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INTEGRATION WITH RESPECT TO PROCESSES OF LINEAR FUNCTIONALS

by M. METIVIER

SUMMARY

The notion of "integrale stochastique radonifiante" as introduced by B. GAVEAU in C.R.A.S. Paris t. 276 - 1973 for the cylindrical brownian motion is studied in the general context of generalized quasi-martingales. This gives use to a theory in which the notions and formulas of the Kunita-Watanabe theory for real square integrable martingales have a natural extension.

INTRODUCTION

For the purpose of studying stochastic partial differential equations it is worth considering perturbations which are "white noise in time and in space". The mathematical expression of such an object is a cylindrical measure, or a linear random functional as studied for example in [1] or [8] . Considering the special case of "cylindrical brownian motion", several authors defined a stochastic integral with respect to such a stochastic process (cf. for example [7] and [11]). In [7] the operator valued processes, which are integrated with respect to the cylindrical brownian motion, are such that the integral process is a (Hilbert valued) Martingale in an ordinary sense ; and in [11], a Girsanov-theorem is obtained in such a situation.

The purpose of this study is to show that in a very general context it is possible to developp a theory of stochastic integration with respect to "cylindrical martingales", which extends in a natural way the classical L^2-stochastic integral with respect to square integrable martingales (real or Hilbert valued) as studied in [10], [14], [18] for

example.

After the necessary definitions we give in §3, a decomposition theorem of Doob-Meyer type for generalized quasi-martingales.

In §4 the stochastic integral is defined, and an isometry formula as in [16] is given. The class of operator-valued processes which are integrated is wide. The values of the process (as in [16]) are not necessairly continuous operators. But if, on the contrary, the processes are Hilbert-Schmidt valued the integral process is an ordinary-sense vector valued martingale.

In §5 an Ito's formula is given for those generalized quasi-martingale. But in this first draft of the work, the formula is stated only in the case of continuous quasi-martingales.

I - LINEAR STOCHASTIC FUNCTIONALS - NOTATIONS

I-1 In all this paper we will assume that T is a closed or open intervall in $\mathbb{R}^+$, a basic probability space $(\Omega, \mathcal{F}, P)$ and an increasing family $(\mathcal{F}_t)_{t \in T \times \mathbb{R}^+}$ of sub-σ-algebras of $\mathcal{F}$ with the usual following completion assumption : $\mathcal{F}$ is P-complete and all the P-null sets in $\mathcal{F}$ are in $\mathcal{F}_t$ for every t.

$\mathcal{R}$ will mean the set of "predictable rectangles" : $]s,t] \times F \subset T \times \Omega$ where $s \leq t$, $s,t \in T$ and $F \in \mathcal{F}_s$.

$\mathcal{A}$ will be the algebra of subsets of $T \times \Omega$ generated by $\mathcal{R}$.

$\mathcal{P}$ is the σ-algebra generated by $\mathcal{A}$, i.e. : the σ-algebra of predictable subsets of $T \times \Omega$.

I-2 $\mathbb{H}$, $\mathbb{G}$, $\mathbb{K}$, will denote real Hilbert spaces, all of them assumed to be separable (in our context this is no restriction). The scalar product in those spaces will be denoted by $<,.>_{\mathbb{H}}$, $<.,.>_{\mathbb{G}}$ ... or simply $<.,.>$, if there is no possible confusion. The norm will be written : $||.||_{\mathbb{H}}$, $||.||_{\mathbb{G}}$, etc... If $\mathbb{B}$ is a Banach space, then $\mathbb{B}'$ will denote the topological dual of B (set of continuous linear form) endowed, if not otherwise specified, with the dual Banach norm. We recall that the algebraic tensor product $\mathbb{H} \otimes \mathbb{G}$ can be endowed with several norms, giving rise to several completions of $\mathbb{H} \otimes \mathbb{G}$:

- $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is the completion for a norm such that every continuous bilinear mapping $b : (\mathbb{H} \times \mathbb{G}) \to \mathbb{K}$ can be factorized in a unique way as $b = u_b \circ \Pi$ where Π is the canonical inbedding $\Pi(x,y) = x \otimes y$ and u_b is a continuous linear mapping from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ into $\mathbb{K}$, with same norm as b. The norm $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is often called the trace-norm and denoted $||.||_{Tr}$. Recall that if $\mathbb{G} = \mathbb{H}$ and $b(x,y) = <x,y>_{\mathbb{H}}$ the corresponding linear form u_b on $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ is called the trace-form and denoted Tr.

- $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ is a Hilbert space with scalar product an extension of $<x \otimes y , x' \otimes y'> = <x,x'>_{\mathbb{H}} . <y,y'>_{\mathbb{G}}$.

- $\mathbb{H} \otimes_\varepsilon \mathbb{G}$ is a Banach space, the norm of which will be more easily described later.

The three topologies induced by the three considered topological tensor product on $\mathbb{H} \otimes \mathbb{G}$ are comparable and we have the canonical continuous injection.

$$\mathbb{H} \hat{\otimes}_1 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_2 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} .$$

Moreover an important theorem of Schatten (see for example [19] chap. 48) says that the function $(x \otimes y, x' \otimes y') \rightsquigarrow <x,x'>_{\mathbb{H}} . <y,y'>_{\mathbb{G}}$ can be extended as a continuous bilinear form on $(\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} \times \mathbb{H} \hat{\otimes}_1 \mathbb{G})$ and that for this duality $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ identifies itself with the Banach dual of $\mathbb{H} \otimes_\varepsilon \mathbb{G}$.

I-3 There is a unique injective linear mapping of $\mathbb{H} \otimes \mathbb{G}$ into the vector space of linear operators with finite range from $\mathbb{H}$ into $\mathbb{G}$, associating to $x \otimes y$ the operator $h \rightsquigarrow <x,h>_{\mathbb{H}} y$. This linear mapping has extensions which are :

1°) isometry from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ onto $\mathcal{L}_1(\mathbb{H} ; \mathbb{G})$, the Banach space of nuclear operators from $\mathbb{H}$ into $\mathbb{G}$ with the trace norm ;

2°) isometry from $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ onto $\mathcal{L}_2(\mathbb{H} ; \mathbb{G})$, the Hilbert space of Hilbert-Schmidt operators from H into $\mathbb{G}$ with the Hilbert-Schmidt scalar product ;

3°) isometry from $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ onto $\mathcal{L}_c(\mathbb{H} ; \mathbb{G})$, the Banach space of compact operators with the usual norm of bounded operators.

In as much $x \otimes y$ can be identified with a bilinear continuous form on $(\mathbb{H} \times \mathbb{G})$ or a continuous linear form on $\mathbb{H} \otimes \mathbb{G}$, through the formula

$$<x \otimes y, x' \otimes y'> = <x,x'>_{\mathbb{H}} . <y,y'>_{\mathbb{G}}$$

there is also a continuous linear extension of the preceding linear mapping, into an isometry from $(\mathbb{H} \hat{\otimes}_1 \mathbb{G})'$ onto $\mathcal{L}(\mathbb{H} ; \mathbb{G})$, the Banach space of linear bounded operators from $\mathbb{H}$ into $\mathbb{G}$ with the usual norm. (This isometry is in fact the one which associates to a bounded bilinear b on $(\mathbb{H} \times \mathbb{G})$ the bounded linear operator $\tilde{b}$ in $\mathcal{L}(\mathbb{H} ; \mathbb{G})$ such that $< \tilde{b}(x), y > = b(x,y)$.) .

I-4 Random variables with values in H will be strongly $\mathcal{F}$-measurable mapping from Ω into $\mathbb{H}$. If such a random variable X has the property $E(||X||^2_{\mathbb{H}}) < \infty$, then $\omega \rightsquigarrow X(\omega) \otimes X(\omega)$ is a strongly measurable random variable with values in $\mathbb{H} \otimes \mathbb{H}$, and as $||x \otimes y||_{Tr} = ||x||_{\mathbb{H}} ||y||_{\mathbb{H}}$,

$X \otimes X$ is an integrable mapping from Ω into $\mathbb{H} \otimes_1 \mathbb{H}$. As a consequence $E(X \otimes X) \in \mathbb{H} \hat{\otimes}_1 \mathbb{H}$ and is called the covariance of the variable X.

If to X is moreover associated the continuous mapping $\tilde{X}$: $h \rightsquigarrow \langle X,h\rangle$ from $\mathbb{H}$ into $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$, this mapping appears to be Hilbert-Schmidt. And it can be shown that conversely to a mapping $\tilde{X}$ linear from $\mathbb{H}$ into $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$ there can be associated a random variable X with values in $\mathbb{H}$, such that $\langle X,h\rangle = \tilde{X}(h)$ a.s, if and only if $\tilde{X}$ is Hilbert-Schmidt. The Hilbert Schmidt norm $||\tilde{X}||_2$ of $\tilde{X}$ is equal then to $\sqrt{E(||X||^2_{\mathbb{H}})}$.

I-5 To abbreviate the writting we will writte

$$\mathcal{H}^p = L^p_{\mathbb{R}}(\Omega, \mathcal{F}, P) \qquad p \geq o$$

$$\mathcal{H}^p_t = L^p_{\mathbb{R}}(\Omega, \mathcal{F}_t, P) \qquad p \geq o$$

I-6 The norm in $\mathcal{L}_1(\mathbb{H} ; \mathbb{C})$ will be written $||.||_{Tr}$, the norm in $\mathcal{L}_2(\mathbb{H} ; \mathbb{C})$: $||.||_{H.S}$, the norm in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$: $||.||_b$.

II - PROCESSES OF LINEAR FUNCTIONALS

II-1 Définition 1

Let us writte $\mathcal{H}^p_t = L^p(\Omega, \mathcal{F}_t, P)$ where $p \geq o$. Let $\mathbb{B}$ be a Banach space. Then a family $\tilde{X} = (\tilde{X}_t)_{t \in T \subset \mathbb{R}}$, where $\tilde{X}_t \in \mathcal{L}(\mathbb{B} ; \mathcal{H}^p_t)$ for every t, will be called a p-process of stochastic linear functionals on $\mathbb{B}$. (we will abbreviate : a process of S.L.F. on $\mathbb{B}$).

If for every $h \in \mathbb{B}$, the real process $(\tilde{X}_t(h))_{t \in T}$ is a martingale, the process $\tilde{X}$ will be called a p-cylindrical martingale.

From what has been said in the first paragraph, if $\tilde{X}_t \in \mathcal{L}_2(\mathbb{H} ; \mathcal{H}_t)$ for all t, then, for every t and h, $\tilde{X}_t(h) = \langle X_t, h\rangle$ where $X = (X_t)_{t \in T}$ is an ordinary square integrable stochastic process with values in $\mathbb{H}$, as considered for example in [10] and [14] .

Definition 2

A p-process $\tilde{X}$ of S.L.F on $\mathbb{B}$ will be said p-continuous (resp. p-right continuous) if, for every $h \in \mathbb{B}$, $t \rightsquigarrow \tilde{X}_t(h)$ is continuous as a mapping from $[0,T]$ into $L^p(\Omega, \mathcal{F}, P)$. It is said continuous (resp. right continuous) if, for every $h \in \mathbb{H}$, there is a version of the process $(\tilde{X}_t(h))_{t \in [0,T]}$ with continuous (resp. right-continuous) paths.

II-2 Doleans' measure of a process of linear functionals

We extend here the concept of Doleans' measure as first defined in [5] for real sub-martingale and extended since then to vector valued quasi-martingales (see for ex. [15]).

To every process $\tilde{X}$ of S.L.F. on the Banach space $\mathbb{B}$, we associate the additive functions $\tilde{\alpha}_X$ with values in $\mathbb{B}'$ defined on the set $\mathcal{R}$ of predictable rectangles by

$$\tilde{\alpha}_X(\,]s,t] \times F) = E\,[1_F \,.(\tilde{X}_t - \tilde{X}_s)] \in \mathbb{B}' \tag{II-2-1}$$

Such a function on $\mathcal{R}$ has clearly an additive extension to the algebra $\mathcal{A}$ generated by $\mathcal{R}$. We call it $\tilde{\alpha}_X$ again.

Definition 3

If the additive function $\tilde{\alpha}_X$ on $\mathcal{A}$ has a bounded variation (for the norm of $\mathbb{B}'$), the process $\tilde{X}$ of stochastic linear functionals, will be called a generalized quasi-martingale.

This clearly generalizes the classical definition (see [15]). We have then the

Proposition 1

For $\tilde{X}$ to be a generalized quasi martingale, it is necessary and sufficient that the family of real additive measures (α^h_X) associated with the real processes $(\tilde{X}(h))_{||h||\leq 1}$ be of bounded variation, and that the set of those variations $|\alpha^h_X|$

has a supremum in the ordered set of bounded positive measures.

Proof

This comes from the fact that the total variation of $\tilde{\alpha}_X$ can be approximated by sums of the type

$$\sum_i E\,(1_{F_i}\,|X_{t_i}(h_i) - X_{s_i}(h_i)|\,) \qquad h_i \in \mathbb{B},\ ||h_i|| \leq 1$$

while the supremum of the variations can be approximated by sums of the type

$$\sum_i |\alpha_X^{h_i}\,]s_i,t_i] \times F_i| = \sum_i |E\,\{1_{F_i}\,[X_{t_i}(h_i) - X_{s_i}(h_i)]\}\,|$$

It is easily seen that both supremum coincide.

II-3 Square integrable cylindrical martingales

Let $\tilde{M}$ be a 2-cylindrical martingale on the Banach space $\mathbb{B}$. We associate to $\tilde{M}$ the 1-process of S.L.F on $\mathbb{B} \hat{\otimes} \mathbb{B}$ defined by

$$\tilde{X}_t(h \otimes g) = \tilde{M}_t(h)\ .\ \tilde{M}_t(g)$$

It is clear that if $\tilde{M}$ is the process of S.L.F associated with an ordinary martingale M with values in $\mathbb{B}$, $\tilde{X}$ is the process of S.L.F. on $\mathbb{B}' \hat{\otimes} \mathbb{B}'$ associated with the ordinary sense process $(M_t \otimes M_t)_{t \in [0,T]}$, taking its values in $\mathbb{B} \hat{\otimes} \mathbb{B}$.

It is known (cf. for example [15]) that, if M is right continuous, $M \otimes M$ is an ordinary sense quasi-martingale, with associated σ-additive Doleans' measure (with values in $\mathbb{B} \hat{\otimes} \mathbb{B}$)

As $\quad ||\alpha_{\tilde{X}}(\,]s,t] \times F)||_{(\mathbb{B}'\hat{\otimes}\mathbb{B}')'} \leq ||E|1_F\,(M_t - M_s)^{\otimes 2}\,|\,||_{\mathbb{B}\hat{\otimes}\mathbb{B}}$

$\tilde{X}$ is then a generalized quasi-martingale.

But the following example shows that for a 2-cylindrical martingale it is not always true that $\tilde{M} \otimes M$ is a generalized quasi-martingale.

Example 1

Let (t_n) be a decreasing sequence to zero. Let $\mathbb{H}$ be an Hilbert space, (e_n) an orthonormal basis of $\mathbb{H}$ and let us define the cylindrical martingale :

$$\tilde{M}_t(h) = \sum_n 1_{[t_{n+1},\infty[}(t)\ (h \mid e_n)\ .\ \frac{1}{\sqrt{t_n - t_{n+1}}}\,(\beta_{t_n \wedge t} - \beta_{t_{n+1}})$$

where (β_t) is a usual real standard Brownian motion. For every n, $(\tilde{M}_t(e_n))_{t \in \mathbb{R}^+}$ is a process with independant increments, zero on $]0,\,t_{n+1}]$, path-wise constant on $[t_n\,,\,\infty[$. For every $h \in \mathbb{H}$, the above series converge in $L^2(\Omega,\,\mathcal{F}_t,\,P)$ with

$$E\,|\tilde{M}_t(h)|^2 \leq ||h||^2_{\mathbb{H}}$$

defining a process with zero-mean independant increments. But considering the partition $(\,]t_{n+1}\,,\,t_n] \times \Omega\,)_{n>o}$ of $]0,t_1] \times \Omega$, it is immediately seen that, $\tilde{X}$ beeing the process of S.L.F. above defined :

$$\sum_n E\,||1_\Omega(\tilde{X}_{t_n} - \tilde{X}_{t_{n+1}})||_{\mathcal{L}(\mathbb{H}\hat{\otimes}\mathbb{H})} \geq \sum_n E\,\frac{1}{\sqrt{t_n - t_{n+1}}}\,(\beta_{t_n} - \beta_{t_{n+1}})^2 = +\infty$$

We will then give the following

Definition 4

A 2-cylindrical martingale $\tilde{M}$, on a Banach space $\mathbb{B}$ will be called a square integrable cylindrical martingale (S.I.C Martingale) if the 1-process of S.L.F. (improperly) denoted by $\tilde{M} \otimes \tilde{M}$, defined on $\mathbb{B} \hat{\otimes} \mathbb{B}$ by

$$\tilde{M} \otimes \tilde{M}(h \otimes g) = \tilde{M}(h)\ .\ \tilde{M}(g)$$

is a quasi-martingale.

The additive measure $\alpha_{\tilde{M} \otimes \tilde{M}}$ on $\mathcal{A}$ will be called the quadratic measure of $\tilde{M}$, and its variation the control measure of $\tilde{M}$.

Example 2

The Brownian process, associated with the unit covariance matrix on a Hilbert space $\mathbb{H}$ is a S.I.C. Martingale : let us recall that this is a process $\tilde{W}$ of S.L.F on $\mathbb{H}$, such that, for every finite set of vectors $(h_1, \ldots, h_n)$ in $\mathbb{H}$, the process $(\tilde{W}_t(h_1), \ldots, \tilde{W}_t(h_n))$ is an n-dimensional gaussian process with independant increments, and the bilinear form $(h,g) \rightsquigarrow E(\tilde{W}_t(h).\tilde{W}_t(g))$ is the scalar product $(h,g) \rightsquigarrow \langle h,g\rangle_{\mathbb{H}}$.

In this case, the quadratic-measure α of $\tilde{W}$ associates to every previsible rectangle $]s,t] \times F$ the linear form on $\mathbb{H} \hat{\otimes} \mathbb{H}$ defined by

$$\alpha\,(\,]s,t] \times F)(h \otimes g) = E\,\{1_F\,[\tilde{W}_t(h).\tilde{W}_t(g) - \tilde{W}_s(h).\tilde{W}_s(g)]\,\}$$

In view of the above properties of $\tilde{W}$:

$$\begin{aligned}\alpha\,(\,]s,t] \times F)(h \otimes g) &= E\,\{1_F\,(W_t(h)-W_s(h)).(W_t(g)-W_s(g))\}\\ &= P(F)\,.\,(t-s)(h|g)\\ &= P(F)\,.\,(t-s)\,\mathrm{Tr}(h \otimes g)\end{aligned}$$

The control measure α is then a one dimensional measure with values in $(\mathbb{H} \otimes \mathbb{H})'$, proportional to the product measure $\ell \otimes P$ where ℓ is the Lebesgue measure on $\mathbb{R}^+$.

Proposition 2 *let us assume that $\mathbb{B}$ is reflexive.*

If $\tilde{M}$ is a right continuous S.I.C. Martingale on the Banach space $\mathbb{B}$, the quadratic measure α of $\tilde{M}$ has a σ-additive extension (in $\mathbb{B}'$) to the σ-algebra of predictable subsets of $\mathbb{R}^+ \times \Omega$.

Proof

It is enough to show that the variation of α has a σ-additive extension. But from proposition 1 and its proof, and the set of σ-additive measures beeing a Riesz-band (cf. [2]) in the ordered set of finitely additive measures with bounded variation, the variation of α will be σ-additive as soon as for every $h \in \mathbb{B}$ the measure α^h is σ-additive. In our case, from the known property of square integrable real martingales (cf. [6], [15]), the measures $\alpha^{h\otimes g}$ are σ-additive, and the same is clearly true for all α^u, with u in $\mathbb{H} \hat{\otimes} \mathbb{H}$.

III - DECOMPOSITION THEOREM FOR A GENERALIZED QUASI-MARTINGALE

For the purpose of transformation formulas in stochastic integration, Doob-Meyer's decomposition theorem play an essential role. We intend to give an extension of such decomposition theorems in our general setting.

All this rests on the following :

III-1 Theorem 1

Let $\mathbb{E}'$ be the Banach dual of a separable Banach space $\mathbb{E}$. Let α be a σ-additive measure on the predictable sets, with values in $\mathbb{E}'$, with bounded variation, such that for every evanescent set A, $\alpha(A) = 0$.

Let $\tau(\mathbb{E}',\mathbb{E})$ denote the Mackey topology on $\mathbb{E}'$ (i.e. the finest locally convex topology on $\mathbb{E}'$ for which $\mathbb{E}$ is the dual of $\mathbb{E}'$).

Then :

1°) There exists a stochastic process V, with values in $\mathbb{E}'$, null in 0, with right continuous paths for $\tau(\mathbb{E}',\mathbb{E})$, unique up to indistingability such that

(i) $\forall t$ V_t is weakly integrable in $\mathbb{E}'$ for the duality $\sigma(\mathbb{E}',\mathbb{E})$.

(ii) For every $\omega \in \Omega$, the intervall-function $]s,t] \rightsquigarrow V_t(\omega) - V_s(\omega)$ can be extended into a measure on the Borels sets of $[0,T]$, with values in $\mathbb{E}'$, with bounded variation (for the norm of $\mathbb{E}'$) and σ-additive for $\tau(\mathbb{E}',\mathbb{E})$.

(iii) If $E(1_F \mid \mathcal{F}_u)^-$ denotes a left continuous (then predictable) version of the real martingale $(E(1_F \mid \mathcal{F}_u))_{u \in [0,T]}$, $\forall s \leq t$ $\forall F \in \mathcal{F}_t$

$$\text{(III-1-1)} \qquad E\,[1_F.(V_t - V_s)] = \int_{]s,t]\times\Omega} E(1_F \mid \mathcal{F}_u)^-(\omega)\ \alpha(du,d\omega)$$

2°) The process V just defined is predictable as a process in $(\mathbb{E}', \sigma(\mathbb{E}',\mathbb{E}))$, and $\|V_t\|_{\mathbb{E}'}$ is predictable and P-integrable.

3°) If for every $A \in \mathcal{P}$ we define (integrating on each path of V) :

$$m(A) = \int 1_A(s,.)\, dV_s(.) \in \Lambda^1_{\mathbf{E}'}(\Omega, \mathcal{F}, P) \quad (*)$$

m is a stochastic measure (cf. [15] or [18]) with bounded variation $|m| = |\alpha|$.

Proof

Let us first remark that, to the difference with the situation in [14] and [18], $\mathbf{E}'$ is not assumed to be separable. The proof, nevertheless goes along the same line. We sketch it, insisting only upon the needed modification.

As in [14] , for every t, the mapping α_t on $\mathcal{F}_t$ defined through

$$\alpha_t(F) = \int_{]0,t]\times\Omega} E(1_F \mid \mathcal{F}_u)^-(\omega)\ \alpha(du,d\omega)$$

is a σ-additive measure with values in $\mathbf{E}'$, with

$$||\alpha_t(F)||_{\mathbf{E}'} \leq \int_{]0,t]\times\Omega} E(1_F \mid \mathcal{F}_u)^-(\omega)\ |\alpha|\ (du,d\omega)$$

The real measure on the right side of this inequality is positive, and then α_t is with bounded variation, and such that $F \in \mathcal{F}_t$, $P(F) = 0$ implies $\alpha_t(F) = 0$. Then there exists (cf. [13]) a density U_t from Ω into $\mathbf{E}'$ measurable for the topology $\sigma(\mathbf{E}',\mathbf{E})$. Because of the separability of $\mathbf{E}$, $||U_t||_{\mathbf{E}}$ is $\mathcal{F}_t$-measurable.

We have too : $\forall f \in L^\infty_{\mathbf{E}}(\Omega, \mathcal{F}_t, P)$, $s \leq t$

$$\text{(III-1-2)} \qquad E\langle f, U_t-U_s\rangle = \int_{]s,t]\times\Omega} E(f \mid \mathcal{F}_u)^-(\omega)\ \alpha(du,d\omega) .$$

and

$$\text{(III-1-3)} \qquad \forall F \in \mathcal{F}_s \qquad E(1_F . ||U_t-U_s||_{\mathbf{E}'}) \leq |\alpha|(\,]s,t] \times F).$$

(*) $\Lambda^1_{\mathbf{E}'}(\Omega, \mathcal{F}, P)$ is the Pettis space of weakly integrable mappings in $\mathbf{E}'$, with norm $||f||_{\Lambda^1_{\mathbf{E}'}} = \sup_{||y||\leq 1, y\in E} E\,|\langle y,f\rangle|$.

Using the separability of $\mathbf{E}$, it is proved, exactly as in [18] that, for each t, U_t can be modified on a P-null set into a process V_t such that for every $y \in \mathbf{E}$, the real process $(\langle y,V_t\rangle)_{t\in[0,T]}$ has right continuous paths with bounded variation. This implies the σ-additivity for every ω , of the intervall-function $]s,t] \rightsquigarrow \langle y,V_t(\omega) - V_s(\omega)\rangle$, and then (cf. [13] chap. III) the extendability of this function into a measure on the Borel sets of $[0,T]$, with values in a ball of $\mathbf{E}'$ (because of the compacity of such a ball for $\sigma(\mathbf{E}', \mathbf{E})$), and σ-additive for all the topologies on $\mathbf{E}'$ which are compatible with the duality $(\mathbf{E}, \mathbf{E}')$, in particular for the topology $\tau(\mathbf{E}', \mathbf{E})$. As a consequence $t \rightsquigarrow V_t(\omega)$ is right continuous for this topology. This proves the part 1) of the theorem, except for bounded variation of the paths.

Part 2) is a mere consequence of the fact mentionned in [14] that (iii) implies the naturality of each real process $(\langle y,V_t\rangle)_{t\in[0,T]}$ and then its predictability. The predictability of $||V||$ follows from the separability of $\mathbf{E}$.

As to part 3) the only thing to prove is the equality $|m| = |\alpha|$. From (III-1-3) it is clear that $|m| \leq |\alpha|$. But the converse inequality is a consequence of : $\forall s < t$, and $F \in \mathcal{F}_s$

$$||m(]s,t]\times F)||_{\Lambda^1_{\mathbf{E}'}} = \sup_{||y||\leq 1} E\{1_F[\langle y,V_t\rangle - \langle y,V_s\rangle]\}$$

$$\geq \sup_{||y||\leq 1} \langle y,\alpha\,]s,t]\times F\rangle = ||\alpha(A)||_{\mathbf{E}'}$$

As now, for every subdivision $0 = t_0 < t_1 < t_2 < t_n = T$ of the intervall $[0,T]$, $E(\sum_i ||V_{t_{i+1}} - V_{t_i}||_{\mathbf{E}'}) \leq |m|(]0,T]\times\Omega) < \infty$, the bounded variation of the paths is immediate and the proof of the theorem is complete.

III-2 Corollary (Decomposition theorem)

Let $\vec{X}$ be a generalized quasi-martingale on the separable Banach space $\mathcal{B}$. Then there exists a uniquely defined (up to indistinguability) process V, with values in $\mathcal{B}'$, with the following properties :

(i) V has paths with bounded variation (for the norm of $\mathbb{B}'$) and right continuous for $\tau(\mathbb{B}', \mathbb{B})$.

(ii) For every $y \in \mathbb{B}$, $< y, V >$ is a real predictable process.

(iii) Denoting by $\tilde{V}$ the 1-process of S.L.F. on $\mathbb{B}$ associated with V, the process $\tilde{X} - \tilde{V}$ is a cylindrical martingale.

Proof

It is an immediate consequence of theorem 1, because saying that the difference $\tilde{X} - \tilde{V}$ of two processes of S.L.F. is a martingale is equivalent to saying that $\tilde{X}$ and $\tilde{V}$ have same Doleans' measure. And V is uniquely determined by theorem 1 as the process with properties (i), (ii) and (iii) of theorem 1, corresponding to the measure $\alpha_{\tilde{X}}$.

Definition 5

Let α be a σ-additive measure as in theorem 1 (resp. $\tilde{X}$ be a generalized quasi-martingale as in corollary). The process V of the theorem (resp. corollary) will be called the natural process of the measure α (resp. of the generalized quasi-martingale $\tilde{X}$).

Proposition 3

Let V be the natural process of the measure α, as in theorem 1, and m the measure as in part 3) of the theorem.

1°) For every $h \in L^{\infty}(\Omega, \mathcal{F}_t, P)$ and every predictable bounded process Φ with values in $\mathcal{L}(\mathbb{B}' ; \mathbb{E})$ where $\mathbb{E}$ is a Banach-space,

$$E(H \, . \int \Phi \, dm) = \int E(h \mid \mathcal{F}_n)^{-}(\omega) \; \Phi(u,\omega) \; \alpha(du,d\omega)$$

2°) For every predictable process Ψ, with values in $\mathcal{L}(\mathbb{B}' ; \mathbb{F}')$ where $\mathbb{F}'$ is the dual of a Banach space $\mathbb{F}$, with the property

$$\int ||\Psi|| \; d\,|\alpha| \; < \; \infty$$

the natural process W of the measure β defined by

$$\beta(A) = \int_A \Psi \, d\alpha$$

is such that for P-almost all ω :

$$W_t(\omega) = \int_{]0,t]} \Psi(s,\omega) \; dV(s,\omega) \qquad \text{(integration with respect to } s\text{).}$$

Proof

The first part of the proposition is immediately true for $\Phi = 1_{]s,t]\times F}$, $s < t$, $F \in \mathcal{F}_s$.

Then it is true, by linearity and density, for any bounded predictable process.

To prove 2°) we have to show that $\forall h \in L^{\infty}(\Omega, \mathcal{F}_t , P)$

$$E(h \, . \int_{]0,t]\times\Omega} \Psi \, dV) = \int_{]0,t]\times\Omega} E(h \mid \mathcal{F}_u)^{-} \; \Psi \, d\alpha$$

But this is 1° with $\mathbb{E} = \mathbb{F}'$.

Definition 6

Let $\tilde{M}$ be a right continuous S.I.C. martingale. We will write $< M >$ the natural process of the quadratic measure α of $\tilde{M}$.

III-3 Local cylindrical quasi-martingale

Let $\tilde{X}$ be a o-process of S.L.F. For a stopping time T, for any $h \in \mathbb{H}$ and $t \in [0,T]$, $h \rightsquigarrow 1_{]0,T]}(t,\cdot)\, \tilde{X}_t(g)$ is a continuous mapping from $\mathbb{H}$ into $L^0(\Omega, \mathcal{F}_t , P)$. The thus defined o-process of S.L.F will be denoted by $1_{]0,T]}\tilde{X}$, we will then have the natural extension of the classical definition.

Definition 7

A o-process of S.L.F $\tilde{X}$ will be called a local cylindrical quasi-martingale (resp. a local S.I.C. martingale) if there exists an increasing sequence $(\tau(n))_{n \in \mathbb{N}}$ of stopping times, and a corresponding sequence $\tilde{X}_n$ of cylindrical quasi-martingales (resp. S.I.C martingales) such that $\lim_n \tau(n) = T$ a.s and

$$\forall n \qquad 1_{]0,\tau(n)]}\, \tilde{X} = 1_{]0,\tau(n)]}\, \tilde{X}_n$$

It is easily seen that the previous results can be extended to local cylindrical quasi-martingales or S.I.C martingales as in the real case.

IV - STOCHASTIC INTEGRAL WITH RESPECT TO A GENERALIZED QUASI-MARTINGALE

As a local generalized quasi-martingale is the sum of a process V, the paths of which define vector measures on T, and of a local generalized martingale , the problem of defining the stochastic integral with respect to a local quasi-martingale, reduces, as in the classical case, to defining the stochastic integral with respect to a local generalized martingale.

Passing over from the integral with respect to a square integrable martingale to a generalized martingale, which can be localized into square-integrable martingales through a suitable increasing sequences (τ_n) of stopping times goes exactly as in the real classical case. So we will omit-it and restrict ourselves to integrating with respect to square integrable cylindrical martingales, extending the isometry formula proved in [16].

IV-1 Theorem 2

Let $\tilde{M}$ be a square integrable cylindrical martingale, on a Hilbert space $\mathbb{H}$, with quadratic measure α and control measure $\lambda = |\alpha|$. Then there exists a process Q with values in the unit ball of $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})'$ with the following properties.

a) Q is weakly predictable and weakly integrable for the duality $\sigma((\mathbb{H} \hat{\otimes}_1 \mathbb{H})' , \mathbb{H} \hat{\otimes}_1 \mathbb{H})$, with values in the cône of symetric positive elements of $(\mathbb{H} \hat{\otimes} \mathbb{H})'$.

b) $\forall A \in \mathcal{P} \qquad \alpha(A) = \int_A Q\, d\lambda$

Q is unique up to a (weak) λ-equivalence.

Proof

This is a mere application of a weak Radon-Nikodym theorem for vector-measures (see [13] th. 7). In fact it is immediate to check that $\alpha(s) \in \lambda(A).\mathbb{B}_1$, when $\mathbb{B}_1$ is the unit ball of $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})'$.

IV-2 Remark

If for every t, $\tilde{M}_t \in \mathcal{L}_2(\mathbb{H} ; \mathcal{H}_t)$, in which case, $\tilde{M}$ is associated with an ordinary $\mathbb{H}$-valued martingale, α takes its values in $(\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{H})' = \mathbb{H} \hat{\otimes} \mathbb{H}$, which identifies itself as such, as a subspace of $(\mathbb{H} \hat{\otimes} \mathbb{H})'$. In this case (cf. [16]), Q takes its values in $\mathbb{H} \tilde{\otimes} \mathbb{H}$ and is even strongly predictable.

IV-3 Definition of spaces $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$ and $\Lambda_T^{*Q\lambda}(\mathbb{H} ; \mathbb{G})$

Let Q be a process in $(\mathbb{H} \hat{\otimes} \mathbb{H})'$, weakly predictable and integrable (for the duality $\sigma((\mathbb{H} \hat{\otimes} \mathbb{H})' , \mathbb{H} \hat{\otimes} \mathbb{H})$), with values in the cône of positive elements of $(\mathbb{H} \hat{\otimes} \mathbb{H})'$. We recall (cf. §1-3 above) that $\tilde{Q}(t,\omega)$ denotes the bounded linear operator associated with

$$Q(t,\omega) : \langle \tilde{Q}(t,\omega)\, h_1 , h_2 \rangle_{\mathbb{H}} = Q(t,\omega)(h_1 \otimes h_2) \quad \text{for every } h_1 \times h_2 \in \mathbb{H} \times \mathbb{H}$$

From the separability of $\mathbb{H}$, it is clear that weak predictability of Q is equivalent to the strong predictability of $\tilde{Q}(.,.)h$ as a process with values in $\mathbb{H}$, for every $h \in \mathbb{H}$.

Let $\mathbb{G}$ be an Hilbert space. We will define the space $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$ of stochastic processes as the space of those processes X, such that :

(Λ1) $\quad \forall (t,\omega) \in [0,T] \times \Omega \qquad X(t,\omega)$ is a linear operator with domain

in $\mathbb{H}$: $\mathcal{D}(X(t,\omega)) \supset$ Range $(\tilde{Q}^{\frac{1}{2}})$ and range in $\mathbb{G}$, such that the linear operator $X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega)$ is extendable into a bounded linear operator from $\mathbb{H}$ into $\mathbb{G}$.

(Λ2) $\forall g \in \mathbb{G}$, $(t,\omega) \to \tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*(t,\omega)\, g$ is strongly predictable

and $\sup\limits_{\|g\|\leq 1} \int \| \tilde{Q}^{\frac{1}{2}} \circ X^*(g)\|^2_{\mathbb{H}}\, d\lambda < \infty$

where X^* denotes the adjoint mapping of X.

We define also the space $\Lambda_T^{*Q\lambda}(\mathbb{H} ; \mathbb{G})$ of stochastic processes as the space of these processes X such that

(Λ^*1) $\forall (t,\omega) \in [0,T]\times\Omega$ $X(t,\omega)$ is a linear operator with domain in $\mathbb{H}$: $\mathcal{D}(X(t,\omega)) \supset$ Range$(\tilde{Q}^{\frac{1}{2}})$ and range in $\mathbb{G}$, such that the linear operator $X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega)$ is extendable into a Hilbert Schmidt operator from $\mathbb{H}$ into $\mathbb{G}$.

(Λ^*2) $\forall h \in \mathbb{H}$, $(t,\omega) \to X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega)\, h$ is strongly predictable

and $\int \|X \circ \tilde{Q}^{\frac{1}{2}}\|^2_{H.S}\, d\lambda < \infty$

IV-4 Properties of spaces $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$ and $\Lambda_T^{*Q\lambda}(\mathbb{H} ; \mathbb{G})$

Theorem 3

Let λ be a bounded positive measure on $([0,T]\times\Omega, \mathcal{P})$

1°) The mapping $X \to \left(\sup\limits_{\|g\|\leq 1} \int_{[0,T]\times\Omega} \|\tilde{Q}^{\frac{1}{2}} \circ X^*(g)\|^2_{\mathbb{H}}\, d\lambda \right)^{\frac{1}{2}}$ *is a complete semi-norm on $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$, for which this space is complete.*

2°) The mapping $X \to \left(\int_{[0,T]\times\Omega} \|X \circ \tilde{Q}^{\frac{1}{2}}\|^2_{H.S}\, d\lambda \right)^{\frac{1}{2}}$, *is an hilbertian semi-norm on $\Lambda_T^{*Q}(\mathbb{H} ; \mathbb{G})$, associated with the positive bilinear form* $(X, Y) \to \int Tr(X \circ \tilde{Q} \circ Y^*)\, d\lambda$.

Proof

The fact that the above mappings are semi-norm (the second one beeing a prehilbertian one) is immediate.

1°) Let us consider now a Cauchy sequence (X_n) in $\Lambda_T^{Q}(\mathbb{H} ; \mathbb{G})$.

Because of the separability of $\mathbb{G}$, it is possible to extract a subsequence (X_{n_k}) and a linear operator

$$\forall g \begin{cases} \tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*_{n_k}(t,\omega)\, g = Y(t,\omega)\, g \quad \lambda \text{ a.c} \\ \sup\limits_k \int \|\tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*_{n_k}(t,\omega)\, g\|^2_{\mathbb{H}}\, \lambda(dt,d\omega) \leq K < \infty \end{cases}$$

From the Banach-Steinhaus theorem $Y(t,\omega)$ is bounded.
The existence of $X(t,\omega)$ (possibly non continuous) defined on Range $\tilde{Q}^{\frac{1}{2}}(t,\omega)$ by $X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega) = Y^*(t,\omega)$ is evident. And moreover, from the inequality

$$\sup_{\|g\|\leq 1} \int \|(\tilde{Q}^{\frac{1}{2}} \circ X^*_n - \tilde{Q}^{\frac{1}{2}} \circ X^*)\, g\|^2_{\mathbb{H}}\, d\lambda = \sup_{\|g\|\leq 1} \liminf_k \int \|(\tilde{Q}^{\frac{1}{2}} \circ X^*_n - \tilde{Q}^{\frac{1}{2}} \circ X^*)\, g\|_{\mathbb{H}}\, d\lambda .$$

and the Cauchy property of the sequence (X_n), it is clear that (X_n) converges to X in $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$.

2°) If (X_n) is a Cauchy sequence in $\Lambda_T^{*Q\lambda}(\mathbb{H} ; \mathbb{G})$, this means that $(X_n \circ \tilde{Q}^{\frac{1}{2}})$ is a Cauchy sequence in the space $L^2_{\mathbb{E}}([0,T]\times\Omega, \mathcal{P}, \lambda)$ where E is the Hilbert space $\mathcal{L}_2(\mathbb{H} ; \mathbb{G})$. Then it has a limit in $L^2_{\mathbb{E}}([0,T]\times\Omega, \mathcal{P}, \lambda)$. Moreover the limit can be written $X \circ \tilde{Q}^{\frac{1}{2}}$ from the first part of the proof, (X_n) beeing a Cauchy sequence in $\Lambda_T^{Q\lambda}(\mathbb{H} ; \mathbb{G})$ too . This proves the theorem.

Remark

In [16] proposition 3, the part 2° of the theorem was proved when $\tilde{Q}^{\frac{1}{2}}$ is Hilbert-Schmidt.

In what follows we will call $\mathcal{E}_T(\mathbb{H} ; \mathbb{C})$ the set of processes of the following form :

$$X(t,\omega) = \sum_{i=1}^{n} 1_{]r_i, s_i] \times F_i}(t,\omega).u_i$$

where n is any integer $r_i \leq s_i \leq T$, $F_i \in \mathcal{F}_{r_i}$ for all i and $u_i \in \mathcal{L}(\mathbb{H};\mathbb{C})$.

The vector spaces $\Lambda_T^{Q,\lambda}$ and $\Lambda_T^{*Q,\lambda}$ are, moreover, always supposed to be endowed with the above semi-norms, and, as usually done, we will consider without changing the name, the associated separated Banach spaces of equivalence classes of processes. So, when speaking of a process X in $\Lambda_T^{Q,\lambda}$ (resp. $\Lambda_T^{*Q,\lambda}$) we will mean a process in $\Lambda_T^{Q,\lambda}$ defined up to an equivalence.

Theorem 4

The closure of $\mathcal{E}_T(\mathbb{H} ; \mathbb{C})$ in $\Lambda_T^{Q,\lambda}(\mathbb{H} ; \mathbb{C})$, (resp. $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$) contains all the processes X with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ strongly predictable for the uniform norm of $\mathcal{L}(\mathbb{H} ; \mathbb{C})$, and such that $\int_{]0,T]\times\Omega} ||\tilde{Q}^{\frac{1}{2}} \circ X^*||_b^2 \, d\lambda < \infty$ (resp. all the processes $X \in \Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$).

Proof

We first remark that if X is strongly predictable as a process with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ (with its operator norm), and such that $||X||_b \leq K$ considering a sequence (X_n) in $\mathcal{E}$ such that $||X_n||_b \leq K$ for all n with $X = \lim_n X_n$ λ.a.e

$$\lim_n \sup_{||g||\leq 1} \int ||Q^{\frac{1}{2}} \circ (X^* - X_n^*)(h)||_{\mathbb{C}}^2 \, d\lambda \leq \lim_n \int ||\tilde{Q}^{\frac{1}{2}} \circ (X^* - X_n^*)||^2 \, d\lambda$$

$$\leq \lim_n \int ||X - X_n||_b^2 \, ||\tilde{Q}^{\frac{1}{2}}||_b^2 \, d\lambda = 0$$

(In view of $||\tilde{Q}||_b \leq 1$)

Newt, if X satisfies the hypothesis of the theorem, it is easily seen to be approximated in $\Lambda_T^{Q,\lambda}(\mathbb{H} ; \mathbb{C})$ by the sequence $1_{[||X||_b \leq n]} X$.

The last same approximation works for a process in $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$, strongly predictable for the uniform norm $\mathcal{L}(\mathbb{H} ; \mathbb{C})$: for any sequence (X_n) in $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$, which λ.a.e converges to X for the norm of $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ and such that $||X_n \circ \tilde{Q}^{\frac{1}{2}}||_{H.S}$ is dominated by a square integrable fonction

$$\lim_n \int ||(X - X_n) \circ \tilde{Q}^{\frac{1}{2}}||_{H.S}^2 \, d\lambda = 0 .$$

Suppose now that X is any process in $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ with $||X||_b \leq K$ and (e_n) is an orthogonal basis of $\mathbb{H}$,

$$X_n = X \circ \Pi_n$$

where Π_n is the orthogonal projection in $\mathbb{H}$ on the subspace generated by $\{e_1,\dots,e_n\}$. The process X_n is clearly strongly predictable as a process with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ (with its uniform norm), and for all (t,ω) and $h \in \mathbb{H}$

$$X(t,\omega) \, \tilde{Q}^{\frac{1}{2}}(t,\omega) \, h = \lim_n X_n(t,\omega) \, \tilde{Q}^{\frac{1}{2}}(t,\omega) \, h$$

with $$||X_n(t,\omega) \, \tilde{Q}^{\frac{1}{2}}(t,\omega)||_{H.S}^2 = \sum_i ||X_n(t,\omega) \, \tilde{Q}^{\frac{1}{2}}(t,\omega) \, e_i||_{\mathbb{C}}^2$$

$$\leq K^2 \, ||\tilde{Q}^{\frac{1}{2}}||_{H.S}^2 < \infty$$

Then $$\lim_n X_n = X \text{ in } \Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C}).$$

Remark

When $\tilde{Q}$ is nuclear the part of this theorem concerning $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ has been proved in [16] prop. 1.

Definition 7

If $\tilde{Q}$ is the process associated with a square integrable-

cylindrical martingale $\tilde{M}$, as in theorem 2, and λ is the control measure of $\tilde{M}$, we will denote $\Lambda_T^{\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$ (resp. $\Lambda_T^{*\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$ instead of $\Lambda_T^{Q,\lambda}(\mathbb{H}\ ;\ \mathbb{G})$ resp. $\Lambda_T^{*Q,\lambda}(\mathbb{H}\ ;\ \mathbb{G})$.

IV-5 Stochastic integral with respect to a square-integrable cylindrical martingale

Theorem 5

Let $\tilde{M}$ be a square-integrable cylindrical martingale, on the separable hilbert space $\mathbb{H}$ and let $\mathbb{G}$ be another separable hilbert space. For every process $X = 1_{]s,t]\times F}\cdot u$ where $]s,t]\times F$ is a predictable rectangle and $u \in \mathcal{L}(\mathbb{H}\ ;\ \mathbb{G})$ we define $\int_0^T X\, d\tilde{M}$ as the L.S.F on $\mathbb{G}$:

$g \rightsquigarrow 1_F\cdot(\tilde{M}_t - \tilde{M}_s)\circ u^*(g)$

Then the mapping $X \rightsquigarrow \int_0^T X\, d\tilde{M}$ has a unique extension as a linear isometry from the closure $\bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$ of $\mathcal{E}_T(\mathbb{H}\ ;\ \mathbb{G})$ in $\Lambda_T^{\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$, into $\mathcal{L}(\mathbb{G};\ L^2(\Omega,\ \mathcal{F}_T\ ,\ P)$.

Proof

It is clear first that there is a unique linear extension to $\mathcal{E}_T(\mathbb{H}\ ;\ \mathbb{G})$ of the mapping $X \rightsquigarrow \int_0^T X\, d\tilde{M}$ given by

$$\forall g \in \mathbb{G},\left(\int_0^T \left(\sum_i 1_{]s_i,t_i]\times F}\cdot u_i\right).d\tilde{M}\right)(g) = \sum_i 1_F\cdot(\tilde{M}_{t_i} - \tilde{M}_{s_i})\circ u_i^*(g)$$

The only thing to check is that the mapping is an isometry. But the norm of $\int_0^T X\, d\tilde{M}$ in $\mathcal{L}_b(\mathbb{G};\ L^2(\Omega,\ \mathcal{F}_T\ ,\ P))$ is given by

$$\left[\sup_{||g||\leq 1} E\left|\sum_i 1_{F_i}\cdot(\tilde{M}_{t_i} - \tilde{M}_{s_i})\circ u_i^*(g)\right|^2\right]^{\frac{1}{2}}$$

, and because of the martingale property of $(\tilde{M}_t(h))_{t\in[0,T]}$ for every h, this is equal to

$$\left[\sup_{||g||\leq 1}\sum_i E\, 1_{F_i}\cdot|(\tilde{M}_{t_i} - \tilde{M}_{s_i})\circ u_i^*(g)|^2\right]^{\frac{1}{2}} =$$

$$= \sup_{||g||\leq 1}\left[\sum_i\int_{]s_i t_i]\times F_i}(\tilde{Q}(\tau,\omega)\circ u_i^*(g)\mid u_i^*(g))\ \lambda(d\tau,\ d\omega)\right]^{\frac{1}{2}}$$

where Q is the process with values $\mathbb{H}\hat{\otimes}\mathbb{H}$ associated with $\tilde{M}$ and λ the control measure of $\tilde{M}$. The last expression can be written

$$\left[\sup_{||g||\leq 1}\int_{]0,T]\times\Omega}||\tilde{Q}^{\frac{1}{2}}(t,\omega)\circ X^*(g)||^2_{\mathbb{H}}\ d\lambda\right]^{\frac{1}{2}}$$

which proves the isometry.

Theorem 6

With the hypotheses of the preceding theorem, the restriction of $X \rightsquigarrow \int_0^T X\, d\tilde{M}$ to the closure $\bar{\Lambda}_T^{2,M}(\mathbb{H}\ ;\ \mathbb{G})$ of $\mathcal{E}_T \cap \Lambda_T^{*\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$ in $\Lambda_T^{*\tilde{M}}(\mathbb{H}\ ;\ \mathbb{G})$ is an isometry from $\bar{\Lambda}_T^{2,M}(\mathbb{H}\ ;\ \mathbb{G})$ into $\mathcal{L}_2(\mathbb{G};\ L^2(\Omega,\ \mathcal{F}_T\ ,\ P)$.

Proof

For $X = \sum_i 1_{]s_i,t_i]\times F_i}\cdot u_i$ we have indeed

$$\left|\left|\int_0^T X.dM^*\right|\right|^2_{\mathbb{H}.S} = \sum_i E\left|\left|\sum_i 1_{F_i}\,(M^*_{t_i} - M^*_{s_i})\circ u_i^*\,(e_n)\right|\right|^2_{\mathbb{H}}$$

where (e_n) is an orthonormal basis of $\mathbb{G}$.

Using again the martingale property of M^* we get

$$\left|\left|\int_0^T X.dM^*\right|\right|^2_{H.S} = \sum_i E\,||1_F(M^*_{t_i} - M^*_{s_i})\circ u_i^*(e_n)||^2_{\mathbb{H}}$$

$$= \sum_i\int_{]s_i,t_i]\times F_i}\sum_n||\tilde{Q}^{\frac{1}{2}}\circ u_i^*(e_n)||^2_{\mathbb{H}}\ d\lambda$$

$$= \int_{]0,T]\times\Omega}|\tilde{Q}^{\frac{1}{2}}\circ X^*||^2_{H.S}\ d\lambda$$

$$= \int_{]0,T]\times\Omega} ||X \circ \tilde{Q}^{\frac{1}{2}}||^2_{H.S}\, d\lambda .$$

This proves the theorem.

Remark

When $\tilde{Q}$ is nuclear this theorem gives the same result as theorem 2 in [16] .

Theorem 7

Let $\tilde{M}$ be a right continuous square integrable cylindrical martingale on the separable Hilbert space $\mathbb{H}$, and let $\mathbb{G}$ be another separable Hilbert space. Let X be a process in $\bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H} ; \mathbb{G})$ (see th. 5) (resp. $\bar{\Lambda}_T^{2,\tilde{M}}(\mathbb{H} ; \mathbb{G})$).

1°). Then $\left(\int_0^t X.d\tilde{M}\right)_{t\in[0,T]}$ is a right continuous square integrable cylindrical martingale (resp. is the process of S.L.F on $\mathbb{G}$ associated with a uniquely defined right continuous martingale with values in $\mathbb{G}$).

2°) If $\alpha_{\tilde{M}}$ (resp. λ) is the quadratic measure (resp. the control measure) of $\tilde{M}$, and if α_Y is the quadratic measure of the cylindrical martingale $\tilde{Y} = \left(\int_0^t X\, d\tilde{M}\right)_{t\in[0,T]}$, then

(IV-5-1) $$\forall A\in\mathcal{P}, \forall g \in \mathbb{G}, \quad \alpha_Y(A)(g\otimes g) = \int_A d\alpha_M \circ (X^*\otimes X^*)(g\otimes g)$$

$$= \int_A Q_M \circ (X^*\otimes X^*)(g\otimes g)\, d\lambda$$

and the natural process $<\tilde{Y}>$ of α_Y is related to the natural process $<\tilde{M}>$ of $\tilde{M}$ by

(IV-5-2) $$<\tilde{Y}>_t = \int_0^t d<\tilde{M}>_s \circ (X_s^*\otimes X_s^*) .$$

This last integral is defined pathwise as the integral with values in $\mathbb{B}_3 = \mathcal{L}(\mathbb{G}\hat{\otimes}_1 \mathbb{G} ; L^1(\Omega, \mathcal{F}_t, P))$ of a mapping $s \rightsquigarrow X_s^*(\omega)\otimes X_s^*(\omega) \in \mathcal{L}(\mathbb{G}\hat{\otimes}_1\mathbb{G} ; \mathbb{H}\hat{\otimes}_1\mathbb{H}) = \mathbb{B}_1$ with respect to a vector valued measure $d<\tilde{M}(\omega)>_s$ with values in $\mathcal{L}(\mathbb{H}\hat{\otimes}\mathbb{H} ; L^1(\Omega, \mathcal{F}_t, P)) = \mathbb{B}_2$, and relatively to the bilinear mapping $f : \mathbb{B}_1\times\mathbb{B}_2 \rightarrow \mathbb{B}_3$ defined by $f(u,m) = m \circ u$.

3°) If $\tilde{M}$ is continuous, then $\tilde{Y}$ is continuous.

Proof

1°) Saying that $\tilde{Y} = \left(\int_0^t X\, d\tilde{M}\right)_{t\in[0,T]}$ is a right continuous square integrable cylindrical martingale comes to saying that, for every $g\in\mathbb{G}$, $(\tilde{Y}_t(g))_{t\in[0,T]}$ is a right continuous real martingale.

We will prove it for a process $X = 1_{]\sigma,T]\times G}\, u$ where $0\leq\sigma\leq T$ and $u\in\mathcal{L}(\mathbb{H} ; \mathbb{G})$, and $G\in\mathcal{F}_\sigma$, and, because of the linearity and continuity of the mapping $X \rightsquigarrow \int_0^t X.d\tilde{M}$, it will appear immediately to be true for any $X\in\bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H} ; \mathbb{G})$.

Let us prove first that $\forall\, 0\leq s\leq t\leq T \quad F\in\mathcal{F}_s$,

(IV-5-3) $$E\left[(\tilde{Y}_t(g) - \tilde{Y}_s(g))\, 1_F\right] = 0$$

For a particular X of the above form :

$$E\left[(\tilde{Y}_t(g) - \tilde{Y}_s(g)).1_F\right] = E\left[1_{F\cap G}\cdot\left[\tilde{M}_{t\vee\sigma} - \tilde{M}_{s\vee\sigma}\right]\circ u^*(g)\right]$$

The martingale property of $\tilde{M}$ gives (IV-5-3) and then the martingale property of $\tilde{Y}$. As, for the same X,

$$\tilde{Y}_t(g) = 1_{F\cap G}\left[M_{t\vee\sigma}(u^*(g)) - M_\sigma(u^*(g))\right]$$

the continuity of the mapping $t \rightsquigarrow \tilde{Y}_t(g)$ is clear.

By linearity and density we gets immediately (IV-5-3) and then the martingale property for a general $X \in \bar{\Lambda}_T^{b,\tilde{M}}(\mathbf{H};\mathbf{G})$.

Let us assume that a sequence $\tilde{Y}^n$ is such that, for every t, $(\tilde{Y}^n_t)_{n\in\mathbf{N}}$ converges to $\tilde{Y}_t$ in $\mathcal{L}(\mathbf{G};\ L^2(\Omega,\ \mathcal{F}_t,P))$. It is clear, using the classical procedures that, for any g, we can deduce the right continuity in L^2 of the real martingale $(\tilde{Y}_t(g))_{t\in[0,T]}$ from the right continuity of the $(\tilde{Y}^n_t(g))_{t\in[0,T]}$. In the particular case when $X \in \bar{\Lambda}_T^{2,\tilde{M}}(\mathbf{H}\ ;\ \mathbf{G})$, it follows from theorem 6 that $\tilde{Y}_t \in \mathcal{L}_2(\mathbf{G};\ L^2(\Omega,\ \mathcal{F}_t,\ P))$, and then there exists an associated martingale $(Y_t)_{t\in[0,T]}$, in the ordinary sense, with values in $\mathbf{G}$. As for every g, the real martingale $(\langle Y_t,g\rangle)_{t\in[0,T]}$ is right continuous in quadratic mean, it is easily deduced that (Y_t) has a version (unique up to indistingability) right continuous in $\mathbf{G}$.

2°) Considering the same $X = 1_{]\sigma,T]\times G}\cdot u$ we get

$$\alpha_Y(\,]s,t]\times F)(g\otimes g) = E\left\{1_{F\cap G}\cdot\left[(\tilde{M}_t - \tilde{M}_{s\vee\sigma})\circ u^*(g)\right]^{\otimes 2}\right\}$$

$$= \alpha_{\tilde{M}}\left(\,]s\vee\sigma,t]\times(F\cap G)\right)(u^*(g)\otimes u^*(g))$$

$$= \int_{]s,t]\times F} d\,\alpha_{\tilde{M}}\circ X^*\otimes X^*(g\otimes g)$$

$$= \int_{]s,t]\times F} Q_{\tilde{M}}\circ(X^*\otimes X^*)(g\otimes g)\,d\lambda\ .$$

This gives the formula (IV-5-1) for a process which is a step process on predictable rectangles, and for $A =]s,t]\times F$

The mapping $X \rightsquigarrow \int_{]s,t]\times F} X\circ Q_{\tilde{M}}\circ X^*\,d\lambda$ beeing continuous from $\bar{\Lambda}_T^{b,\tilde{M}}(\mathbf{H}\ ;\ \mathbf{G})$ into $\mathcal{L}(\mathbf{G}\hat{\otimes}\mathbf{G};\ L^1(\Omega,\ \mathcal{F}_T,P))$ in view of

$$\sup_{\|g\|\leq 1} E\left\|\int_{]s,t]\times F}\langle X\circ\tilde{Q}_M\circ X^*(g),\ g\rangle_{\mathbf{H}}\,d\lambda\right\| \leq \sup_{\|g\|\leq 1}\int \|X\circ\tilde{Q}_M\circ X^*(g)\|\,d\lambda$$

$$= \sup_{\|g\|\leq 1}\int \|\tilde{Q}_M^{\frac{1}{2}}\circ X^*(g)\|^2\,d\lambda$$

we get formula (IV-5-1) for all $X \in \bar{\Lambda}_T^{b,\tilde{M}}(\mathbf{H}\ ;\ \mathbf{G})$ and $A =]s,t]\times F$.

The formula for all predictable A's follows from the σ-additivity in A of both members of (IV-5-1).

The Pettis integrability of $X^*\otimes X^*$ with respect to α_M, proves that $X^*\otimes X^*$ is Pettis integrable on P-almost all paths ω with respect to $\langle\tilde{M}\rangle$. The cylindrical process $\Phi^*_t = \int_0^t d\,\langle\tilde{M}\rangle_s\circ(X^*_s\otimes X^*_s)$ is weak predictable and satisfies

$$\alpha_Y(\,]s,t]\times F\,)(g\otimes g) = E\,\{1_F\,(\Phi^*_t - \Phi^*_s)(g\otimes g)\}$$

which proves formula (IV-5-2).

3°) It is sufficient to prove that, for all $g \in \mathbf{G}$, Y(g) is continuous since $\tilde{Y}(g)$ is a martingale. If $\tilde{M}$ is continuous the real process $\tilde{Y}(g) = (\int X.d\tilde{M})(g)$ is continuous when $X = u.1_A$ where $A\in\mathcal{P}$ and $u \in \mathcal{L}(\mathbf{H}\ ;\ \mathbf{G})$. Then, the same is true in the general case by linearity and continuity.

<u>Remark</u>

If $\tilde{M}$ is a local cylindrical square integrable martingale and if $X \in \Lambda_T^{\tilde{M}}(\mathbf{H}\ ;\ \mathbf{G})$ (resp. $X \in \Lambda_T^{\tilde{M}}(\mathbf{H}\ ;\ \mathbf{G})$), the process $(\int_0^t X.d\tilde{M})_{t\in T}$ is a local cylindrical martingale (resp. there is a local square integrable G-valued martingale Y associated to the local cylindrical martingale $\tilde{Y}$ where $\tilde{Y}_t = \int_{]0,t]} X.d\tilde{M}$).

V - FORMULAS OF CHANGE OF VARIABLE. ITO'S FORMULA

In [7] an Ito's formula has been given for the stochastic integral with respect to a cylindrical Brownian motion. We want only to note that this formula is immediate consequence of known general "Ito's formulas" as stated for example in [9]. We will restrict ourselves to the continuous case. We recall a general "Ito's formula" as stated in [9] and [14]. Let V be a process with values in a reflexive Banach space $\mathbb{F}$, with continuous paths of bounded variation. Let Y be a square integrable martingale with values in $\mathbb{G}$, with continuous path. If Φ is a mapping of $\mathbb{F} \times \mathbb{G}$, into a Hilbert space $\mathbb{K}$ once continuously differentiable in the first variable, with derivative denoted by $D_x \Phi(x,y) \in \mathcal{L}(\mathbb{F} ; \mathbb{K})$ bounded on any bounded set in $\mathbb{F} \times \mathbb{H}$, and twice continuously differentiable in the second variable, with derivatives denoted by $D_y \Phi(x,y) \in \mathcal{L}(\mathbb{G} ; \mathbb{K})$ and $D^2_{y^2} \Phi(x,y) \in \mathcal{L}(\mathbb{G} \times \mathbb{G} ; \mathbb{K})$ equally bounded on any bounded set in $\mathbb{F} \times \mathbb{G}$, then we have the following formula, expressing equality up to indistingability between two processes :

$$\Phi(V_t , Y_t) = \Phi(V_o, Y_o) + \int_o^t D_x \Phi(V_s , Y_s) \, d V_s + \int_o^t D_y \Phi(V_s , Y_s) \, d Y_s$$
$$+ \frac{1}{2} \int_o^t D^2_{y^2} \Phi(V_s , Y_s) \, d \langle Y\rangle_s$$

(The first and third integrals beeing taken pathwise, while the second is a stochastic integral).

If now M^* is a continuous S.I.C Martingale on $\mathbb{H}$ and $X \in \bar{\Lambda}^{2,M}_T(\mathbb{H};\mathbb{G})$, and Y is the continuous martingale version of the S.I.C Martingale $\int X^*.dM^*$, then, using the formulas of theorem 6 we get immediately the following :

$$\Phi(V_t,Y_t) = \Phi(V_o,Y_o) + \int_o^t D_x \Phi(V_s,Y_s) \, d V_s + \int_o^t D_y \Phi(V,Y) \circ X \circ dM^*$$
$$+ \frac{1}{2} \int_o^t d \langle M\rangle^*_s \circ (X^*_s \otimes X^*_s) \circ D^2_{y^2} \Phi^*(V_s,Y_s)$$

If instead of the bilinear from $D^2_{y^2} \Phi^*(x,y)$ we consider the associated continuous linear mapping $\widetilde{D^2_{y^2}} \Phi^*(x,y) \in \mathcal{L}(\mathbb{G} ; \mathcal{L}(\mathbb{G} ; \mathbb{K})$ the

last integral can be written

$$\frac{1}{2} \int_o^t d \widetilde{\langle M^*\rangle}_s \circ X^*_s \circ \widetilde{D^2_{y^2}}\Phi^*(V,Y) \circ X_s$$

In the particular case when M^* is a cylindrical brownian motion B^* as in [7] , the process of S.L.F $\langle M^*\rangle_s$ reduces to a non stochastic linear mapping $s.\tilde{C}$ (the covariance) of $\mathcal{L}(\mathbb{H} ; \mathbb{H})$. The above formula reduces, when moreover V = 0, to the formula of [7] :

$$\Phi(Y_t) = \Phi(Y_o) + \int_o^t D_y \Phi(Y) \circ X \circ dB^* + \frac{1}{2} \int_o^t X_s \circ \widetilde{D^2_{y^2}} \Phi(V,Y) \circ X^*_s \circ \tilde{C} \, ds$$

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