & -\frac{1}{2}uz^2 + \left(\frac{1}{16}u^3 - \frac{1}{8}\partial_t^2 u\right) \end{pmatrix}.$$

Substituting into formula (8.3.8), we obtain:

$$(8.3.10) \quad \partial_{t_3} u = \frac{3}{8}u^2 \partial_t u - \frac{1}{4}\partial_t^3 u.$$

This is the mKdV equation up to a slight redefinition of variables. One can check that the corresponding equation on $v = \frac{1}{4}u^2 + \frac{1}{2}\partial_t u$ (applying the Miura transformation) is the KdV equation (8.1.3).

8.4. Generalized Affine Opers

We now introduce generalized affine opers, which appear to provide the broadest setting in which we can obtain integrable systems of KdV type using the Drinfeld-Sokolov construction, Proposition 7.3.7. In particular, their moduli spaces will turn out to be isomorphic to the appropriate abelianized Grassmannians \mathcal{G}_r^X .

8.4.1. Let $\mathfrak{a} \subset \mathbf{L}\mathfrak{g}$ be a Heisenberg algebra with the canonical filtration $\{\mathfrak{a}^{\geq j}\}$, and let $p \in \mathfrak{a}^{\geq l}$, $l < 0$, be an element with regular symbol \bar{p} (then p is automatically regular as well). Let $\{\mathbf{L}\mathfrak{g}^{\geq j}\}$ be a filtration compatible with \mathfrak{a} , in the sense of § 5.6. We consider this as chosen once and for all, so that all superscripts refer to this filtration, and we otherwise suppress it in the notation.

Recall the notion of tautological relative position from Definition 2.3.6, and the notation $\widehat{A}_p = \{e^p\}$ for the one-dimensional formal additive subgroup of A/A_+ generated by p . We then have the following generalization of the notion of an affine oper.

8.4.2. Definition. — An *affine* (A, p) -oper is a quintuple $(\mathcal{Z}, \nabla, \mathcal{Z}_-, \mathcal{Z}_+, \mathcal{Z}^+)$, where \mathcal{Z} is an LG-torsor on \widehat{A}_p with a flat connection ∇ , a flat reduction \mathcal{Z}_- to LG_-^X , and compatible reductions \mathcal{Z}_+ and \mathcal{Z}^+ to LG_+ and LG^+ , respectively. We require that \mathcal{Z}^+ is in tautological relative position with ∇_p .

In the case when $X = \mathbf{P}^1$, we call an affine (A, p) -oper *generic* if the reductions \mathcal{Z}_- and \mathcal{Z}_+ are in general position.

The moduli stack classifying affine (A, p) -opers is denoted by $\mathcal{A}\mathcal{O}_{A, p}^X$, and the open substack of $\mathcal{A}\mathcal{O}_{A, p}^{\mathbf{P}^1}$ classifying generic affine (A, p) -opers is denoted by $\mathcal{A}\mathcal{O}_{A, p}^\circ$.

8.4.3. Remarks.

- (1) When \mathfrak{a} is a smooth Heisenberg, so that $\mathfrak{a}^+ \subset \mathfrak{a}_+$ (Definition 5.3.5), we may require $\mathrm{LG}^+ \subset \mathrm{LG}_+$ (e.g. for a filtration coming from a compatible gradation), and hence the LG_+ -reduction is redundant. But in general we do not have such an inclusion, making the above definition somewhat cumbersome.
- (2) Since LG^+ preserves the \mathfrak{a} -filtration, the \mathcal{Z}^+ -twist $(\mathrm{Lg})_{\mathcal{Z}^+}$ carries a canonical filtration and hence the relative position condition makes sense. Also note that compatible reductions to LG_+ and LG^+ amount to a reduction \mathcal{Z}_+^+ to $\mathrm{LG}_+ \cap \mathrm{LG}^+$.
- (3) It is possible to define affine (A, p) opers on an arbitrary differential scheme (S, ∂) . Theorem 8.5.2 will then identify $\mathcal{G}r_A^X$ (with the action of p) with the moduli stack of such objects.
- (4) For general Heisenbergs, there seems to be no simple generalization of the notion of G-oper. It is however possible to identify generic versions of affine opers with flat connections on appropriate finite-dimensional bundles. If A is of Coxeter type (i.e. LG-conjugate to the principal Heisenberg), then we can always define a map from an open subset of the moduli of “generic” affine opers to the moduli of ordinary opers.

The following key result follows from the generalized Drinfeld-Sokolov gauge (Proposition 7.3.7):

8.4.4. Proposition. — Let $(\mathcal{Z}, \nabla, \mathcal{Z}_-, \mathcal{Z}^+)$ be an affine oper on \widehat{A}_{-1} . Then \mathcal{Z} has a unique reduction $\mathcal{Z}^{A_+} \subset \mathcal{Z}^+$ to A_+ , such that

- (1) the induced A -torsor $\mathcal{Z}^A \subset \mathcal{Z}$ is flat;

(2) \mathcal{Z}^{Λ^+} has the tautological relative position with respect to ∇ .

8.4.5. Proof. — First recall that the above relative position condition means that the action of the vector field p on \widehat{A}_p lies in the \mathcal{Z}^{Λ^+} -twist of the A_+ -orbit $[-p] \in \mathfrak{a}/\mathfrak{a}_+ \subset \mathrm{Lg}/\mathfrak{a}_+$.

We now use Proposition 7.3.7 to reduce \mathcal{Z}^+ to A^+ .

Pick an arbitrary trivialization $t^+ : \mathrm{LG}^+ \rightarrow \mathcal{Z}^+$ of the LG^+ -bundle \mathcal{Z}^+ . By the assumption on the connection ∇ , we may pick this trivialization (after possibly using the LG^+ -action) so that $t^+(\nabla_p)t^{+1} = \partial_p + p + q$, where q is a section of the trivial LG^+ -bundle on \widehat{D}_t . By Lemma 7.3.7, we may find an $M \in \mathrm{LG}^{>0}$ so that $M^{-1}t^{-1}\nabla_p tM = \partial + p + p_+$, with $p_+ \in \mathfrak{a}^{>l}$. Define $\mathcal{Z}^{\Lambda^+} = tM(A^+)$, the image of $A^+ \subset \mathrm{LG}$ under the isomorphism $tM : \mathrm{LG} \rightarrow \mathcal{Z}$. This is an A^+ -torsor in \mathcal{Z} , since if $x = tM(a) \in \mathcal{Z}^{\Lambda^+}$ and $b \in A^+$, then $b \cdot x = tM(ab^{-1}) \in \mathcal{Z}^{\Lambda^+}$.

Now suppose we pick a different trivialization $t' : \mathrm{LG} \rightarrow \mathcal{Z}$, and define M' as above. It follows that the composition $N = M^{-1}t^{-1}t'M'$ conjugates the connection $\partial + p + p'_+$ to the form $\partial + p + p_+$, where both $p_+, p'_+ \in \mathfrak{a}^{>l}$. By the uniqueness statement of Proposition 7.3.7, it follows that $N \in A^+$. Hence $x = tMa = t'M'd'$, where $a = Na'$, so that the two definitions of \mathcal{Z}^{Λ^+} agree. Furthermore, the A^+ -torsor structures agree as well: $b \cdot x = t'M'(a'b^{-1}) = tM(Na'b^{-1}) = tM(ab^{-1})$.

Now the intersection $\mathcal{Z}^{\Lambda^+} \cap \mathcal{Z}_+$ is an A_+ -reduction \mathcal{Z}^{Λ^+} , since by definition the LG_+ - and LG^+ -reductions are compatible. It is clear that \mathcal{Z}^{Λ^+} satisfies the above conditions.

8.5. Generalized Drinfeld-Sokolov Hierarchies

In this subsection we extend the construction of integrable systems to general Heisenberg subalgebras.

8.5.1. Recall the definition of $\mathcal{M}_{A,p}^X$ from § 6.3 and the isomorphism between the abelianized Grassmannian $\mathcal{G}r_A^X$ and $\mathcal{M}_{A,p}^X$ from Proposition 6.3.1. The following results are proved in the same way as in the case of the principal Heisenberg. (See Remark 5.6.10 concerning strongly regular elements.)

8.5.2. Theorem. — Let $p \in \mathfrak{a}^{>l}$, $l < 0$, be a strongly regular element.

- (1) There is a natural morphism from $\mathcal{M}_{A,p}^X$ to $\mathcal{A}\mathcal{O}_{A,p}^X$, and hence from $\mathcal{G}r_A^X$ to $\mathcal{A}\mathcal{O}_{A,p}^X$.
- (2) The morphism between the abelianized Grassmannian $\mathcal{G}r_A^X$ and the moduli stack $\mathcal{A}\mathcal{O}_{A,p}^X$ of affine (A, p) -opers on \widehat{A}_p is an isomorphism.

- (3) The above isomorphism identifies the big cell $\mathcal{G}r_A^\circ$ of $\mathcal{G}r_A^{\mathbf{P}^1}$ and the moduli stack $\mathcal{AO}_{\Lambda, p}^\circ$ of generic affine opers on \widehat{A}_p .

8.5.3. Recall that $\mathcal{G}r_A^\circ$ is the quotient of the scheme $Gr_A^\circ = G \backslash LG_+ / A_+$ by the trivial action of a group A_0 (see § 6.1.5). On the other hand, the argument of Lemma 6.1.6 applied to $\mathcal{AO}_{\Lambda, p}^\circ$ shows that it is the quotient of a scheme $AO_{\Lambda, p}^\circ$ by the trivial action of A_0 . Therefore we obtain

8.5.4. Corollary. — For each strongly regular $p \in \mathfrak{a}^{\geq l}$, $l < 0$, there is a canonical isomorphism $Gr_A^\circ \simeq AO_{\Lambda, p}^\circ$.

8.5.5. Canonical Form. — Choosing a canonical form of an affine (A, p) -oper allows us to identify the ring of functions on $AO_{\Lambda, p}^\circ$ (and hence on Gr_A°) with the ring of differential polynomials. This can be done as follows.

Given an (A, p) -oper, we can choose a trivialization of the underlying LG-bundle, which preserves the reductions \mathcal{Z}^+ , \mathcal{Z}^- . Such a trivialization is unique up to the action of the finite-dimensional group $LG^+ \cap LG_-$. Requiring the trivialized connection to have symbol p reduces gauge group to the unipotent group $R = LG^{>0} \cap LG_-$. The connection operator of the affine oper then reads:

$$(8.5.1) \quad \partial_t + p + b(t), \quad b(t) \in (\mathfrak{Lg}^{>l} \cap \mathfrak{Lg}_-)[[t]].$$

Denote the space of all such operators by $\widetilde{AO}_{\Lambda, p}^\circ$. Then $AO_{\Lambda, p}^\circ$ is the quotient of $\widetilde{AO}_{\Lambda, p}^\circ$ by the gauge action of $R[[t]]$. Denote by \mathfrak{r} the (nilpotent) Lie algebra of the group R . The following lemma, which is a generalization of Lemma 7.1.5, is proved along the lines of Proposition 7.3.7.

8.5.6. Lemma. — The action of $R[[t]]$ on $\widetilde{AO}_{\Lambda, p}^\circ$ is free. Moreover, each $R[[t]]$ -orbit in $\widetilde{AO}_{\Lambda, p}^\circ$ contains a unique operator of the form

$$(8.5.2) \quad \partial_t + p + v(t), \quad v(t) \in V[[t]],$$

where $V \subset \mathfrak{r}$ is such that $\mathfrak{r} = V \oplus \text{Im ad } p$. Thus, we can identify $AO_{\Lambda, p}^\circ$ with $V[[t]]$.

8.5.7. Choosing a basis $\{v_i\}$ of V , we identify $\mathbf{C}[AO_{\Lambda, p}^\circ]$, and hence $\mathbf{C}[Gr_A^\circ]$, with the ring of differential polynomials $\mathbf{C}[v_i^{(n)}]$. Moreover, the action of p on Gr_A° corresponds to the standard action of ∂_t on $\mathbf{C}[v_i^{(n)}]$.

The action of A/A_+ on Gr_A° gives rise to an infinite hierarchy of commuting flows on $AO_{\Lambda, p}^\circ$, and hence commuting evolutionary derivations on $\mathbf{C}[v_i^{(n)}]$. They form the *generalized Drinfeld-Sokolov hierarchy* associated to A and the strongly regular element

$p \in \mathfrak{a}$. One can write these equations down explicitly in the *zero curvature form* following § 8.2.1. Namely, for each $q \in \mathfrak{a}/\mathfrak{a}_+$ we have the equation

$$(8.5.3) \quad [\partial_s + (\mathbf{K}_+(t)q\mathbf{K}_+(t)^{-1})_-, \partial_t + p + v(t)] = 0.$$

Here $\mathbf{K}_+(t)$ comes from the factorization $\mathbf{K}_-(t)\mathbf{K}_+(t) = \mathbf{K}_+e^{-\psi}$ along the $\widehat{\mathbf{A}}_p$ -orbit of $\mathbf{K}_+ \in \mathbf{G} \backslash \mathbf{LG}_+/A'_+ \simeq \text{Gr}_A^\circ$. Note that

$$\partial_t + p + v(t) = \partial_t + (\mathbf{K}_+(t)p\mathbf{K}_+(t)^{-1})_-.$$

The equation (8.5.3) is invariant under the residual gauge group $\mathbf{R}[[t]]$. Hence the above equations with q running over $\mathfrak{a}/\mathfrak{a}_+$ really define commuting evolutionary derivations on the ring of differential polynomials $\mathbf{C}[v_i^{(n)}]$.

8.5.8. Remark: How Many Hierarchies? — Suppose A and A' are \mathbf{LG}_+ -conjugate Heisenbergs, so that $gA_+g^{-1} = A'_+$ for some fixed $g \in \mathbf{LG}_+$. Then there is an isomorphism $\mathcal{G}r_{A'}^X \simeq \mathcal{G}r_A^X$ intertwining the actions of A'/A'_+ and A/A_+ . Namely, we have a well-defined map on double quotients $i_K : \mathbf{LG}_- \cdot \mathbf{M} \cdot A'_+ \mapsto \mathbf{LG}_- \cdot \mathbf{M}g \cdot A_+$. It follows that the integrable systems associated with A and $p \in \mathfrak{a}$ and A' and $p' = gp g^{-1} \in \mathfrak{a}'$ are equivalent.

Thus our construction associates an integrable system to each \mathbf{LG}_+ -conjugacy class of pairs (\mathfrak{a}, p) , where \mathfrak{a} is an arbitrary Heisenberg subalgebra of \mathbf{Lg} , and p is a strongly regular element of $\mathfrak{a}/\mathfrak{a}_+$. (Different strongly regular p of the same \mathfrak{a} give different presentations of the same underlying integrable system.)

Let $\mathbf{LH}^{[w]}$ denote a graded Heisenberg of the same type $[w]$ as A . As we remarked in § 5.3.6 and § 5.3.8, the “difference” between A and $\mathbf{LH}^{[w]}$ is measured by the finite-dimensional group scheme A^+/A_+ . In particular, this group acts on $\mathcal{G}r_A^X$ (commuting with the flows) and the quotient is isomorphic to $\mathcal{G}r_{\mathbf{LH}^{[w]}}^X$ (with its natural flows). However this isomorphism does *not* lead to an isomorphism of the big cells, and so the resulting integrable systems on $\mathcal{G}r_A^\circ$ and $\mathcal{G}r_{\mathbf{LH}^{[w]}}^\circ$ can be quite different.

8.5.9. There is an obvious version of the above construction, in which the abelianized Grassmannian Gr_A° is replaced by its flag manifold version \mathbf{Fl}_A° , as in § 8.3. In particular, we obtain an identification between \mathbf{Fl}_A° and an appropriate moduli space of affine Miuraopers. The corresponding flows form the generalized \mathbf{mKdV} hierarchy.

One can also introduce “partially modified” hierarchies by considering moduli spaces that are intermediate between $\mathcal{G}r_A^{\mathbf{P}^1}$ and $\mathcal{F}l_A$, namely, the moduli space of \mathbf{G} -bundles \mathcal{E} on \mathbf{X} with a reduction of $\mathcal{E}|_{\mathbf{D}^\times}$ to A_+ and a reduction of $\mathcal{E}|_{0 \in \mathbf{X}}$ to a parabolic subgroup \mathbf{P} of \mathbf{G} . (The partial flag at 0 is chosen so as to provide a partial splitting of the A -filtration.)

8.5.10. Examples. — In the case when \mathfrak{a} is a graded Heisenberg subalgebra, the corresponding generalized Drinfeld-Sokolov hierarchy was introduced in [dGHM].

In this case, one can promote the canonical filtration on \mathbf{Lg} into a \mathbf{Z} -gradation, which simplifies the study of the equations. Delduc and Feher [DF, Fe] have described explicitly the strongly regular elements p of the graded Heisenberg subalgebras.

The most widely known example is of course the case when \mathfrak{a} is the principal Heisenberg subalgebra and $p = p_{-1}$, which corresponds to the generalized KdV hierarchies discussed above. In [Ba], Balan studies, along the lines of [FF1, FF2, EF1], the hierarchy corresponding to $p = p_{-3}$ in the case of the principal Heisenberg subalgebra of \mathbf{Lsl}_2 . The other well-known example is the generalized AKNS (or non-linear Schrödinger) hierarchy, which corresponds to the case of the homogeneous Heisenberg subalgebra (see [FF3]).

Finally, we have worked out explicitly the case of the simplest non-smooth Heisenberg subalgebra, which was introduced in § 5.5. We plan to present this hierarchy elsewhere.

Acknowledgments

We would like to acknowledge very helpful conversations with A. Beilinson, P. Burchard, D. Gaiatsgory, A. Genestier, M. Rothstein, and especially R. Donagi, who explained to us the theory of spectral curves, alerted us to the role of singular curves, and provided useful comments on an early draft.

REFERENCES

- [AB] M. ADAMS and M. BERGVELT, The Krichever map, vector bundles over algebraic curves, and Heisenberg algebras, *Comm. Math. Phys.*, **154** (1993), 265-305.
- [AMP] A. ÁLVAREZ, J. MUÑOZ and F. PLAZA, The algebraic formalism of soliton equations over arbitrary base fields, *Workshop on Abelian Varieties and Theta Functions* (Morelia, 1996), *Aportaciones Mat. Investig.*, **13**, Soc. Mat. Mexicana (3-40), 1998 (alg-geom/9606009).
- [Ba] A. BALAN, *Les équations mKdV généralisées*, Preprint école Polytechnique, No. 98-5, April 1998.
- [BL] A. BEAUVILLE and Y. LASZLO, Conformal blocks and generalized theta functions, *Comm. Math. Phys.*, **164** (1993), 385-419.
- [BB] A. BEILINSON and J. BERNSTEIN, A Proof of Jantzen Conjectures, *Advances in Soviet Mathematics*, Vol. **16**, Part 1 (1-50), 1993.
- [BD1] A. BEILINSON and V. DRINFELD, *Opers*, Preprint, 1994.
- [BD2] A. BEILINSON and V. DRINFELD, *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*, in preparation.
- [BS] A. BLUM and U. STUHLER, Drinfeld modules and elliptic sheaves, *Vector bundles on curves — new directions* (eds S. Kumar et al.) LNM 1649, 1997.
- [Ch1] I. CHEREDNIK, Group interpretation of Baker functions and τ -function, *Uspekhi Mat. Nauk* **38**, No. 6 (1983), 133-134.
- [Ch2] I. CHEREDNIK, Determination of τ -functions for generalized affine Lie algebras, *Funct. Anal. Appl. Prilozhen*, **17**, No. 3 (1983), 93-95.
- [Ch3] I. CHEREDNIK, Functional realizations of basic representations of factorizable Lie groups and Lie algebras, *Funct. Anal. Appl.*, **19**, 193-206 (1985).
- [CC] C. CONTOU-CARRÈRE, Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré, *C. R. Acad. Sci. Paris, série I* **318** (1994), 743-746.

- [DJKM] E. DATE, M. JIMBO, M. KASHIWARA, T. MIWA, *Transformation groups for soliton equations*, in: M. Jimbo, T. Miwa (eds.), *Non-linear Integrable Systems — Classical Theory and Quantum Theory*, pp. 39-120, Singapore: World Scientific, 1983.
- [dGHM] M. de GROOT, T. HOLLOWOOD and L. MIRAMONTES, Generalized Drinfeld-Sokolov hierarchies, *Comm. Math. Phys.*, **145** (1992), 57-84.
- [DF] F. DELDUC and L. FEHÉR, Regular conjugacy classes in the Weyl group and integrable hierarchies, *J. Phys.*, **A28** (1995), 5843-5882.
- [D] R. DONAGI, Decomposition of spectral covers, *Astérisque*, **218** (1993), 145-175.
- [DG] R. DONAGI and D. GAITSGORY, *The gerbe of Higgs bundles*, Preprint math. AG/0005132.
- [DM] R. DONAGI and E. MARKMAN, Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles, Integrable systems and quantum groups, *Lecture Notes in Math.*, **1620**, Springer 1995.
- [Dr] V. DRINFELD, Commutative subrings of certain noncommutative rings, *Funct. Anal Appl.*, **11**, No. 1 (1977), 11-14.
- [DSi] V. DRINFELD and C. SIMPSON, B-structures on G-bundles and local triviality, *Math. Res. Lett.*, **2**, No. 6 (1995), 823-829.
- [DS] V. DRINFELD and V. SOKOLOV, Lie algebras and equations of Korteweg-de Vries type, *Journal of Soviet Mathematics*, vol. **30** (1985), 1975-2035.
- [EF1] B. ENRIQUEZ and E. FRENKEL, Equivalence of two approaches to integrable equations of KdV type, *Comm. Math. Phys.*, **185** (1997), 211-230.
- [EF2] B. ENRIQUEZ and E. FRENKEL, Geometric interpretation of the Poisson structure in affine Toda field theories, *Duke Math. J.*, **92** (1998), 459-495.
- [Fa] G. FALTINGS, Stable G-bundles and projective connections, *J. Alg. Geom.*, **2** (1993), 507-568.
- [Fe] L. FEHÉR, *KdV type systems and \mathcal{W} -algebras in the Drinfeld-Sokolov approach*, Preprint hep-th/9510001.
- [FHM] L. FEHÉR, J. HARNAD and I. MARSHALL, Generalized Drinfeld-Sokolov reductions and KdV type hierarchies, *Comm. Math. Phys.*, **154** (1993), 181-214.
- [FF1] B. FEIGIN, E. FRENKEL, Integrals of motion and quantum groups, in *Lect. Notes in Math.*, **1620**, pp. 349-418, Springer Verlag, 1995.
- [FF2] B. FEIGIN and E. FRENKEL, Kac-Moody groups and integrability of soliton equations, *Invent. Math.*, **120** (1995), 379-408.
- [FF3] B. FEIGIN and E. FRENKEL, Integrable hierarchies and Wakimoto modules, in *Differential Topology, Infinite-Dimensional Lie Algebras, and Applications: D. B. Fuchs' 60th Anniversary Collection*, A. Astashkevich and S. Tabachnikov (eds), pp. 27-60, AMS, 1999.
- [F] E. FRENKEL, Five lectures on soliton equations, in *Integral Systems*, C.-L. Terng and K. Uhlenbeck (eds.), *Surveys in Differential Geometry*, vol. 4, pp. 131-180, International Press, 1998.
- [Gin] V. GINZBURG, *Perverse Sheaves on a Loop Group and Langlands' Duality*, Preprint alg-geom/9511007.
- [Ha] G. HARDER, Halbeinfache Gruppenschemata über Dedekindringen, *Invent. Math.*, **4** (1967), 165-191.
- [Hi] N. HITCHIN, Stable bundles and integrable systems, *Duke Math. J.*, **54** (1990), 91-114.
- [Kac1] V. G. KAC, Infinite-dimensional algebras, Dedekind η -function, classical Möbius function and the very strange formula, *Adv. Math.*, **30** (1978), 85-136.
- [Kac2] V. G. KAC, *Infinite-dimensional Lie algebras*, third edition, Cambridge University Press, 1990.
- [KP] V. G. KAC and D. PETERSON, 112 constructions of the basic representation of the loop group of E8, *Proceedings of "Anomalies, geometry, topology"* (Argonne, 1985), pp. 276-298, World Scientific, 1985.
- [KSU] T. KATSURA, Y. SHIMIZU and K. UENO, Formal groups and conformal field theory over \mathbf{Z} , in integrable systems in quantum field theory and statistical mechanics, *Adv. Stud. Pure Math.*, **19**, Boston (1989), Academic Press, 347-366.
- [KL] D. KAZHDAN and G. LUSZTIG, Fixed point varieties on affine flag manifolds, *Israel J. of Math.*, **62**, No. 1 (1988), 129-168.
- [Kos] B. KOSTANT, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.*, **81** (1959), 973-1032.
- [Kr] I. KRICHEVER, Algebro-geometric construction of the Zakharov-Shabat equations and their periodic solutions, *Dokl. Akad. Nauk SSSR*, **227**, No. 2 (1976), 291-294.

- [LS] Y. LASZLO and C. SORGER, The line bundles on the stack of parabolic G-torsors over curves and their sections, *Ann. Sci. Éc. Norm. Sup.*, **30**, No. 4 (1997), 499-525.
- [Lau1] G. LAUMON, *Transformation de Fourier généralisée*, Preprint alg-geom/9603004.
- [Lau2] G. LAUMON, *Sur les modules de Krichever*, Preprint, Université de Paris-Sud, 86 T 20.
- [LMB] G. LAUMON and L. MORET-BAILLY, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge 39. Springer-Verlag, Berlin, 2000.
- [LM] Y. LI and M. MULASE, Prym varieties and integrable systems, *Comm. Anal. Geom.*, **5**, No. 2 (1997), 279-332.
- [McK] H. MCKEAN, Is there an infinite-dimensional algebraic geometry? Hints from KdV, *Proc. Symp. Pure Math.*, **49**, vol. I, pp. 27-37, AMS, 1989.
- [MV] I. MIRKOVIC and K. VILONEN, Perverse sheaves on loop Grassmannians and Langlands duality, *Math. Res. Lett.*, **7** (2000), 13-24.
- [M1] M. MULASE, Cohomological structure in soliton equations and Jacobian varieties, *J. Diff. Geom.*, **19** (1984), 403-430.
- [M2] M. MULASE, Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians, *Int. J. Math.*, **1** (1990), 293-342.
- [M3] M. MULASE, Algebraic theory of the KP equations. Perspectives in mathematical physics, *Conf. Proc. Lect. Notes Math. Phys. III.*, Cambridge MA (1994), Internat. Press, 151-217.
- [Mum] D. MUMFORD, An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations, *Int. Symposium on Algebraic Geometry*, Kyoto, pp. 115-153, Kinokuniya Book Store, 1977.
- [N1] A. NAKAYASHIKI, Structure of Baker-Akhiezer modules of principally polarized abelian varieties, commuting partial differential operators and associated integrable systems, *Duke Math J.*, **62** (1991), 315-358.
- [N2] A. NAKAYASHIKI, Commuting partial differential operators and vector bundles over abelian varieties, *Amer. J. Math.*, **116** (1994), 65-100.
- [PS] A. PRESSLEY and G. SEGAL, *Loop Groups*, Oxford University Press, 1986.
- [Ro1] M. ROTHSTEIN, Connections on the total Picard sheaf and the KP hierarchy, *Acta Applicandae Mathematicae*, **42** (1996), 297-308.
- [Ro2] M. ROTHSTEIN, Sheaves with connection on abelian varieties, *Duke Math J.*, **84**, No. 3 (1996), 565-598.
- [Sor] C. SORGER, Lectures on moduli of principal G-bundles over algebraic curves. School on Algebraic Geometry (Trieste, 1999) ICTP, *Lect. Notes* **1**, Abdus Salam ICTP, (Trieste, 2000), 1-57.
- [Tel] C. TELEMAN, Borel-Weil-Bott theory on the moduli stack of G-torsors over a curve, *Invent. Math.*, **134** (1998), 1-57.
- [SW] G. SEGAL and G. WILSON, Loop groups and equations of KdV type, *Publ. Math. IHES*, **61** (1985), 5-65.
- [W] G. WILSON, Habillage et fonctions τ , *C. R. Acad. Sci. Paris*, **299**, Sér. I, No. 13 (1984), 587-590.

D. B.-Z.

Department of Mathematics,
University of Chicago,
Chicago IL 60637 USA

E. F.

Department of Mathematics,
University of California,
Berkeley, CA 94720 USA

Manuscrit reçu le 7 décembre 1999.