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L_∞-CONVERGENCE OF FINITE ELEMENT APPROXIMATION

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0. Introduction

Projections such as L_2 - or Ritz-approximations are formulated in a Hilbert-space-setting, therefore error estimates are primarily available in Sobolev norms. The derivation of L_∞-estimates is a famous question. We mention the case in which the approximation space consists of trigonometric polynomials of degree n and the result of Faber (see Lorentz [1], p.96): The L_∞-norm of any such projection is (at least) of order ln n.

The situation is better if splines or finite elements are used instead of trigonometric functions, as was shown in Nitsche [1] for linear splines in one dimension and generalized by Douglas-Dupont-Wheeler [1], Wahlbin [1], and Wheeler [1]. In higher dimensions some results are known for uniform resp. rectangular meshes, see Bramble-Nitsche-Schatz, Bramble-Schatz [1], [2], Douglas-Dupont-Wheeler [1] and Strang-Fix [1].

A first L_∞-error estimate for Ritz-approximations and linear finite elements on a general mesh was given by Nitsche [2], but with a loss of convergence-rate depending on the dimension. This was improved by Ciarlet-Raviart [2], still with a loss but independent of the dimension. Natterer [1] gets in two dimensions convergence with a power $2-\varepsilon$ of the mesh-size h . Also in two dimensions but for general finite elements Scott [1] derives optimal L_∞-error estimates. His method consists in a careful analysis of the approximability of Green's function in the L_1-norm.

In this paper we show in §2 the uniform boundedness of L_2-projections in the L_∞-norm. The idea is - similar to Natterer - first to work with weighted Sobolev-norms. In order to illustrate the power of this method we derive in §3 for Ritz-approximations a corresponding boundedness result from which optimal L_∞-estimates follow. The case of linear finite elements (for Ritz-approximations to second order problems) is excluded here, in this case logarithmic factors appear.

1. Notations, finite elements

In the following $\Omega \subseteq R^N$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteq \Omega$ in the Sobolev spaces $H_k(\Omega') = W_2^k(\Omega')$ we will consider besides the usual norms also weighted semi-norms - with a positive weight-factor p -

$$(1) \qquad \|\nabla^1 v\|_{p.\Omega'} = \Big\{ \sum_{|\alpha|=1} \int\!\!\int_{\Omega'} p\ |D^\alpha v|^2\ dx \Big\}^{1/2} \quad .$$

We omit the subscript Ω' in case of $\Omega' = \Omega$. By Γ_h a subdivision of Ω into generalized simplices Δ_i is meant, i.e. Δ_i is a simplex in case Δ_i and $\partial\Omega$ have in common at most a finite number of points and otherwise one of the faces may be curved. Γ_h is called $\varkappa$-regular if for any $\Delta_i \in \Gamma_h$ there are two spheres with radii $\varkappa^{-1}h$ and $\varkappa h$ such that Δ_i contains the one and is contained in the other.

By $H_k' = H_k'(\Gamma_h)$ we denote the space of functions $v \in L_2(\Omega)$ such that the restriction of v to any $\Delta_i \in \Gamma_h$ is in $H_k(\Delta_i)$. Parallel to (1) we introduce the 'broken' semi-norms

$$(2) \qquad \|\nabla^1 v\|_p' = \Big\{ \sum_{\Delta_i \in \Gamma_h} \|\nabla^1 v\|^2_{p.\Delta_i} \Big\}^{1/2} \quad .$$

We will consider finite element spaces $S_h = S_h(\Gamma_h)$ of order m : Any $\chi \in S_h$ is in $C^0(\Omega)$ and the restriction to $\Delta_i \in \Gamma_h$ is a polynomial of degree $\leq m-1$. In case of essential boundary conditions as discussed in section 3 we think of isoparametric elements as discussed by Ciarlet-Raviart [1] and Zlamal [1] .

A family of weight-factors $\{p_\rho(x)\}$ is said to be in class $\mathfrak{C}_\gamma = \mathfrak{C}_{\gamma.m}$ if

$$(3) \qquad \sup_{x\in\Omega} p_\rho^{-1}(x) \mid D^\alpha p_\rho(x)\mid \leq \gamma \rho^{-|\alpha|} \quad \text{for} \quad |\alpha| \leq m \quad .$$

The standard approximation and inverse properties of finite elements may be transferred to estimates in weighted norms (see Natterer [1]).

<u>Lemma 1:</u> Let Γ_h be a $\varkappa$-regular subdivision and $\{p_\rho\}$ belong to class $\mathfrak{C}_\gamma$. There are constants $C_\nu = C_\nu(\gamma,\varkappa,m)$ $(\nu = 1,2,3)$ such that whenever $h \leq C_1\rho$ the statements are true:

(i) To any $v \in H_l^{'}$ with $l \leq m$ there is a $\chi \in S_h$ according to

$$(4) \qquad \|\nabla^k(v-\chi)\|_p^{'} \leq C_2 h^{l-k} \|\nabla^l v\|_p^{'} \qquad (0 \leq k < l) \quad .$$

(ii) For any $\chi \in S_h$ Bernstein-type inequalities hold:

(5) $$\|\nabla^l \chi\|_p' \le C_3 h^{k-l} \|\nabla^k l\|_p' \qquad (0 \le k < l < m) \quad .$$

With a proper approximation resp. interpolation χ it is well-known

$$\|\nabla^k (v-\chi)\|^2_{L_2(\Delta_i)} \le c_1 h^{2(l-k)} \|\nabla^l v\|^2_{L_2(\Delta_i)}$$

with a constant $c_1 = c_1(\varkappa, m)$. From this we get with

$$\underline{p}_i = \inf \{p(x) | x \in \Delta_i\} \quad ,$$

$$\overline{p}_i = \sup \{p(x) | x \in \Delta_i\}$$

immediately

(6) $$\|\nabla^k (v-\chi)\|^2_p \le (\overline{p}_i / \underline{p}_i)^2 c_1 h^{2(l-k)} \|\nabla^l v\|^2_p \quad .$$

Now there are $\underline{x}, \overline{x} \in \Delta_i$ with $\underline{p}_i = p(\underline{x})$, $\overline{p}_i = p(\overline{x})$. Since $|\overline{x}-\underline{x}| \le \varkappa h$ and $|Dp| \le \gamma \rho^{-1} \overline{p}_i$ in Δ_i we get

$$\overline{p}_i \le \underline{p}_i + \varkappa \gamma h \rho^{-1} \overline{p}_i \quad .$$

The choice $C_1 = (2\varkappa\gamma)^{-1}$ guarantees $\overline{p}_i / \underline{p}_i \le 2$ if $h \le C_1 \rho$. Summation over all $\Delta_i \in \Gamma_h$ in (6) gives (4). The proof of (5) follows the same lines.

Remark 1: In the proof condition (3) was used only with $|\alpha| = 1$.

Remark 2: If $p > 0$ fulfills (3) with $|\alpha| = 1$ then also p^{-1} does.

2. L_2-projections

Let P_h be the L_2-projection onto the finite element space S_h defined by

$$u_h = P_h u \in S_h \quad ,$$

$$(u_h, \chi) = (u, \chi) \qquad \text{for} \quad \chi \in S_h \quad .$$

We first show the boundedness of P_h with respect to weighted norms. With any $\chi \in S_h$ we have

$$\|u_h\|_p^2 = \iint pu_h^2$$

$$= \iint u_h(pu_h - \chi) - \iint u(pu_h - \chi) + \iint pu\, u_h \quad .$$

With the help of

$$\iint pu\, u_h \le \tfrac{1}{2} \iint pu^2 + \tfrac{1}{2} \iint pu_h^2$$

and

$$|\iint v\, w| \le \{\iint p\, v^2\}^{1/2} \{\iint p^{-1} w^2\}^{1/2}$$

$$\le \delta \iint pv^2 + \frac{1}{4\delta} \iint p^{-1} w^2$$

for any $\delta > 0$ we get $(\delta = \frac{1}{4})$

$$\|u_h\|_p^2 \le \tfrac{3}{4}(\|u\|_p^2 + \|u_h\|_p^2) + 2\ \|pu_h - \chi\|_{p^{-1}}^2$$

or

$$\|u_h\|_p^2 \le 3\ \|u\|_p^2 + 8\ \|pu_h - \chi\|_{p^{-1}}^2 \quad .$$

Now let χ be an approximation to pu_h according to lemma 1 with the weight-factor p^{-1} . Then

$$\|pu_h - \chi\|_{p^{-1}} \le c_2\ h^m\ \|\nabla^m (pu_h)\|'_{p^{-1}} \quad .$$

In $\Delta_1 \in \Gamma_h$ the function u_h is a polynomial of degree $\le m-1$ *). By Leibniz' rule we get

$$D^\alpha(pu_h) = \sum_{\substack{\beta\le\alpha \\ |\beta|\le m-1}} c_\beta\ (D^{\alpha-\beta}p)\ D^\beta\ u_h$$

and because of (3)

$$(7) \qquad \|\nabla^m (pu_h)\|'_{p^{-1}} \le c_2 \sum_{k=0}^{m-1} \rho^{-m+k}\ \|\nabla^k u_h\|'_p \quad .$$

Now we make use of the inverse properties (5) and come to

$$\|pu_h - \chi\|_{p^{-1}} \le c_3\ h\ \rho^{-1}\ \|u_h\|_p$$

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*) In case of isoparametric elements the m-th derivatives are linear-combinations of the lower ones, therefore (7) is valid.

with $c_3 = c_3(\gamma,\varkappa,m)$. If we choose $h \leq C_4\ \rho$ with $C_4 = \mathrm{Min}\ (C_1, \frac{1}{6}\ c_3^{-1})$ we get therefore

$$(8) \qquad \|u_h\|_p \leq 2\ \|u\|_p \quad .$$

Theorem 1: Let Γ_h be a $\varkappa$-regular subdivision and the weight-factor be of class τ_γ . For $h \leq C_4(\gamma,\varkappa,m)\ \rho$ the L_2-projection $P_h : L_2 \to S_h$ is uniformly bounded.

In order to derive L_∞-estimates for P_h we specify the functions p . Let $x_o \in \overline{\Omega}$ be such that

$$(9) \qquad u_h(x_o) = \pm\ \|u_h\|_{L_\infty(\Omega)}$$

(without loss of generality we can take the positive sign). Then we define

$$(10) \qquad p_\rho(x) = (|x-x_o|^2 + \rho^2)^{-a}$$

with any $a > N/2$, f.i. we take $a = N$. Obviously condition (3) is met with $\gamma = \gamma(m)$. Because of

$$p_\rho(x) \leq (|x-x_o|^N + \rho^N)^{-2}$$

we can estimate

$$\|u\|_p^2 \leq c_4 \|u\|_{L_\infty(\Omega)}^2 \int_0^\infty \frac{d\tau}{(\tau+\rho^N)^2}$$

or

(11) $$\|u\|_p \leq c_5\ \rho^{-N/2}\ \|u\|_{L_\infty(\Omega)} \quad .$$

Because of the standard inverse inequality

$$\|\nabla\chi\|_{L_\infty(\Omega)} \leq c_6 h^{-1} \|\chi\|_{L_\infty(\Omega)}$$

$(c_6 = c_6(\varkappa,m))$ we find using (9)

(12) $$u_h(x) \geq \|u_h\|_{L_\infty(\Omega)} \left\{1 - c_6\ h^{-1}|x-x_o|\right\} \quad .$$

The volumen of the intersection between Ω and the sphere with center in x_o and radius $c_6^{-1}\ h$ is bounded from below by $c_7\ h^N$. With the help of (12) we get therefore with $c_8 = c_8(\gamma,\varkappa,m) > 0$

$$c_8\ \rho^{-2N}\ h^N\ \|u_h\|^2_{L_\infty(\Omega)} \leq \|u_h\|^2_p \quad .$$

Comparing this and (11) with (8) we get finally

$$\|u_h\|_{L_\infty(\Omega)} \leq c_9(\varkappa,m)\ (\tfrac{\rho}{h})^{N/2}\ \|u\|_{L_\infty(\Omega)} \quad .$$

For any given h we take now $\rho = C_4^{-1}h$. Then the factor ρ/h depends only on $\varkappa,m$:

<u>Theorem 2:</u> Let Γ_h be a $\varkappa$-regular subdivision. Then the L_2-projection $P_h : L_2 \to S_h$ has uniformly bounded norm in $L_\infty(\Omega)$.

The inequality - see Alexitis [1]

$$\|u-P_h u\| \leq (1 + \|P\|) \inf_{\chi \in S_h} \|u-\chi\|$$

in connection with the approximation properties of finite elements gives

<u>Corollary 1:</u> Let $u \in W^n_\infty(\Omega)$ with $n \leq m$. Then

(13) $$\|u-P_h u\|_{L_\infty(\Omega)} \leq C_5 \, h^n \, \|u\|_{W^n_\infty(\Omega)}$$

with $C_5 = C_5(\varkappa, m)$.

3. Ritz-approximations

Now we consider the Dirichlet problem

$$(14) \qquad \begin{aligned} -\Delta u &= f \qquad && \text{in } \Omega \;, \\ u &= 0 && \text{on } \partial\Omega \;. \end{aligned}$$

Let S_h fulfill the boundary condition. Then the Ritz-approximation R_h is defined by

$$(15) \qquad \begin{aligned} \Phi &= R_h u \in S_h \;, \\ D(\Phi, \chi) &= (f, \chi) \qquad && \text{for } \chi \in S_h \end{aligned}$$

with the Dirichlet-form

$$D(\Phi, \chi) = \iint_\Omega \left\{ \sum_{i=1}^{N} \Phi_{|i} \, \chi_{|i} \right\} dx \quad .$$

We may also insert u and write

$$(16) \qquad D(\Phi, \chi) = D(u, \chi) \qquad \text{for } \chi \in S_h \quad .$$

Besides the weight-factor p_ρ (10) with $x_o \in \Omega$ to be fixed later we introduce

$$(17) \qquad q_\rho(x) = (|x - x_o|^2 + \rho^2)^{-a-1} \quad .$$

We will also write only p and q .

In analogy to theorem 1 we have in the present situation

Theorem 3: Assume

(i) Γ_h is a $\varkappa$-regular subdivision of Ω ,

(ii) S_h is of order $m \geq 3$ and contained in $\overset{\circ}{H}_1(\Omega)$,

(iii) p,q are defined by (10), (17) with $N/2 < a < N/2+1$

(iv) h and ρ are connected by $h \leq C_6\, \rho$ with $C_6 = C_6(\varkappa,m)$.

Then the Ritz-approximation $\Phi = R_h u$ is bounded in the sense

(18) $$\|\Phi\|_q + \|\nabla\Phi\|_p \leq C_7(\|u\|_q + \|\nabla u\|_p)$$

with $C_7 = C_7(\varkappa,m)$.

The proof is divided in three steps. First in estimating the gradient of Φ we use the identity

$$\|\nabla\Phi\|_p^2 = D(\Phi,p\Phi) + \frac{1}{2} \iint \Phi^2 \Delta p \quad .$$

The second term on the right hand side is bounded by $c_{10}\|\Phi\|_q^2$ because of $\Delta p \leq c_{10} q$. The first term is handled similar to the L_2-case. So we find

$$(19) \qquad \|\nabla\Phi\|_p^2 \le c_{11}(\varkappa,m)\ \left\{\|\Phi\|_q^2 + \|\nabla u\|_p^2\right\} \quad .$$

Next we introduce an auxiliary function w defined by

$$(20) \qquad \begin{aligned} -\Delta w &= q\Phi \qquad &\text{in} \quad \Omega \\ w &= 0 \qquad &\text{on} \quad \partial\Omega \end{aligned}$$

by means of which we get

$$\|\Phi\|_q^2 = (\Phi,\ -\Delta w) = D(\Phi,w) \quad .$$

Because of (16) the equation

$$\|\Phi\|_q^2 = D(\Phi,w-\chi) - D(u,w-\chi) + D(u,w)$$

holds with $\chi \in S_h$ arbitrarily chosen. The last term can be replaced by $\iint qu\Phi$. Applying Schwarz' inequality in an appropriate way we come to

$$\|\Phi\|_q^2 \le \|u\|_q^2 + \|\nabla u\|_p^2 + \delta\|\nabla\Phi\|_p^2 + (1+\delta^{-1})\ \|\nabla(w-\chi)\|_{p^{-1}}^2 \quad .$$

Now we choose $\delta < c_{11}^{-1}$ and combine the last inequality with (19):

$$(21) \quad \|\Phi\|_q^2 + \|\nabla\Phi\|_p^2 \le c_{12}\ \left\{\|u\|_q^2 + \|\nabla u\|_p^2 + \|\nabla(w-\chi)\|_{p^{-1}}^2\right\} \quad .$$

Since $m \geq 3$ we get with lemma 1

$$(22) \qquad \|\nabla(w-\chi)\|_{p^{-1}} \leq c_{13}\, h^2 \|\nabla^3 w\|_{p^{-1}} \quad .$$

Remark 3: Since $\Phi \in \overset{o}{H}_1(\Omega)$ and $q \in C^\infty(\Omega)$ we have $w \in H_3(\Omega)$.

The final step from (21) to (18) is done by the

Lemma 2: Assume $N/2 < a < N/2+1$. Let w be defined by (20). Then

$$(23) \qquad \|\nabla^3 w\|_{p^{-1}} \leq c_{14}\, \rho^{-2}(\|\Phi\|_q + \|\nabla\Phi\|_p)$$

with a numerical constant c_{14} .

The proof of the lemma is highly technical and is not given here. It only remains to couple ρ and h in order to have - see (21) - (23) - $c_{12}\, c_{13}^2\, c_{14}^2 \quad h^4 \, \rho^{-4} < 1$.

Parallel to the L_2-case we derive now from (18)

$$\|\Phi\|_{L_\infty(\Omega)} + h\|\nabla\Phi\|_{L_\infty(\Omega)} \leq C_8(\varkappa,m)\left\{\|u\|_{L_\infty(\Omega)} + h\|\nabla u\|_{L_\infty(\Omega)}\right\} \quad .$$

The final result is

Corollary 2: Assume the order m of the finite elements used is at least 3 . Let $u \in W^n_\infty(\Omega)$ with $1 \leq n \leq m$.

Then

$$\|u-R_h u\|_{L_\infty(\Omega)} \leq c_8\ h^n \|u\|_{W^n_\infty(\Omega)}$$

$$\|u-R_h u\|_{W^1_\infty(\Omega)} \leq c_8\ h^{n-1} \|u\|_{W^n_\infty(\Omega)} \quad .$$

<u>Remark 4:</u> In case $m = 2$ the best choice is $a = N/2$.

Inequality (22) is to be replaced by

$$(22') \qquad \|\nabla(w-\chi)\|_{p^{-1}} \leq c_{13}\ h \|\nabla^2 w\|_{p^{-1}}$$

and similarily (23) by

$$(23') \qquad \|\nabla^2 w\|_{p^{-1}} \leq c'_{14}\ \rho^{-1} / \ln \rho /^{1/2}\ \|\Phi\|_q \quad .$$

In this way logarithmic terms "come in" .

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