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INTEGRATION WITH RESPECT TO PROCESSES OF LINEAR FUNCTIONALS

by M. METIVIER

of "integrale stochastique radonifiante" as introduced
.S. Paris t. 276 - 1973 for the cylindrical brownian
the general context of generalized quasi-martingales.
heory in which the notions and formulas of the Kunita-
eal square integrable martingales have a natural ex-

rpose of studying stochastic partial differential
considering perturbations which are "white noise in
he mathematical expression of such an object is a
or a linear random functional as studied for example
dering the special case of "cylindrical brownian motion",
ed a stochastic integral with respect to such a sto-
for example [7] and [11]). In [7] the operator valued
integrated with respect to the cylindrical brownian
the integral process is a (Hilbert valued) Martingale
; and in [11], a Girsanov-theorem is obtained in such

e of this study is to show that in a very general
e to developp a theory of stochastic integration with
al martingales", which extends in a natural way the
ic integral with respect to square integrable martin-
t valued) as studied in [10], [14], [18] for

example.

After the necessary definitions we give in §3
theorem of Doob-Meyer type for generalized quasi-martinga

In §4 the stochastic integral is defined, and
mula as in [16] is given. The class of operator-value
are integrated is wide. The values of the process (as in
necessairly continuous operators. But if, on the contrary
Hilbert-Schmidt valued the integral process is an ordinar
valued martingale.

In §5 an Ito's formula is given for those gen
martingale. But in this first draft of the work, the form
only in the case of continuous quasi-martingales.

IC FUNCTIONALS - NOTATIONS

all this paper we will assume that T is a closed or $\mathbb{R}^+$, a basic probability space $(\Omega, \mathcal{F}, P)$ and an increasing $\mathbb{R}^+$ of sub-σ-algebras of $\mathcal{F}$ with the usual following on : $\mathcal{F}$ is P-complete and all the P-null sets in $\mathcal{F}$ ery t.

mean the set of "predictable rectangles" : $]s,t] \times F \subset T\times\Omega$ T and $F \in \mathcal{F}_s$.

be the algebra of subsets of $T \times \Omega$ generated by $\mathcal{R}$. he σ-algebra generated by $\mathcal{B}$, i.e. : the σ-algebra of of $T \times \Omega$.

, $\mathbb{G}$, $\mathbb{K}$, will denote real Hilbert spaces, all of them able (in our context this is no restriction). The scalar aces will be denoted by $< ,. >_{\mathbb{H}}$, $< .,. >_{\mathbb{G}}$ ... or simply is no possible confusion. The norm will be written : tc... If $\mathbb{B}$ is a Banach space, then $\mathbb{B}'$ will denote the B (set of continuous linear form) endowed, if not ,with the dual Banach norm. We recall that the algebraic $\mathbb{G}$ can be endowed with several norms, giving rise to of $\mathbb{H} \otimes \mathbb{G}$:

$\mathbb{G}$ is the completion for a norm such that every continuous : $(\mathbb{H} \times \mathbb{G}) \to \mathbb{K}$ can be factorized in a unique way as I is the canonical inbedding $\Pi(x,y) = x \otimes y$ and u_b is mapping from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ into $\mathbb{K}$, with same norm as b. The n called the trace-norm and denoted $||.||_{Tr}$. Recall $b(x,y) = <x,y>_{\mathbb{H}}$ the corresponding linear form u_b on he trace-form and denoted Tr.

$\mathbb{G}$ is a Hilbert space with scalar product an extension $>= <x,x'>_{\mathbb{H}} \cdot <y,y'>_{\mathbb{G}}$.

$\mathbb{G}$ is a Banach space, the norm of which will be more ater.

The three topologies induced by the three co tensor product on $\mathbb{H} \otimes \mathbb{G}$ are comparable and we have the injection.

$$\mathbb{H} \hat{\otimes}_1 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_2 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} .$$

Moreover an important theorem of Schatten (s chap. 48) says that the function $(x \otimes y,\ x' \otimes y') \rightsquigarrow <x,x'$ be extended as a continuous bilinear form on $(\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} \times \mathbb{H}$ this duality $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ identifies itself with the Banach d

I-3 There is a unique injective linear mapp the vector space of linear operators with finite range f associating to $x \otimes y$ the operator $h \rightsquigarrow <x,h>_{\mathbb{H}}\, y$. This l extensions which are :

1°) isometry from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ onto $\mathcal{L}_1(\mathbb{H} ; \mathbb{G})$, th nuclear operators from $\mathbb{H}$ into $\mathbb{G}$ with the

2°) isometry from $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ onto $\mathcal{L}_2(\mathbb{H} ; \mathbb{G})$, th Hilbert-Schmidt operators from H into $\mathbb{G}$ Schmidt scalar product ;

3°) isometry from $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ onto $\mathcal{L}_c(\mathbb{H} ; \mathbb{G})$, th compact operators with the usual norm of

In as much $x \otimes y$ can be identified with a bi form on $(\mathbb{H} \times \mathbb{G})$ or a continuous linear form on $\mathbb{H} \otimes \mathbb{G}$, th

$$<x \otimes y,\ x' \otimes y'> = <x,x'>_{\mathbb{H}} \cdot <y,y$$

there is also a continuous linear extension of the prece into an isometry from $(\mathbb{H} \hat{\otimes}_1 \mathbb{G})'$ onto $\mathcal{L}(\mathbb{H} ; \mathbb{G})$, the Bana bounded operators from $\mathbb{H}$ into $\mathbb{G}$ with the usual norm. (Th fact the one which associates to a bounded bilinear b or ded linear operator $\tilde{b}$ in $\mathcal{L}(\mathbb{H} ; \mathbb{G})$ such that $< \tilde{b}(x),$

I-4 Random variables with values in H will measurable mapping from Ω into $\mathbb{H}$. If such a random varia property $E(||X||^2_{\mathbb{H}}) < \infty$, then $\omega \rightsquigarrow X(\omega) \otimes X(\omega)$ is a st random variable with values in $\mathbb{H} \otimes \mathbb{H}$, and as $||x \otimes y||_{Tr}$

able mapping from Ω into $\mathbb{H} \otimes_1 \mathbb{H}$. As a consequence
and is called the covariance of the variable X.

X is moreover associated the continuous mapping $\tilde{X}$:
into $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$, this mapping appears to be Hilbert-
n be shown that conversely to a mapping $\tilde{X}$ linear from $\mathbb{H}$ into
can be associated a random variable X with values in $\mathbb{H}$,
$\tilde{X}(h)$ a.s, if and only if $\tilde{X}$ is Hilbert-Schmidt. The Hilbert
$_2$ of $\tilde{X}$ is equal then to $\sqrt{E(||X||^2_{\mathbb{H}})}$.

To abbreviate the writting we will writte

$$\mathcal{H}^p = L^p_{\mathbb{R}}(\Omega, \mathcal{F}, P) \qquad p \geq o$$

$$\mathcal{H}^p_t = L^p_{\mathbb{R}}(\Omega, \mathcal{F}_t, P) \qquad p \geq o$$

The norm in $\mathcal{L}_1(\mathbb{H} ; \mathbb{C})$ will be written $||.||_{Tr}$, the norm
$.||_{H.S}$, the norm in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$: $||.||_b$.

LINEAR FUNCTIONALS

Définition 1

Let us writte $\mathcal{H}^p_t = L^p(\Omega, \mathcal{F}_t, P)$ *where* $p \geq o$. *Let* $\mathbb{B}$
Then a family $\tilde{X} = (\tilde{X}_t)_{t \in T \subset \mathbb{R}}$, *where* $\tilde{X}_t \in \mathcal{L}(\mathbb{B} ; \mathcal{H}^p_t)$
be called a p-process of stochastic linear functionals on
iate : a process of S.L.F. on $\mathbb{B}$ *).*

r every $h \in \mathbb{B}$, *the real process* $(\tilde{X}_t(h))_{t \in T}$ *is a martin-*
$\tilde{X}$ *will be called a p-cylindrical martingale.*

what has been said in the first paragraph, if $\tilde{X}_t \in \mathcal{L}_2(\mathbb{H} ; \mathcal{H}_t)$
or every t and h, $\tilde{X}_t(h) = \langle X_t, h \rangle$ *where* $X = (X_t)_{t \in T}$
are integrable stochastic process with values in $\mathbb{H}$, *as*
mple in [10] *and* [14] .

Definition 2

A p-process $\tilde{X}$ *of S.L.F on* $\mathbb{B}$ *will be sai*
p-right continuous) if, for every $h \in \mathbb{B}$, $t \rightsquigarrow \tilde{X}_t(h)$
mapping from $[0,T]$ *into* $L^p(\Omega, \mathcal{F}, P)$. *It is said co*
continuous) if, for every $h \in \mathbb{H}$, *there is a version o*
$(\tilde{X}_t(h))_{t \in [0,T]}$ *with continuous (resp. right-continu*

II-2 Doleans' measure of a process of lin

We extend here the concept of Doleans'
defined in [5] for real sub-martingale and extended s
valued quasi-martingales (see for ex. [15]).

To every process $\tilde{X}$ of S.L.F. on the Banach
the additive functions $\tilde{\alpha}_X$ with values in $\mathbb{B}'$ defined
predictable rectangles by

$$\text{(II-2-1)} \qquad \tilde{\alpha}_X(\,]s,t] \times F) = E\left[1_F \,.(\tilde{X}_t - \tilde{X}_s)\right] \in \mathbb{B}$$

Such a function on $\mathcal{R}$ has clearly an addi
algebra $\mathcal{A}$ generated by $\mathcal{R}$. We call it $\tilde{\alpha}_X$ agair

Definition 3

If the additive function $\tilde{\alpha}_X$ *on* $\mathcal{A}$ *has*
(for the norm of $\mathbb{B}'$*), the process* $\tilde{X}$ *of stochastic li*
will be called a generalized quasi-martingale.

This clearly generalizes the classical def
We have then the

Proposition 1

For $\tilde{X}$ *to be a generalized quasi ma*
sary and sufficient that the family of rea
(α^h_X) *associated with the real processes*
bounded variation, and that the set of tho

ordered set of bounded positive measures.

from the fact that the total variation of $\tilde{\alpha}_X$ can be
f the type

$$|X_{t_i}(h_i) - X_{s_i}(h_i)| \;) \qquad h_i \in \mathbb{B},\ ||h_i|| \leq 1$$

the variations can be approximated by sums of the

$$]s_i,t_i] \times F_i| = \sum_i |E\{1_{F_i}[X_{t_i}(h_i) - X_{s_i}(h_i)]\}|$$

both supremum coincide.

are integrable cylindrical martingales

$\tilde{M}$ be a 2-cylindrical martingale on the Banach space
he 1-process of S.L.F on $\mathbb{B} \hat{\otimes} \mathbb{B}$ defined by

$$h \otimes g) = \tilde{M}_t(h) \,.\, \tilde{M}_t(g)$$

ar that if $\tilde{M}$ is the process of S.L.F associated with
M with values in $\mathbb{B}$, $\tilde{X}$ is the process of S.L.F. on
h the ordinary sense process $(M_t \otimes M_t)_{t \in [0,T]}$,
$\hat{\otimes}\, \mathbb{B}$.

wn (cf. for example [15]) that, if M is right conti-
inary sense quasi-martingale, with associated σ-addi-
(with values in $\mathbb{B} \hat{\otimes} \mathbb{B}$)

$$s,t] \times F)||_{(\mathbb{B}'\hat{\otimes}\mathbb{B}')'} \leq ||E|1_F\,(M_t - M_s)^{\otimes 2}\;|\;||_{\mathbb{B}\hat{\otimes}\mathbb{B}}$$

d quasi-martingale.

ollowing example shows that for a 2-cylindrical martin-
true that $\tilde{M} \otimes M$ is a generalized quasi-martingale.

Example 1

Let (t_n) be a decreasing sequence to zero. Le
space, (e_n) an orthonormal basis of $\mathbb{H}$ and let us define th
martingale :

$$\tilde{M}_t(h) = \sum_n 1_{[t_{n+1},\infty[}(t)\ (h \mid e_n) \cdot \frac{1}{\sqrt{t_n - t_{n+1}}}\ (\beta$$

where (β_t) is a usual real standard Brownian motion. For e
is a process with independant increments, zero on $]0, t_{n+1}$
constant on $[t_n, \infty[$. For every $h \in \mathbb{H}$, the above series
$L^2(\Omega, \mathcal{F}_t, P)$ with

$$E\,|\tilde{M}_t(h)|^2 \leq ||h||^2_{\mathbb{H}}$$

defining a process with zero-mean independant increments.
the partition $(\,]t_{n+1}, t_n] \times \Omega\,)_{n>0}$ of $]0,t_1] \times \Omega$, it
seen that, $\tilde{X}$ beeing the process of S.L.F. above defined :

$$\sum_n E\,||1_\Omega(\tilde{X}_{t_n} - \tilde{X}_{t_{n+1}})||_{\mathcal{L}(\mathbb{H}\hat{\otimes}\mathbb{H})} \geq \sum_n E\,\frac{1}{\sqrt{t_n - t_{n+}}}$$

We will then give the following

Definition 4

*A 2-cylindrical martingale $\tilde{M}$, on a Banach
called a square integrable cylindrical martingale (S.I.C M
the 1-process of S.L.F. (improperly) denoted by $\tilde{M} \otimes \tilde{M}$, de*

$$\tilde{M} \otimes \tilde{M}(h \otimes g) = \tilde{M}(h) \,.\, \tilde{M}(g)$$

is a quasi-martingale.

*The additive measure $\alpha_{\tilde{M} \otimes \tilde{M}}$ on $\mathcal{A}$ will be c
quadratic measure of $\tilde{M}$, and its variation the control meas*

Example 2

The Brownian process, associated with the uni
matrix on a Hilbert space $\mathbb{H}$ is a S.I.C. Martingale : let u

of S.L.F on $\mathbb{H}$, such that, for every finite set of vectors
the process $(\tilde{W}_t(h_1),\ldots, \tilde{W}_t(h_n))$ is an n-dimensional
h independant increments, and the bilinear form
g)) is the scalar product $(h,g) \rightsquigarrow \langle h,g\rangle_{\mathbb{H}}$.

case, the quadratic-measure α of $\tilde{W}$ associates to every
$]s,t] \times F$ the linear form on $\mathbb{H} \hat{\otimes} \mathbb{H}$ defined by

$$t] \times F)(h \otimes g) = E\{1_F \left[\tilde{W}_t(h).\tilde{W}_t(g) - \tilde{W}_s(h).\tilde{W}_s(g)\right]\}$$

properties of $\tilde{W}$:

$$t] \times F)(h \otimes g) = E\{1_F\,(W_t(h)-W_s(h)).(W_t(g)-W_s(g))\}$$

$$= P(F)\,.\,(t-s)(h|g)$$

$$= P(F)\,.\,(t-s)\,\mathrm{Tr}(h \otimes g)$$

α is then a one dimensional measure with values in
nal to the product measure $\ell \otimes P$ where ℓ is the Lebesgue

tion 2   *let us assume that $\mathbb{B}$ is reflexive.*

If $\tilde{M}$ is a right continuous S.I.C. Martingale on the space $\mathbb{B}$, the quadratic measure α of $\tilde{M}$ has a σ-additive on (in $\mathbb{B}'$) to the σ-algebra of predictable subsets of
.

nough to show that the variation of α has a σ-additive
proposition 1 and its proof, and the set of σ-additive
esz-band (cf. [2]) in the ordered set of finitely
th bounded variation , the variation of α will be
s for every $h \in \mathbb{B}$ the measure α^h is σ-additive. In
own property of square integrable real martingales
e measures $\alpha^{h\otimes g}$ are σ-additive, and the same is clearly
ith u in $\mathbb{H} \hat{\otimes} \mathbb{H}$.

III - DECOMPOSITION THEOREM FOR A GENERALIZED QUASI-MART

For the purpose of transformation formulas
integration, Doob-Meyer's decomposition theorem play an
We intend to give an extension of such decomposition the
setting.

All this rests on the following :

III-1 Theorem 1

Let $\mathbb{E}'$ be the Banach dual of a separable B
α be a σ-additive measure on the predictable sets, with
bounded variation, such that for every evanescent set A,

Let $\tau(\mathbb{E}',\mathbb{E})$ denote the Mackey topology
finest locally convex topology on $\mathbb{E}'$ for which $\mathbb{E}$ is the

Then :

1°) There exists a stochastic process V, with valu
with right continuous paths for $\tau(\mathbb{E}',\mathbb{E})$, uniq
gability such that

(i) $\forall t$ $\quad V_t$ is weakly integrable in $\mathbb{E}'$ for

(ii) For every $\omega \in \Omega$, the intervall-functi
$]s,t] \rightarrow V_t(\omega) - V_s(\omega)$ can be extend
on the Borels sets of $[0,T]$, with values in
variation (for the norm of $\mathbb{E}'$) and σ-additive

(iii) If $E(1_F \mid \mathfrak{F}_u)^-$ denotes a left contin
table) version of the real martingale $(E(1_F \mid$
$\forall s \leqslant t \quad \forall F \in \mathfrak{F}_t$

$$(III\text{-}1\text{-}1) \qquad E\left[1_F.(V_t-V_s)\right] = \int_{]s,t]\times\Omega} E(1_F \mid$$

2°) The process V just defined is predictable as a
$(\mathbb{E}', \sigma(\mathbb{E}',\mathbb{E}))$, and $\|V_t\|_{\mathbb{E}'}$ is predictable a

ry $A \in \mathcal{P}$ we define (integrating on each path of V) :

$$\int 1_A(s,.)\, dV_s(.) \in \Lambda^1_{\mathbb{E}'}\, (\Omega, \mathcal{F}, P) \quad (*)$$

chastic measure (cf. [15] or [18]) with bounded varia-
$= |\alpha|$.

rst remark that, to the difference with the situation
is not assumed to be separable. The proof, nevertheless
line. We sketch it, insisting only upon the needed mo-

] , for every t, the mapping α_t on $\mathcal{F}_t$ defined through

$$= \int_{]0,t]\times\Omega} E(1_F \mid \mathcal{F}_u)^-(\omega)\ \alpha(du,d\omega)$$

ure with values in $\mathbb{E}'$, with

$$|_{\mathbb{E}'} \leq \int_{]0,t]\times\Omega} E(1_F \mid \mathcal{F}_u)^-(\omega)\ |\alpha|\ (du,d\omega)$$

measure on the right side of this inequality is positive,
h bounded variation, and such that $F \in \mathcal{F}_t$, $P(F) = 0$
Then there exists (cf. [13]) a density U_t from Ω
for the topology $\sigma(\mathbb{E}',\mathbb{E})$. Because of the separability
$\mathcal{F}_t$-measurable.

too : $\forall f \in L^\infty_{\mathbb{E}}(\Omega, \mathcal{F}_t , P)$, $s \leq t$

$$E \langle f, U_t - U_s\rangle = \int_{]s,t]\times\Omega} E\ (f \mid \mathcal{F}_u)^-(\omega)\ \alpha(du,d\omega)\ .$$

and

$$) \quad \forall F \in \mathcal{F}_s \qquad E(1_F . ||U_t - U_s||_{\mathbb{E}'}) \leq |\alpha|(\ ]s,t] \times F).$$

---

is the Pettis space of weakly integrable mappings in

$$f||_{\Lambda^1_{\mathbb{E}'}} = \sup_{||y||\leq 1,\, y\in E} E\ |\langle y,f\rangle|\ .$$

Using the separability of $\mathbb{E}$, it is proved,
that, for each t, U_t can be modified on a P-null set int
that for every $y \in \mathbb{E}$, the real process $(\langle y,V_t\rangle)_{t\in[0,T]}$
paths with bounded variation. This implies the σ-additiv
of the intervall-function $]s,t] \rightsquigarrow \langle y,V_t(\omega) - V_s(\omega)\rangle$,
chap. III) the extendability of this function into a me
sets of $[0,T]$, with values in a ball of $\mathbb{E}'$ (because of
such a ball for $\sigma(\mathbb{E}', \mathbb{E})$), and σ-additive for all the to
which are compatible with the duality $(\mathbb{E}, \mathbb{E}')$, in partic
logy $\tau(\mathbb{E}', \mathbb{E})$. As a consequence $t \rightsquigarrow V_t(\omega)$ is right c
topology. This proves the part 1) of the theorem, except
tion of the paths.

Part 2) is a mere consequence of the fact
that (iii) implies the naturality of each real process (<
and then its predictability. The predictability of $||V||$
separability of $\mathbb{E}$.

As to part 3) the only thing to prove is th
From (III-1-3) it is clear that $|m| \leq |\alpha|$. But the conve
a consequence of : $\forall s < t$, and $F \in \mathcal{F}_s$

$$||\ m\ (]s,t]\times F)||_{\Lambda^1_{\mathbb{E}'}} = \sup_{||y||\leq 1} E\ \{1_F[\langle y,V_t\rangle - \langle y$$

$$\geq \sup_{||y||\leq 1} \langle y,\alpha\ ]s,t]\times F\rangle = ||\alpha($$

As now, for every subdivision $0 = t_0 < t_1$
of the intervall $[0,T]$, $E(\sum_i ||V_{t_{i+1}} - V_{t_i}||_{\mathbb{E}'}) \leq |m|(]$
bounded variation of the paths is immediate and the proo
is complete.

III-2 Corollary (Decomposition theorem)

Let $\vec{X}$ be a generalized quasi-martingale o
Banach space $\mathbb{B}$. Then there exists a uniquely defined (up
lity) process V, with values in $\mathbb{B}'$, with the following p

ths with bounded variation (for the norm of B') and tinuous for $\tau(B', B)$.

$y \in B$, $< y, V >$ is a real predictable process.

by $\tilde{V}$ the 1-process of S.L.F. on B associated with V, ss $\tilde{X} - \tilde{V}$ is a cylindrical martingale.

immediate consequence of theorem 1, because saying that of two processes of S.L.F. is a martingale is equiva- $\tilde{X}$ and $\tilde{V}$ have same Doleans' measure. And V is uniquely m 1 as the process with properties (i), (ii) and (iii) ponding to the measure $\alpha_{\tilde{X}}$.

on 5

be a σ-additive measure as in theorem 1 (resp. $\tilde{X}$ be martingale as in corollary). The process V of the lary) will be called the natural process of the measure eralized quasi-martingale $\tilde{X}$).

ion 3

Let V be the natural process of the measure α , as in 1, and m the measure as in part 3) of the theorem.

every $h \in L^\infty(\Omega, \mathcal{F}_t, P)$ and every predictable bounded ess Φ with values in $\mathcal{L}(B' ; E)$ where E is a Banach- e,

$$. \int \Phi \, dm) = \int E(h \mid \mathcal{F}_n)^-(\omega) \; \Phi(u,\omega) \; \alpha(du,d\omega)$$

every predictable process Ψ, with values in $\mathcal{L}(B' ; F')$ e F' is the dual of a Banach space F, with the property

$$\int ||\Psi|| \; d \, |\alpha| \; < \; \infty$$

natural process W of the measure β defined by

$$\beta(A) = \int_A \Psi \, d\alpha$$

is such that for P-almost all ω :

$$W_t(\omega) = \int_{]0,t]} \Psi(s,\omega) \; dV(s,\omega) \quad \textit{(integra} \atop \textit{t}$$

Proof

The first part of the proposition is immedia $\Phi = 1_{]s,t]\times F}$, $s < t$, $F \in \mathcal{F}_s$.

Then it is true, by linearity and density, f predictable process.

To prove 2°) we have to show that $\forall h \in L^\infty(\Omega$

$$E(h \, . \int_{]0,t]\times\Omega} \Psi \; dV) = \int_{]0,t]\times\Omega} E(h \mid \mathcal{F}_u)^- \; \Psi \; d\alpha$$

But this is 1° with $E = F'$.

Definition 6

Let $\tilde{M}$ be a right continuous S.I.C. marti $< M >$ the natural process of the quadratic measure α of

III-3 Local cylindrical quasi-martingale

Let $\tilde{X}$ be a o-process of S.L.F. For a stopp any $h \in H$ and $t \in [0,T]$, $h \rightsquigarrow 1_{]0,T]}(t,\cdot) \; \tilde{X}_t(g)$ is a from H into $L^0(\Omega, \mathcal{F}_t, P)$. The thus defined o-process denoted by $1_{]0,T]}\tilde{X}$, we will then have the natural exte sical definition.

ition 7

-process of S.L.F $\tilde{X}$ *will be called a local cylindrical* *resp. a local S.I.C. martingale) if there exists an in-* $(\tau(n))_{n \in \mathbb{N}}$ *of stopping times, and a corresponding se-* *indrical quasi-martingales (resp. S.I.C martingales)* $(n) = T$ *a.s and*

$$\forall\, n \qquad 1_{]0,\tau(n)]}\,\tilde{X} = 1_{]0,\tau(n)]}\,\tilde{X}_n$$

that the previous results can be extended to local cylin- *gales or S.I.C martingales as in the real case.*

TEGRAL WITH RESPECT TO A GENERALIZED QUASI-MARTINGALE

ocal generalized quasi-martingale is the sum of a process ich define vector measures on T, and of a local generalized roblem of defining the stochastic integral with respect artingale, reduces, as in the classical case, to defining egral with respect to a local generalized martingale.

over from the integral with respect to a square inte- to a generalized martingale, which can be localized into martingales through a suitable increasing sequences (τ_n) goes exactly as in the real classical case. So we will ct ourselves to integrating with respect to square inte- martingales, extending the isometry formula proved in

Theorem 2

Let $\tilde{M}$ *be a square integrable cylindrical martingale, on* *with quadratic measure* α *and control measure* $\lambda = |\alpha|$. *process Q with values in the unit ball of* $(H \hat{\otimes}_1 H)'$ *with* *rties.*

a) Q is weakly predictable and weakly integr *duality* $\sigma((H \hat{\otimes}_1 H)', H \hat{\otimes}_1 H)$, *with value* *symetric positive elements of* $(H \hat{\otimes} H)'$.

b) $\forall\, A \in \mathcal{P} \qquad \alpha(A) = \int_A Q\, d\lambda$

Q is unique up to a (weak) λ*-equivalence.*

Proof

This is a mere application of a weak Radon-N for vector-measures (see [13] th. 7). In fact it is imm that $\alpha(s) \in \lambda(A).B_1$, when B_1 is the unit ball of $(H \hat{\otimes}$

IV-2 Remark

If for every t, $\tilde{M}_t \in \mathcal{L}_2(H ; \mathcal{H}_t)$, i associated with an ordinary H-valued martingale, α takes $(H \hat{\otimes}_\varepsilon H)' = H \hat{\otimes} H$, which identifies itself as such, as a $(H \hat{\otimes} H)'$. In this case (cf. [16]), Q takes its values i even strongly predictable.

IV-3 Definition of spaces $\Lambda_T^{Q\lambda}(H ; G)$ and

Let Q be a process in $(H \hat{\otimes} H)'$, *weakl* *integrable (for the duality* $\sigma((H \hat{\otimes} H)', H \hat{\otimes} H)$), *with* *of positive elements of* $(H \hat{\otimes} H)'$. *We recall (cf. §1-3 abo* *denotes the bounded linear operator associated with*

$$Q(t,\omega) : \langle \tilde{Q}(t,\omega)\, h_1 , h_2 \rangle_H = Q(t,\omega)(h_1 \otimes h_2) \quad f$$

From the separability of H, *it is clear that weak predict* *equivalent to the strong predictability of* $\tilde{Q}(.,.)h$ *as a* *values in* H, *for every* $h \in H$.

Let G *be an Hilbert space. We will define th* *of stochastic processes as the space of those processes X*

(Λ1) $\quad \forall (t,\omega) \in [0,T] \times \Omega \qquad X(t,\omega)$ *is a linear op*

) $\supset$ *Range* ($\tilde{Q}^{\frac{1}{2}}$) *and range in* $\mathbb{G}$*, such that the linear*

$\tilde{Q}^{\frac{1}{2}}(t,\omega)$ *is extendable into a bounded linear operator*

, $(t,\omega) \to \tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*(t,\omega)\, g$ *is strongly predictable*

$\frac{1}{2} \circ X^*(g)||^2_{\mathbb{H}}\, d\lambda < \infty$

the adjoint mapping of X.

ine also the space $\Lambda_T^{*Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$ *of stochastic processes*
ese processes X such that

$\omega) \in [0,T]\times\Omega$ $X(t,\omega)$ *is a linear operator with domain*

) $\supset$ *Range*($\tilde{Q}^{\frac{1}{2}}$) *and range in* $\mathbb{G}$*, such that the linear*

$\tilde{Q}^{\frac{1}{2}}(t,\omega)$ *is extendable into a Hilbert Schmidt operator*

$\mathbb{H}$, $(t,\omega) \rightsquigarrow X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega)\, h$ *is strongly predictable*

$\int ||X \circ \tilde{Q}^{\frac{1}{2}}||^2_{H.S}\, d\lambda < \infty$

<u>Properties of spaces $\Lambda_T^{Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$ and $\Lambda_T^{*Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$</u>

<u>m 3</u>

λ *be a bounded positive measure on* ($[0,T]\times\Omega$,$\mathcal{P}$)

e mapping $X \rightsquigarrow \left(\sup_{||g||\leq 1} \int_{[0,T]\times\Omega} ||\tilde{Q}^{\frac{1}{2}} \circ X^*(g)||^2_{\mathbb{H}}\, d\lambda \right)^{\frac{1}{2}}$ *is*

complete semi-norm on $\Lambda_T^{Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$*, for which this space*
complete.

e mapping $X \to \left(\int_{[0,T]\times\Omega} ||X \circ \tilde{Q}^{\frac{1}{2}}||^2_{H.S}\, d\lambda \right)^{\frac{1}{2}}$*, is an*

lbertian semi-norm on $\Lambda_T^{*Q}(\mathbb{H}\,;\,\mathbb{G})$*, associated with the*
sitive bilinear form $(X, Y) \rightsquigarrow \int Tr(X \circ \tilde{Q} \circ Y^*)\, d\lambda$.

<u>Proof</u>

The fact that the above mappings are semi-n
beeing a prehilbertian one) is immediate.

1°) Let us consider now a Cauchy sequence (

Because of the separability of $\mathbb{G}$, it is
a subsequence (X_{n_k}) and a linear operat

$$\forall g \begin{cases} \tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*_{n_k}(t,\omega)\, g = Y(t,\omega)\, g \\ \sup_k \int ||\tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*_{n_k}(t,\omega)\, g||^2_{\mathbb{H}}\, \lambda(d \end{cases}$$

From the Banach-Steinhaus theorem $Y(t,\omega)$
The existence of $X(t,\omega)$ (possibly non continuous) de
by $X(t,\omega) \circ \tilde{Q}^{\frac{1}{2}}(t,\omega) = Y^*(t,\omega)$ is evident. And moreov
lity

$$\sup_{||g||\leq 1} \int ||(\tilde{Q}^{\frac{1}{2}} \circ X^*_n - \tilde{Q}^{\frac{1}{2}} \circ X^*)\, g||^2_{\mathbb{H}}\, d\lambda = \sup_{||g||\leq 1} \lim_k$$

$$- \tilde{Q}$$

and the Cauchy property of the sequence (X_n), it is cl
to X in $\Lambda_T^{Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$.

2°) If (X_n) is a Cauchy sequence in $\Lambda_T^{*Q\lambda}(\mathbb{H}$
$(X_n \circ \tilde{Q}^{\frac{1}{2}})$ is a Cauchy sequence in the sp
where E is the Hilbert space $\mathcal{L}_2(\mathbb{H}\,;\,\mathbb{G}$
in $L^2_E([0,T]\times\Omega, \mathcal{P}, \lambda)$. Moreover the limi
$X \circ \tilde{Q}^{\frac{1}{2}}$ from the first part of the proo
Cauchy sequence in $\Lambda_T^{Q\lambda}(\mathbb{H}\,;\,\mathbb{G})$ too . Th

<u>Remark</u>

In [16] proposition 3, the part 2° of the
when $\tilde{Q}^{\frac{1}{2}}$ is Hilbert-Schmidt.

at follows we will call $\mathcal{E}_T(\mathbb{H} ; \mathbb{C})$ the set of processes
orm :

$$) = \sum_{i=1}^{n} 1_{]r_i,s_i]\times F_i}(t,\omega).u_i$$

nteger $r_i \leq s_i \leq T$, $F_i \in \mathcal{F}_{r_i}$ for all i and $u_i \in \mathcal{L}(\mathbb{H};\mathbb{C})$.

ector spaces $\Lambda_T^{Q,\lambda}$ and $\Lambda_T^{*Q,\lambda}$ are, moreover, always
owed with the above semi-norms, and, as usually done, we
out changing the name, the associated separated Banach
nce classes of processes. So, when speaking of a process
$\Lambda_T^{*Q,\lambda}$) we will mean a process in $\Lambda_T^{Q,\lambda}$ defined up to

em 4

losure of $\mathcal{E}_T(\mathbb{H} ; \mathbb{C})$ in $\Lambda_T^{Q,\lambda}(\mathbb{H} ; \mathbb{C})$, (resp. $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$)
rocesses X with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ strongly predicta-
n norm of $\mathcal{L}(\mathbb{H} ; \mathbb{C})$, and such that $\int_{]0,T]\times\Omega} ||\tilde{Q}^{\frac{1}{2}} \circ X^*||_b^2 \, d\lambda < \infty$

cesses $X \in \Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$).

irst remark that if X is strongly predictable as a process
$(\mathbb{H} ; \mathbb{C})$ (with its operator norm), and such that $||X||_b \leq K$
ence (X_n) in $\mathcal{E}$ such that $||X_n||_b \leq K$ for all n with
.e

$$\circ (X^* - X_n^*)(h)||_{\mathbb{C}}^2 \, d\lambda \leq \lim_n \int ||\tilde{Q}^{\frac{1}{2}} \circ (X^* - X_n^*)||^2 \, d\lambda$$

$$\leq \lim_n \int ||X - X_n||_b^2 \, ||\tilde{Q}^{\frac{1}{2}}||_b^2 \, d\lambda = 0$$

iew of $||\tilde{Q}||_b \leq 1$)

Newt, if X satisfies the hypothesis of the t
seen to be approximated in $\Lambda_T^{Q,\lambda}(\mathbb{H} ; \mathbb{C})$ by the sequence

The last same approximation works for a proc
with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$, strongly predictable for the
$\mathcal{L}(\mathbb{H} ; \mathbb{C})$: for any sequence (X_n) in $\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$, wh
to X for the norm of $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ and such that $||X_n \circ \tilde{Q}^{\frac{1}{2}}|$
by a square integrable fonction

$$\lim_n \int ||(X - X_n) \circ \tilde{Q}^{\frac{1}{2}}||_{H.S}^2 \, d\lambda = 0 .$$

Suppose now that X is any process in $\Lambda_T^{*Q,}$
$||X||_b \leq K$ and (e_n) is an orthogonal basis of $\mathbb{H}$,

$$X_n = X \circ \Pi_n$$

where Π_n is the orthogonal projection in $\mathbb{H}$
generated by $\{e_1,\dots,e_n\}$. The process X_n is clearly st
as a process with values in $\mathcal{L}(\mathbb{H} ; \mathbb{C})$ (with its unifor
all (t,ω) and $h \in \mathbb{H}$

$$X(t,\omega)\, \tilde{Q}^{\frac{1}{2}}(t,\omega)\, h = \lim_n X_n(t,\omega)\, \tilde{Q}^{\frac{1}{2}}(t$$

with $$||X_n(t,\omega)\, \tilde{Q}^{\frac{1}{2}}(t,\omega)||_{H.S}^2 = \sum_i ||X_n(t,\omega)\, \tilde{Q}^{\frac{1}{2}}(t,\omega)$$

$$\leq K^2 \, ||\tilde{Q}^{\frac{1}{2}}||_{H.S}^2 < \infty$$

Then $$\lim_n X_n = X \text{ in } \Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C}).$$

Remark

When $\tilde{Q}$ is nuclear the part of this theore
$\Lambda_T^{*Q,\lambda}(\mathbb{H} ; \mathbb{C})$ has been proved in [16] prop. 1.

Definition 7

If $\tilde{Q}$ is the process associated with a

gale $\tilde{M}$, as in theorem 2, and λ is the control measure
te $\Lambda_T^{\tilde{M}}(\mathbb{H} ; \mathbb{G})$ (resp. $\Lambda_T^{*\tilde{M}}(\mathbb{H} ; \mathbb{G})$ instead of $\Lambda_T^{Q,\lambda}(\mathbb{H} ; \mathbb{G})$
).

<u>Stochastic integral with respect to a square-integrable</u>
<u>cylindrical martingale</u>

<u>em 5</u>

$\tilde{}$ be a square-integrable cylindrical martingale, on the
space $\mathbb{H}$ and let $\mathbb{G}$ be another separable hilbert space.
$X = 1_{]s,t]\times F} \cdot u$ where $]s,t]\times F$ is a predictable rec-
$(\mathbb{H} ; \mathbb{G})$ we define $\int_0^T X\, d\tilde{M}$ as the L.S.F on $\mathbb{G}$:
$\circ\, u^*(g)$

$X \rightsquigarrow \int_0^T X\, d\tilde{M}$ has a unique extension as a linear isometry
$\bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H} ; \mathbb{G})$ of $\mathcal{E}_T(\mathbb{H} ; \mathbb{G})$ in $\Lambda_T^{\tilde{M}}(\mathbb{H} ; \mathbb{G})$, into
P).

clear first that there is a unique linear extension to
mapping $X \rightsquigarrow \int_0^T X\, d\tilde{M}$ given by

$$1_{]s_i,t_i]\times F} \cdot u_i).d\tilde{M} \Big) (g) = \sum_i 1_F.(\tilde{M}_{t_i} - \tilde{M}_{s_i}) \circ u_i^*(g)$$

nly thing to check is that the mapping is an isometry. But
$d\tilde{M}$ in $\mathcal{L}_b(\mathbb{G}; L^2(\Omega, \mathcal{F}_T , P))$ is given by

$$_{F_i} .(\tilde{M}_{t_i} - \tilde{M}_{s_i}) \circ u_i^*(g)|^2 \Big]^{\frac{1}{2}}$$, and because of the martin-

$(\tilde{M}_t(h))_{t \in [0,T]}$ for every h, this is equal to

$$\Big[\sup_{||g||\leq 1} \sum_i E\, 1_{F_i} . |(\tilde{M}_{t_i} - \tilde{M}_{s_i}) \circ u_i^*(g)|^2 \Big]^{\frac{1}{2}} =$$

$$= \sup_{||g||\leq 1} \Big[\sum_i \int_{]s_i t_i]\times F_i} (\tilde{Q}(\tau,\omega) \circ u_i^*(g)$$

where Q is the process with values $\mathbb{H} \hat{\otimes} \mathbb{H}$ associated
control measure of $\tilde{M}$. The last expression can be writt

$$\Big[\sup_{||g||\leq 1} \int_{]0,T]\times\Omega} ||\tilde{Q}^{\frac{1}{2}}(t,\omega) \circ X^*(g)||^2_{\mathbb{H}}\, d\lambda \Big]^{\frac{1}{2}}$$ which pr

<u>Theorem 6</u>

With the hypothesesof the preceding theor
of $X \rightsquigarrow \int_0^T X\, d\tilde{M}$ to the closure $\bar{\Lambda}_T^{2,M}(\mathbb{H} ; \mathbb{G})$ of $\mathcal{E}_T$
$\Lambda_T^{*\tilde{M}}(\mathbb{H} ; \mathbb{G})$ is an isometry from $\bar{\Lambda}_T^{2,M}(\mathbb{H} ; \mathbb{G})$ into $\mathcal{L}_2($

<u>Proof</u>

For $X = \sum_i 1_{]s_i,t_i]\times F_i} .u_i$ we have ind

$$|| \int_0^T X.dM^* ||^2_{\mathbb{H}.S} = \sum_i E\, || \sum_i 1_{F_i} (M^*_{t_i} - M^*_{s_i}) \circ u_i^* ($$

where (e_n) is an orthonormal basis of $\mathbb{G}$.

Using again the martingale property of M^* we get

$$|| \int_0^T X.dM^* ||^2_{H.S} = \sum_i E\, ||1_F(M^*_{t_i} - M^*_{s_i}) \circ u_i^*(e_n)||^2_{\mathbb{H}}$$

$$= \sum_i \int_{]s_i,t_i]\times F_i} \sum_n ||\tilde{Q}^{\frac{1}{2}} \circ u_i^*(e_n)||^2_{\mathbb{H}}$$

$$= \int_{]0,T]\times\Omega} |\tilde{Q}^{\frac{1}{2}} \circ X^*||^2_{H.S}\, d\lambda$$

$$= \int_{]0,T]\times\Omega} ||X \circ \tilde{Q}^{\frac{1}{2}}||^2_{H.S} \, d\lambda \, .$$

es the theorem.

$\tilde{Q}$ is nuclear this theorem gives the same result as

.

em 7

be a right continuous square integrable cylindrical separable Hilbert space $\mathbb{H}$, and let $\mathbb{G}$ be another separable X be a process in $\tilde{\Lambda}^{b,\tilde{M}}_T(\mathbb{H} \, ; \, \mathbb{G})$ (see th. 5) (resp.

hen $\left(\int_0^t X.d\tilde{M} \right)_{t \in [0,T]}$ *is a right continuous square ntegrable cylindrical martingale (resp. is the process f S.L.F on $\mathbb{G}$ associated with a uniquely defined right ontinuous martingale with values in $\mathbb{G}$).*

f $\alpha_{\tilde{M}}$ (resp. λ) is the quadratic measure (resp. the ontrol measure) of $\tilde{M}$, and if α_Y is the quadratic measure f the cylindrical martingale $\tilde{Y} = \left(\int_0^t X \, d\tilde{M} \right)_{t \in [0,T]}$, *hen*

$$, \forall \, g \in \mathbb{G}, \quad \alpha_Y(A)(g \otimes g) = \int_A d\alpha_M \circ (X^* \otimes X^*)(g \otimes g)$$

$$= \int_A Q_M \circ (X^* \otimes X^*)(g \otimes g) \, d\lambda$$

nd the natural process $<\tilde{Y}>$ of α_Y is related to the atural process $<\tilde{M}>$ of $\tilde{M}$ by

$$< \tilde{Y} >_t \; = \; \int_0^t d < \tilde{M} >_s \circ (X^*_s \otimes X^*_s) \, .$$

This last integral is defined pathwise values in $\mathbb{B}_3 = \mathcal{L}(\mathbb{G} \hat{\otimes}_1 \mathbb{G} \, ; \, L^1(\Omega,$ $s \rightsquigarrow X^*_s(\omega) \otimes X^*_s(\omega) \in \mathcal{L}(\mathbb{G} \hat{\otimes}_1 \mathbb{G} \, ; \, \mathbb{H} \hat{\otimes}_1$ respect to a vector valued measure values in $\mathcal{L}(\mathbb{H} \hat{\otimes} \mathbb{H} \, ; \, L^1(\Omega, \mathcal{F}_t, P) =$ to the bilinear mapping $f : \mathbb{B}_1 \times \mathbb{B}_2$ $f(u,m) = m \circ u$.

3°) If $\tilde{M}$ is continuous, then $\tilde{Y}$ is cont

Proof

1°) Saying that $\tilde{Y} = \left(\int_0^t X \, d\tilde{M} \right)_{t \in [0,T]}$ i square integrable cylindrical martinga that, for every $g \in \mathbb{G}$, $(\tilde{Y}_t(g))_{t \in [0,T}$ nuous real martingale.

We will prove it for a process X $0 \leq \sigma \leq T$ and $u \in \mathcal{L}(\mathbb{H} \, ; \, \mathbb{G})$, and G of the linearity and continuity of the $X \rightsquigarrow \int_0^t X.d\tilde{M}$, it will appear immedia any $X \in \tilde{\Lambda}^{b,\tilde{M}}_T(\mathbb{H} \, ; \, \mathbb{G})$.

Let us prove first that $\forall \, 0 \leq s \leq$

(IV-5-3) $\quad E \left[(\tilde{Y}_t(g) - \tilde{Y}_s(g)) \, 1_F \right] = 0$

For a particular X of the above fo

$$E \left[(\tilde{Y}_t(g) - \tilde{Y}_s(g)).1_F \right] = E \left[1_{F \cap G} \cdot \right.$$

The martingale property of $\tilde{M}$ gives the martingale property of $\tilde{Y}$. As, for

$$\tilde{Y}_t(g) = 1_{F \cap G} \left[M_{t \vee \sigma} (u^*(g)) - M_\sigma(u^*(g \right.$$

inuity of the mapping $t \rightsquigarrow \tilde{Y}_t(g)$ is clear.

linearity and density we gets immediately (IV-5-3)
the martingale property for a general $X \in \bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H};\mathbb{G})$.

us assume that a sequence $\tilde{Y}^n$ is such that, for
$(\tilde{Y}^n_t)_{n\in\mathbb{N}}$ converges to $\tilde{Y}_t$ in $\mathcal{L}(\mathbb{G}; L^2(\Omega, \mathfrak{F}_t, P))$.
ear, using the classical procedures that, for any
n deduce the right continuity in L^2 of the real
le $(\tilde{Y}_t(g))_{t\in[0,T]}$ from the right continuity of
g))$_{t\in[0,T]}$. In the particular case when
($\mathbb{H}$; $\mathbb{G}$), it follows from theorem 6 that
($\mathbb{G}$; $L^2(\Omega, \mathfrak{F}_t, P)$), and then there exists an asso-
artingale $(Y_t)_{t\in[0,T]}$, in the ordinary sense,
ues in $\mathbb{G}$. As for every g, the real martingale
>)$_{t\in[0,T]}$ is right continuous in quadratic mean,
sily deduced that (Y_t) has a version (unique up to
gability) right continuous in $\mathbb{G}$.

ing the same $X = 1_{]\sigma,T]\times G}\cdot u$ we get

$$]\times F)(g \otimes g) = E\left\{ 1_{F\cap G}\cdot\left[(\tilde{M}_t - \tilde{M}_{s\vee\sigma})\circ u^*(g)\right]^{\otimes 2}\right\}$$

$$= \alpha_{\tilde{M}}\left( ]s\vee\sigma,t]\times(F\cap G)\right)(u^*(g)\otimes u^*(g))$$

$$= \int_{]s,t]\times F} d\,\alpha_{\tilde{M}} \circ X^* \otimes X^*(g\otimes g)$$

$$= \int_{]s,t]\times F} Q_{\tilde{M}} \circ (X^*\otimes X^*)(g\otimes g)\, d\lambda \quad .$$

es the formula (IV-5-1) for a process which is a
cess on predictable rectangles, and for $A =]s,t]\times F$
mapping $X \rightsquigarrow \int_{]s,t]\times F} X\circ Q_{\tilde{M}}\circ X^*\, d\lambda$ beeing con-
from $\bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H} ; \mathbb{G})$ into $\mathcal{L}(\mathbb{G}\hat{\otimes}\mathbb{G}; L^1(\Omega, \mathfrak{F}_T, P))$
of

$$\sup_{\|g\|\leq 1} E\left\| \int_{]s,t]\times F} \langle X\circ \tilde{Q}_M \circ X^*(g), g\rangle_{\mathbb{H}}\, d\lambda \right\| \leq \sup_{\|g\|\leq 1} \int \|X \circ$$

$$= \sup_{\|g\|\leq 1} \int \|\tilde{Q}_M^{\frac{1}{2}}$$

we get formula (IV-5-1) for all $X \in \bar{\Lambda}_T^{b,\tilde{M}}(\mathbb{H} ; \mathbb{G})$ and $A =]s,t$

The formula for all predictable A's follows from
vity in A of both members of (IV-5-1).

The Pettis integrability of $X^*\otimes X^*$ with respect
that $X^*\otimes X^*$ is Pettis integrable on P-almost all paths ω wit
$<\tilde{M}>$. The cylindrical process $\Phi_t^* = \int_0^t d\,\langle\tilde{M}\rangle_s \circ (X_s^*\otimes X_s^*)$
table and satisfies

$$\alpha_Y( ]s,t]\times F)(g \otimes g) = E\,\{1_F\, (\Phi_t^* - \Phi_s^*)(g$$

which proves formula (IV-5-2).

3°) It is sufficient to prove that, for all $g \in \mathbb{G}$
continuous since $\tilde{Y}(g)$ is a martingale. If $\tilde{M}$ i
the real process $\tilde{Y}(g) = (\int X.d\tilde{M})(g)$ is cor
$X = u.1_A$ where $A\in\mathcal{P}$ and $u \in \mathcal{L}(\mathbb{H} ; \mathbb{G})$. The
is true in the general case by linearity and

<u>Remark</u>

If $\tilde{M}$ is a local cylindrical square integrable m
and if $X \in \Lambda_T^{\tilde{M}}(\mathbb{H} ; \mathbb{G})$ (resp. $X \in \Lambda_T^{\tilde{M}}(\mathbb{H} ; \mathbb{G})$), the process (
is a local cylindrical martingale (resp. there is a local squa
G-valued martingale Y associated to the local cylindrical mart
where $\tilde{Y}_t = \int_{]o,t]} X.d\tilde{M}$).

NGE OF VARIABLE. ITO'S FORMULA

an Ito's formula has been given for the stochastic inte-
o a cylindrical Brownian motion. We want only to note
is immediate consequence of known general "Ito's formu-
example in [9]. We will restrict ourselves to the conti-
ll a general "Ito's formula" as stated in [9] and [14].
with values in a reflexive Banach space $\mathbb{F}$, with continuous
riation. Let Y be a square integrable martingale with
ontinuous path. If Φ is a mapping of $\mathbb{F} \times \mathbb{G}$, into a
e continuously differentiable in the first variable, with
by $D_x \Phi(x,y) \in \mathcal{L}(\mathbb{F} ; \mathbb{K})$ bounded on any bounded set in
ntinuously differentiable in the second variable, with
by $D_y \Phi(x,y) \in \mathcal{L}(\mathbb{G} ; \mathbb{K})$ and $D^2_{y^2} \Phi(x,y) \in \mathcal{L}(\mathbb{G}\times\mathbb{G} ; \mathbb{K})$
any bounded set in $\mathbb{F} \times \mathbb{G}$, then we have the following
equality up to indistingability between two processes :

$$, Y_o) + \int_o^t D_x \Phi(V_s , Y_s) \, d V_s + \int_o^t D_y \Phi(V_s , Y_s) \, d Y_s$$

$$+ \frac{1}{2} \int_o^t D^2_{y^2} \Phi(V_s , Y_s) \, d \langle Y\rangle_s$$

d integrals beeing taken pathwise, while the second is
al).

M^* is a continuous S.I.C Martingale on $\mathbb{H}$ and $X \in \bar{\Lambda}^{2,M}_T(\mathbb{H};\mathbb{G})$,
uous martingale version of the S.I.C Martingale
sing the formulas of theorem 6 we get immediately the

$$_o) + \int_o^t D_x \Phi(V_s,Y_s) \, d V_s + \int_o^t D_y \Phi(V,Y) \circ X \circ dM^*$$

$$+ \frac{1}{2} \int_o^t d \langle M\rangle^*_s \circ (X^*_s \otimes X^*_s) \circ D^2_{y^2} \Phi^*(V_s,Y_s)$$

tead of the bilinear from $D^2_{y^2} \Phi^*(x,y)$ we consider the
us linear mapping $\widetilde{D^2_{y^2}} \Phi^*(x,y) \in \mathcal{L}(\mathbb{G} ; \mathcal{L}(\mathbb{G} ; \mathbb{K})$ the

last integral can be written

$$\frac{1}{2} \int_o^t d \, \widetilde{\langle M^*\rangle}_s \circ X^*_s \circ \widetilde{D^2_{y^2}} \Phi^*(V,Y) \circ X_s$$

In the particular case when M^* is a cylind
B^* as in [7] , the process of S.L.F $\langle M^*\rangle_s$ reduces to
linear mapping $s.\tilde{C}$ (the covariance) of $\mathcal{L}(\mathbb{H} ; \mathbb{H})$. The
ces, when moreover V = 0, to the formula of [7] :

$$\Phi(Y_t) = \Phi(Y_o) + \int_o^t D_y \Phi(Y) \circ X \circ dB^* + \frac{1}{2} \int_o^t X_s \circ D$$

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