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ON THE GENERAL STOCHASTIC EPIDEMIC

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I. Introduction

The purpose of this paper is to survey some recent results obtained in the solution of the model for the general stochastic epidemic, which was originally proposed by Bartlett (1949). Various aspects of the general epidemic, particularly in the stationary state, have previously been considered in detail by Bailey (1953, 1957), Whittle (1955), Foster (1955) and Kendall (1956), among others. Around October 1964, Siskind at University College, London, and I at Michigan State University , Lansing, independently arrived at explicit time-dependent solutions for this model ; our complementary results, which differ in various details, have appeared in Biometrika(1965; Vol. 52, Parts 3 and 4). What I shall attempt to outline here is an improved method of solution for the general stochastic epidemic ; this is, I believe, simpler than any so far proposed, and provides greater insight into the structure of the model. The same approach can also be used to attack recurrent epidemic processes for which a solution has been sketched (cf. Gani, 1965b).

The stochastic epidemic model considered is that for which at time $t \geq 0$, there are in circulation in a closed population of size $n + a$ $(n,a \geq 1)$,

$0 \leq r \leq n$ uninfected susceptibles,

$0 \leq s \leq n + a - r$ infectives,

the remaining $n + a - r - s \geq 0$ individuals having been removed through immunity or death. At time $t = 0$, the population is known to consist of n susceptibles and a infectives.

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Let the probabilities of possible transitions in the interval $(t, t + \delta t)$ be

$$\Pr \{ r, s \longrightarrow r - 1, s + 1 \} = rs\, \delta t + O(\delta t),$$

$$\Pr \{ r, s \longrightarrow r , s - 1 \} = \rho s\, \delta t + O(\delta t),$$

where for convenience the usual infection parameter β is set equal to 1 and ρ denotes the (relative) removal parameter. The process $\{ r, s \}$ is Markovian and the transition probabilities of r susceptibles and s infectives at time $t \geq 0$,

$$P_{rs}(t) = \Pr \{ r, s \text{ at time } t \mid n, a \text{ at time } 0 \}$$

satisfy the equations

(I.I)

$$\frac{dp_{rs}}{dt} = (r+1)(s-1)\, p_{r+1,\, s-1} - s(r+\rho)\, p_{rs} + \rho(s+1)\, p_{r,\, s+1}$$

$(0 \leq s \leq n + a - r \,;\, 0 \leq r \leq n)$,

where the $p_{ij} = 0$ if i or j are outside their appropriate ranges.

The initial condition is $p_{na}(0) = 1$.

It is well - known that the associated probability generating function (p.g.f.)

(I.2)

$$\Pi(z,w,t) = \sum_{r,s} p_{rs}(t)\, z^r w^s \qquad (|z|,|w| \leq 1)$$

satisfies the partial differential equation

(I.3)

$$\frac{\partial \Pi}{\partial t} = w(w-z)\frac{\partial^2 \Pi}{\partial z \partial w} + \rho(1-w)\frac{\partial \Pi}{\partial w}$$

with the initial condition $\Pi(z,w,0) = z^n w^a$.

The essence of both Siskind's and my own methods of solution consists of noting that if we write

(I.4)

$$\Pi\,(z, w, t) = \sum_{r=0}^{n} z^r f_r\,(w, t)$$

where $f_r\,(w, t) = \sum_{s=0}^{n+a-r} w^s p_{rs}\,(t)$, then the order of the partial differential equation (I.3) may be reduced to the first. Substituting (I.4) in (I.3) and equating coefficients of z^r on right and left hand sides, we obtain

(I.5)

$$\frac{\partial f_n}{\partial t} = -\left((n+\rho)\, w - \rho \right) \frac{\partial f_n}{\partial w},$$

$$\frac{\partial f_r}{\partial t} = w^2 (r+1) \frac{\partial f_{r+1}}{\partial w} - \left((r+\rho)\, w - \rho \right) \frac{\partial f_r}{\partial w}$$

$(r = 0, 1, \ldots, n-1)$.

At this stage Siskind proceeds by direct recursive integration of the $f_r\,(w, t)$. My own approach makes use of Laplace transforms

(I.6)

$$F_r\,(w, s) = \int_0^\infty e^{-st} f_r\,(w, t)\, dt$$

$(\operatorname{Re}(s) > 0)$

to reduce the equations (I.5) to

(I.7)

$$sF_n\,(w,s) - w^a = -\left((n+\rho)\, w - \rho \right) \frac{\partial F_n}{\partial w},$$

$$sF_r\,(w,s) = w^2 (r+1) \frac{\partial F_{r+1}}{\partial w} - \left((r+\rho)\, w - \rho \right) \frac{\partial F_r}{\partial w}$$

$(r = 0, \ldots, n-1)$,

for which recursive solutions are also found. I think it could be fairly said of both methods that they involve a good deal of untidy algebra ; the following approach may simplify the solution while at the same time clarifying the structure of the process.

2 . The solution in matrix form

Let us write (I.7) in the matrix form

(2.I)

$$A(w) \frac{\partial F}{\partial w} + sF = w^{a} E$$

where F (w,s) and E are column vectors whose transposes are $F'(w,s) = \{ F_n(w,s), \ldots, F_0(w,s) \}$, $E' = \{ 1, 0, \ldots, 0 \}$, and

(2.2)

$$A(w) = \begin{bmatrix} (n+\rho)w-\rho & & & & & \\ -n w^2 & (n-1+\rho)w-\rho & & & & \\ & -(n-1)w^2 & (n-2+\rho)w-\rho & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -2w^2 & (1+\rho)w-\rho & \\ & & & & -w^2 & \rho w-\rho \end{bmatrix}$$

Then, we may write Taylor's theorem for F(w,s) in the form

(2.3)

$$F(w,s) = F(0,s) + wF^{(1)}(0,s) + \frac{w^2 F}{2\,!}^{(2)}(0,s) + \ldots + \frac{w^{n+a}}{(n+a)!} F^{(n+a)}(0,s),$$

where, since the highest degree (of the polynomial $F_0(w,s)$) in w is n + a, the series must terminate with the term involving the (n+a) - th derivative of F (w,s) at w = 0, namely $F^{(n+a)}(0,s)$.

From (2.I) it is possible to express all higher derivatives in terms of F (0,s). For, it is seen directly that

$$A(0)\,F^{(1)}(0,s) = -sF(0,s)$$

or from (2.2), since $A(0) = -\rho I$

(2.4)

$$F^{(1)}(0,s) = \frac{s}{\rho} F(0,s).$$

Differentiating (2.I) with respect to w, we obtain

(2.5)

$$\{ A^{(1)}(w) + sI \} F^{(1)}(w,s) + A(w)\,F^{(2)}(w,s) = aw^{a-1}E$$

$$(a \geq 1),$$

whence setting w = 0,

(2.6)

$$F^{(2)}(0,s) = \frac{1}{\rho} \{ A^{(1)}(0) + sI\} F^{(1)}(0,s) - a!\,\delta_{Ia}E$$

with δ_{ij} as the Kronecker delta, and $A^{(1)}(0)$ the diagonal matrix

$$A^{(1)}(0) = \begin{bmatrix} n+\rho & & & & \\ & n-1+\rho & & & \\ & & \cdots & & \\ & & & 2\rho & \\ & & & & \rho \end{bmatrix}$$

The next derivative can be found from (2.5) as

$$A^{(2)}(w)\,F^{(1)}(w,s) + \{ 2A^{(1)}(w) + sI \} F^{(2)}(w,s) + A(w)\,F^{(3)}(w,s) = a(a-1)w^{a-2}E$$

or, setting w = 0

(2.7)

$$F^{(3)}(0,s) = \frac{1}{\rho} \left[\{ 2A^{(1)}(0) + sI \} F^{(2)}(0,s) + A^{(2)}(0)F^{(1)}(0,s) - a!\,\delta_{2a}E \right],$$

where

$$A^{(2)}(0) = -2\begin{bmatrix} 0 & & & \\ n & 0 & & \\ & & \cdots & \\ & & 1 & 0 \end{bmatrix}$$

We may show in general that the following (2n + 2) -rowed vectors satisfy the equations

(2.8)

$$\begin{bmatrix} F^{(i+1)}(0,s) \\ F^{(i)}(0,s) \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho}\{iA^{(1)}(0) + sI\} & \frac{i(i-1)}{2\rho} A^{(2)}(0) \\ I & 0 \end{bmatrix} \begin{bmatrix} F^{(i)} \\ F^{(i-1)} \end{bmatrix} - \frac{a!}{\rho} \delta_{ia} \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$(i = 0, 1, \dots),$

where $F^{(-1)} \equiv 0$, $F^{(0)} = F(0,s)$. This may be simplified by rewriting the vectors in the form

(2.9)

$$\varphi^{(i+1)}(0,s) = B_i \varphi^{(i)} - \frac{a!}{\rho} \delta_{ia} E \qquad (i = 0, 1, \dots)$$

where E is now a (2n + 2) X 1 column vector.

It follows that for $a \geq 1$, we can write

(2.10)

$$\varphi^{(i)} = \{ \prod_{j=0}^{i-1} B_j \} \varphi^{(0)} \qquad (i = 1, \dots , a),$$

$$\varphi^{(i)} = \{ \prod_{j=0}^{i-1} B_j \} \varphi^{(0)} - \frac{a!}{\rho} \{ \prod_{j=a+1}^{i-1} B_j \} E$$

$(i = a+1, \dots, n+a+1),$

where $\prod_{j=a+1}^{a} B_j$ is defined as I, and the products $\prod_{j=0}^{i-1} B_j = B_{i-1} \dots B_0$, and $\prod_{j=a+1}^{i-1} B_j = B_{i-1} \dots B_{a+1}$ must be carried out in the particular order indicated.

Thus, since

(2.11)

$$\sum_{i=0}^{n+a+1} \frac{w^i}{i!} \varphi^{(i)} = \sum_{i=0}^{n+a+1} \frac{w^i}{i!} \begin{bmatrix} F^{(i)} \\ F^{(i-1)} \end{bmatrix} = \begin{bmatrix} F(w, s) \\ \int_0^w F(v, s)\, dv \end{bmatrix}$$

we obtain that

(2.12)

$$\begin{bmatrix} F(w,s) \\ \int_0^w F(v,s)\,dv \end{bmatrix} = \sum_{i=0}^{n+a+1} \frac{w^i}{i!} \{ \prod_{j=0}^{i-1} B_j \} \varphi^{(0)} - \sum_{i=a+1}^{n+a+1} \frac{w^i}{i!} \frac{a!}{\rho} \{ \prod_{j=a+1}^{i-1} B_j \} E,$$

where, as above, $\prod_{j=k}^{k-1} B_j$ is defined as I. The unknown $\varphi^{(0)} = \begin{bmatrix} F(0,s) \\ 0 \end{bmatrix}$ may be found by equating the first n + 1 rows of (2.12) to zero since these are coefficients of w^{n+a+1}, which is a degree higher than that of any of the polynomials in F (w,s).

We see that this gives

$$\{ \prod_{j=0}^{n+a} B_j \} \begin{bmatrix} F(0,s) \\ 0 \end{bmatrix} - \frac{a\,!}{\rho} \{ \prod_{j=a+1}^{n+a} B_j \} \quad E = \begin{bmatrix} 0 \\ F^{(n+a)}(0,s) \end{bmatrix},$$

so that

(2.13)

$$F(0,s) = \{ \prod_{j=0}^{n+a} B_j \}^{-1}_{n+1} \left[\frac{a\,!}{\rho} \{ \prod_{j=a+1}^{n+a} B_j \}_{n+1} E \right]$$

where $\left[\cdot \right]_{n+1}$ indicates the truncated (n+1) X (n+1) matrix of the first n+1 rows and columns. It is clear that

$$\{ \prod_{j=0}^{n+a} B_j \}_{n+1}$$

is non - singular, since from the structure of $A^{(1)}(0)$, $A^{(2)}(0)$ this product is seen to be a triangular matrix with non - zero eigenvalues for Re (s) > 0.

3. An illustration of the method : the 2-person family

Let $a = 1$, $n = 1$; then

(3.1)

$$A^{(1)}(0) = \begin{bmatrix} 1+\rho & 0 \\ 0 & \rho \end{bmatrix} , \quad A^{(2)}(0) = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} .$$

The matrices B_i are readily seen to be

(3.2)

$$B_0 = \begin{bmatrix} \frac{s}{\rho} & 0 & 0 & 0 \\ 0 & \frac{s}{\rho} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \quad B_1 = \begin{bmatrix} \frac{1+\rho+s}{\rho} & 0 & 0 & 0 \\ 0 & \frac{\rho+s}{\rho} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \quad B_2 = \begin{bmatrix} \frac{2(1+\rho)+s}{\rho} & 0 & 0 & 0 \\ 0 & \frac{2\rho+s}{\rho} & \frac{-2}{\rho} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ,$$

and the required products B_1B_0 and $B_2B_1B_0$ are therefore

(3.3)

$$B_1B_0 = \begin{bmatrix} \frac{s(1+\rho+s)}{\rho^2} & 0 & 0 & 0 \\ 0 & \frac{s(\rho+s)}{\rho^2} & 0 & 0 \\ \frac{s}{\rho} & 0 & 0 & 0 \\ 0 & \frac{s}{\rho} & 0 & 0 \end{bmatrix} , \quad B_2B_1B_0 = \begin{bmatrix} \frac{s(1+\rho+s)(2+2\rho+s)}{\rho^3} & 0 & 0 & 0 \\ -\frac{2s}{\rho^2} & \frac{s(\rho+s)(2\rho+s)}{\rho^3} & 0 & 0 \\ \frac{s(1+\rho+s)}{\rho^2} & 0 & 0 & 0 \\ 0 & \frac{s(\rho+s)}{\rho^2} & 0 & 0 \end{bmatrix} .$$

Following the theory outlined in the previous section exactly, we find that

(3.4)

$$F(0,s) = \left[B_2B_1B_0\right]_2^{-1} \left[\frac{1}{\rho} B_2\right]_2 E$$

$$= \frac{\rho^6}{s^2(1+\rho+s)(2+2\rho+s)(\rho+s)(2\rho+s)} \begin{bmatrix} \frac{s(\rho+s)(2\rho+s)}{\rho^3} & 0 \\ \frac{2s}{\rho^2} & \frac{s(1+\rho+s)(2+2\rho+s)}{\rho^3} \end{bmatrix} \begin{bmatrix} \frac{2(1+\rho)+s}{\rho^2} \\ 0 \end{bmatrix}$$

$$F(0,s) = \begin{bmatrix} \dfrac{\rho}{s(1+\rho+s)} \\ \dfrac{2\rho^2}{s(\rho+s)(2\rho+s)(1+\rho+s)} \end{bmatrix} .$$

The full solution to the 2-person epidemic may then be obtained by taking only the appropriate parts of the upper left (2 X 2) matrix in $\{ I + wB_0 + \frac{w^2}{2!} B_1 B_0 \} \varphi^{(0)}$, and for simplicity (instead of carrying out in detail the algebra involved in the right hand part of Equation (2.12)) deleting any terms in powers of w which are known not to appear in any $F_r(w,s)$. Hence we find that

$$F(w,s) = \begin{bmatrix} 1+\dfrac{ws}{\rho} & 0 \\ 0 & 1+\dfrac{ws}{\rho}+\dfrac{w^2}{2!}\dfrac{s(\rho+s)}{\rho^2} \end{bmatrix} \begin{bmatrix} \dfrac{\rho}{s(1+\rho+s)} \\ \dfrac{2\rho^2}{s(\rho+s)(2\rho+s)(1+\rho+s)} \end{bmatrix}$$

This method has been successfully applied to higher values of n and a by J. Moreno of Michigan State University.

4. Total size of the epidemic

One of the advantages of the previous analysis of the epidemic process is the simplicity of the resulting formulae for the distribution of the total size of the epidemic. These have already been discussed in several different (algebraically complex) ways by Bailey (1953), Whittle (1955), Foster (1955) and Siskind (1965).

Consider the probabilities $\{ P_{n-r} \}$ of an epidemic of total size n-r, not counting the initial cases ; $0 \leq r \leq n$ will then be the number of susceptibles remaining after the epidemic is over. It is clear that

(4.1) $$P_{n-r} = \lim_{t \to \infty} p_{r0}(t)$$

$$= \lim_{s \to 0} sF_r(0,s).$$

In matrix terms

(4.2)

$$P = \begin{bmatrix} P_0 \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \lim_{s \to 0} s \begin{bmatrix} F_n\ (0,s) \\ \cdot \\ \cdot \\ F_0\ (0,s) \end{bmatrix} = \lim_{s \to 0} sF\ (0,s).$$

We have seen in (2.13) that

(4.3)

$$\{ \prod_{j=0}^{n+a} B_j \}_{n+1}\ F\ (0,s) = \frac{a\ !}{\rho} \{ \prod_{j=a+1}^{n+a} B_j \}_{n+1}\ E\ ,$$

and since

$$B_0 = \begin{bmatrix} \frac{s}{\rho} I & 0 \\ I & 0 \end{bmatrix} ,$$

we may write (4.3) as

(4.4)

$$\{ \prod_{j=1}^{n+a} B_j \}_{n+1}\ sF\ (0,s) = a\ !\ \{ \prod_{j=a+1}^{n+a} B_j \}_{n+1}\ E\ .$$

It is readily found by taking limits as $s \to 0$ that this leads to

(4.5)

$$\{ \prod_{j=1}^{n+a} \begin{bmatrix} \frac{j}{\rho} A^{(1)}(0) & \frac{j(j-1)}{2\rho} A^{(2)}(0) \\ I & 0 \end{bmatrix} \}_{n+1} \lim_{s \to 0} sF(0,s) =$$

$$a!\{ \prod_{j=a+1}^{n+a} \begin{bmatrix} \frac{j}{\rho} A^{(1)}(0) & \frac{j(j-1)}{2\rho} A^{(2)}(0) \\ I & 0 \end{bmatrix} \}_{n+1} E$$

from which the vector P of probabilities of total epidemic size can be expressed as

(4.6)

$$P = a!\left\{\prod_{j=1}^{n+a}\begin{bmatrix}\frac{j}{\rho}A^{(1)}(0) & \frac{j(j-1)}{2\rho}A^{(2)}(0)\\ I & 0\end{bmatrix}\right\}^{-1}_{n+1} \times \left\{\prod_{j=a+1}^{n+a}\begin{bmatrix}\frac{j}{\rho}A^{(1)}(0) & \frac{j(j-1)}{2\rho}A^{(2)}(0)\\ I & 0\end{bmatrix}\right\}_{n+1} E.$$

This result involves only a set of direct matrix operations. It is clear, as it was earlier at the end of Section 2, that

$$\left\{\prod_{j=1}^{n+a}\begin{bmatrix}\frac{j}{\rho}A^{(1)}(0) & \frac{j(j-1)}{2\rho}A^{(2)}(0)\\ I & 0\end{bmatrix}\right\}_{n+1}$$

is non-singular, since this product results in a triangular matrix with non-zero eigenvalues.

In the case of the 2-person epidemic, for example, we readily obtain from (4.6) the known result (cf. Bailey, 1957)

$$P = \left\{\begin{bmatrix}\frac{2}{\rho}(1+\rho) & 0 & 0 & 0\\ 0 & 2 & -\frac{2}{\rho} & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\rho}(1+\rho) & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\right\}^{-1}_{2}\begin{bmatrix}\frac{2}{\rho}(1+\rho)\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{\rho}(1+\rho)^2 & 0\\ \frac{-2}{\rho} & 2\end{bmatrix}^{-1}\begin{bmatrix}\frac{2}{\rho}(1+\rho)\\ 0\end{bmatrix} = \begin{bmatrix}\frac{\rho^2}{2(1+\rho)^2} & 0\\ \frac{\rho}{2(1+\rho)^2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{2}{\rho}(1+\rho)\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\rho}{1+\rho}\\ \frac{1}{1+\rho}\end{bmatrix}$$

The simplicity of the equation (4.6) for P, provides a straight forward method for the numerical evaluation of probabilities of total epidemic size for large n and a, given any suitable numerical values of ρ.

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