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ON NUMERICAL METHODS FOR THE STOKES PROBLEM[1]

R. GLOWINSKI and O. PIRONNEAU

1. INTRODUCTION

A great deal of work has been done already for the numerical solution of the Stokes and Navier Stokes equations. Since it is impossible to review all the papers on this subject, we shall mention only those which we feel are related to the methods developped in this Chapter. For a more complete study we send the reader to TEMAM [1] and the bibliography therein.

The following study can be roughly divided into two parts :

- In the first part we shall review briefly the Stokes and Navier-Stokes equations and some classical methods for the solution of the stationary Stokes problem. The cost of the numerical solution of the approximated problem will be our point of view.

- In the second section we shall introduce a new method for the approximation of Stokes problem ; it is based upon a new variational formulation. This approach allows the use of Lagrangian conforming elements of low order (quadratic for the velocity and linear for the pressure). The errors of approximation are shown of optimal order. Then we shall describe several methods for the solution of the approximated problem which are based upon the very peculiar structure of the problem.

The main purpose behing this study is to obtain an efficient "Stokes Solver" for an iterative solution of the Navier-Stokes equations.

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(1) This chapter follows the text of a lecture given at the VIIth GATLINBURG Meeting on Numerical Algebra and Optimization (Asilomar, California ; December 11, 1977- December 17, 1977).

2. THE STOKES AND THE NAVIER STOKES EQUATIONS

Several Sobolev spaces will be used ; for their definitions and properties we send to ADAMS [2], LIONS-MAGENES [3], NECAS [4], ODEN-REDDY [5].

Let Ω be an open set of $\mathbb{R}^N$ (N=2 or 3). Let $\Gamma = \partial\Omega$ be its boundary that we assume smooth. The non stationary flows of incompressible viscous Newtonian fluids are governed in Ω by the Navier-Stokes equations :

$$(1)\quad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{f} & \text{in } \Omega, \\ \nabla\cdot\vec{u} = 0 & \text{in } \Omega, \\ \vec{u}|_\Gamma = \vec{u}_\beta & \left(\text{with } \displaystyle\int_\Gamma \vec{u}_\beta\cdot\vec{n}\, d\Gamma = 0\right). \end{cases}$$

In (1) and in a suitable system of units :

- $\vec{u}$ is the velocity of the flow and p is the pressure (which is defined up to a constant),
- ν (>0) is the (Kinematic) viscosity,
- $\vec{n}$ is the unitary normal vector to Γ, exterior to Ω,
- $\vec{u}_\beta$ (given) is the velocity of the flow on Γ,
- $\vec{f}$ is the density of external forces,
- the condition $\nabla\cdot\vec{u} = 0$ comes from the incompressibility of the fluid.

In this Chapter we shall study the homogeneous stationary Stokes problem :

$$(2)\quad \begin{cases} -\nu\Delta\vec{u} + \nabla p = \vec{f} & \text{in } \Omega, \\ \nabla\cdot\vec{u} = 0 & \text{in } \Omega, \\ \vec{u}|_\Gamma = \vec{0}. \end{cases}$$

The following results and methods are very easy to extend to the non stationary and/or non homogeneous flows (see GLOWINSKI-PIRONNEAU [6]).

Let us recall a theorem of existence whose proof and extention to the case Ω unbounded can be found in [1], [7] :

Theorem 2.1 : If Ω is bounded (in one direction at least) and if $\vec{f} \in (H^{-1}(\Omega))^N$ then 2 has a unique solution in $(H_o^1(\Omega))^N \times (L^2(\Omega)/\mathbb{R})$.

3. REVIEW OF SOME STANDARD NUMERICAL METHODS FOR STOKES PROBLEM.

It follows from $\vec{v}|_\Gamma = 0$ that

$$\int_\Omega q \, \nabla\cdot\vec{v} \, dx = - < \nabla q, \vec{v}> \quad \forall q \in L^2(\Omega) \quad \forall \vec{v} \in (H_o^1(\Omega))^N,$$

where $<\cdot,\cdot>$ stands for the duality pairing between $(H^{-1}(\Omega))^N$ and $(H_o^1(\Omega))^N$. In other words,

$$- \nabla : L^2(\Omega) \to (H^{-1}(\Omega))^N \text{ is the adjoint operator to}$$

$$\nabla\cdot : (H_o^1(\Omega))^N \to L^2(\Omega).$$

This shows that 2 is of the form

$$3 \qquad \begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}.$$

In 3, $A \in \mathcal{L}(V,V')$, $B \in \mathcal{L}(H,H)$ where V (resp. H) is a Hilbert space whose dual is V' (resp. H' that we identify with H). Moreover A is self-adjoint and V-elliptic, i.e.,

$$<Av,v> \geq \alpha \|v\|_V^2 \quad \forall v \in V.$$

where $<\cdot,\cdot>$ is the duality pairing between V' and V.
For the Stokes problem 2 we have :

$$A = -\nu\Delta \ , \quad B = -\nabla\cdot \ , \quad B^t = \nabla$$

$$H = L^2(\Omega) \ , \quad V = (H_o^1(\Omega))^N \ , V' = (H^{-1}(\Omega))^N.$$

It is desirable that this structure be preserved when .2 is approximated by finite differences or finite elements.

Example : On a 2-D example we shall exhibit some of the properties of the linear system approximating Stokes problem.

We take $\Omega =]0,1[^2$ and 2 is discretized by finite differences. Let M be a positive integer and let $h = 1/M$. On $\bar{\Omega}$ we define the nets (see Figure 1

$$\mathcal{U}_h = \{M_{ij} | M_{ij} = \{ih, jh\} \ , \ 0 \le i,j \le M\} \ ,$$

$$\mathcal{U}_h^o = \{M_{ij} | M_{ij} \in \mathcal{U}_h \ , \ 1 \le i,j \le M-1\} = \mathcal{U}_h \cap \Omega \ ,$$

$$\mathcal{P}_h = \{M_{i+1/2,j+1/2} | M_{i+1/2,j+1/2} = \{(i+\tfrac{1}{2})h, (j+\tfrac{1}{2})h\} \ , \ 0 \le i,j \le M-1\} \ .$$

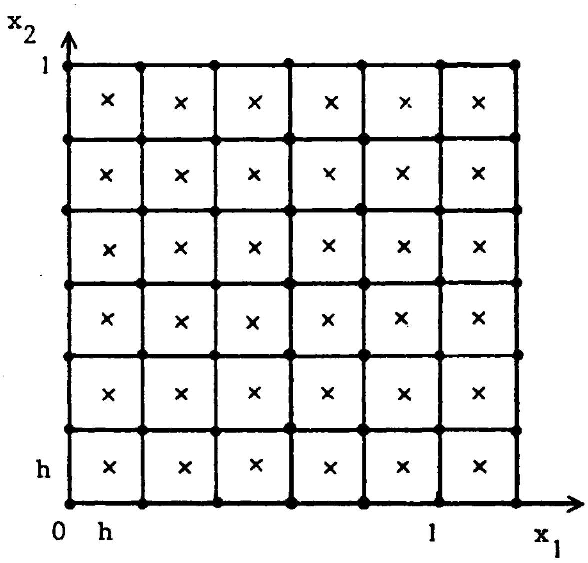

Figure 1

• Nodes of $\mathcal{U}_h$,
× Nodes of $\mathcal{P}_h$.

The velocity is approximated on the net $\mathcal{U}_h$ by the vector $\{\vec{u}_{ij}\}_{0 \le i,j \le M}$ while the pressure is approximated on the net $\mathcal{P}_h$ by

$\{p_{i+1/2,j+1/2}\}_{0 \le i \le j \le M-1}$ (do not forget that $\vec{u}_{ij} \in \mathbb{R}^2$, $\vec{u}_{ij} = \{u^1_{ij}, u^2_{ij}\})$.

Then Δ is discretized by the classical 5 point formula and $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ by centered 4 point formulae.

Therefore the approximate Stokes problem is the linear system :

$$4_1 \begin{cases} -\frac{\nu}{h^2}(u^1_{i+1j}+u^1_{i-1j}+u^1_{ij+1}+u^1_{ij-1}-4u^1_{ij})+\frac{1}{2h}(p_{i+1/2j+1/2}-p_{i-1/2j+1/2}+ \\ +p_{i+1/2j-1/2}-p_{i-1/2j-1/2}) = f^1_{ij}\ ,\ 1 \le i,j \le M-1\ , \end{cases}$$

$$4_2 \begin{cases} -\frac{\nu}{h^2}(u^2_{i+1j}+u^2_{i-1j}+u^2_{ij+1}+u^2_{ij-1}-4u^2_{ij})+\frac{1}{2h}(p_{i+1/2j+1/2}-p_{i+1/2j-1/2}+ \\ +p_{i-1/2j+1/2}-p_{i-1/2j-1/2}) = f^2_{ij}\ ,\ 1 \le i,j \le M-1, \end{cases}$$

$$.5 \begin{cases} \frac{1}{2h}(u^1_{i+1j+1}-u^1_{ij+1}+u^1_{i+1j}-u^1_{ij})+\frac{1}{2h}(u^2_{i+1j+1}-u^2_{i+1j}+u^2_{ij+1}-u^2_{ij}) = 0\ , \\ 0 \le i,j \le M-1\ . \end{cases}$$

In .4 we assume $\vec{u}_{k\ell} = 0$ if $M_{k\ell} \in \Gamma$.

Remark 3.1 : Eq. 4 (resp. 5) are derived by discretizing the first equation of 2 (resp. the second equation of .2) at the points of $\mathcal{U}_h$ (resp. $\mathcal{P}_h$).

Remark 3.2 : If $\vec{f}$ is continuous one takes $\vec{f}_{ij} = \vec{f}(M_{ij})$.

Remark 3.3 : Formulae .4 , .5 can also be obtained from a finite element discretization with rectangles and piecewise bilinear approximation for $\vec{u}$ and piecewise constant pressures. Let us mention by the way that the above method is a variant of the MAC (Markers And Cells) method developped at Los Alamos.

Some Properties of the linear system 4 and 5 - If the unknowns $\{u^1_{ij}\}$, $\{u^2_{ij}\}$, $\{p_{i+1/2j+1/2}\}$ are numbered properly and if .5 is multiplied by -1, then we obtain a linear system of type 3 with A positive, definite and symmetric. It is instructive to compare some properties of this system with the system arising from the Dirichlet problem

$$(6)\quad \begin{cases} -\Delta u = f, \\ u|_\Gamma = 0 \end{cases}$$

(see Table 1 below) .

If (6) is discretized with the 5 point formula we have

$$(7)\quad \begin{cases} -\dfrac{u_{i+1j}+u_{i-1j}+u_{ij+1}+u_{ij-1}-4u_{ij}}{h^2} = f_{ij}, \\ 1 \le i,j \le M-1 \; ; \; u_{k\ell} = 0 \text{ if } M_{k\ell} \in \Gamma. \end{cases}$$

PROBLEM	DISCRETE STOKES'	DISCRETE DIRICHLET'S
NUMBER OF UNKNOWNS	$2(N-1)^2+N^2$	$(N-1)^2$
NUMBER OF NON ZERO MATRIX ELEMENTS	2(13N-17)(N-1)	(5N-9)(N-1)
PROPERTIES OF THE MATRIX	- SPARSE - SYMMETRIC - INDEFINITE	- SPARSE - SYMMETRIC - POSITIVE DEFINITE
BANDWIDTH	BANDWIDTH STOKES >>	BANDWIDTH DIRICHLET

Table 1

By inspection of this table it appears that the numerical solution of Stokes problem may cost much more that the one of Difichlet's problem. This comparison is even worse in the 3-D case.

Orientation : It appears from the short analysis above that two directions may be pursued for the solution of Stokes problem :

(1) Use the general methods for symmetric, indefinite, linear systems.
Either the recent direct methods of DUFF-MUNKSGAARD-NIELSEN-REID [8] which seems very interesting for sparse matrices ; or use the iterative methods of Lanczos type like e.g. PAIGE-SAUNDERS [9], WIDLUND [10] (some recent tests done by THOMASSET and WIDLUND at IRIA and at the Courant Institute, demonstrate the interesting properties of Lanczos methods for the Stokes and Navier Stokes problems).

(2) Use specific methods based upon the particular structure of the problem.

In the sequel we shall focus on the second approach. In particular we whall break down Stokes' problem into a finite number of Dirichlet problems for $-\Delta$ (for which a very sophisticated methodology can be used either with finite differences or finite elements).

3.2 Gradient and Conjugate Gradient methods .

3.2.1. Generalities.

From now on Ω is bounded and Γ is regular (Lipschitz continuous).
We define $H \subset L^2(\Omega)$ by

$$H = \{q \in L^2(\Omega) \mid \int_\Omega q(x)\,dx = 0\} .$$

The iterative methods below are based upon the following result :

Theorem 3.1 : Let $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ be defined by

8 $\quad q \in L^2(\Omega),$

9 $\quad \begin{cases} \Delta\vec{v} = \nabla q \text{ in } \Omega, \\ \vec{v} \in (H_0^1(\Omega))^N \quad \text{(which implies } \vec{v}|_\Gamma = \vec{0}), \end{cases}$

10 $\quad \mathcal{A}q = \nabla\cdot\vec{v}.$

Then $\mathcal{A}$ is H-elliptic, self adjoint, automorphic from H onto H (i.e. $\exists\, \alpha > 0$ such that $(\mathcal{A}q,q)_{L^2} \geq \alpha \|q\|^2_{L^2}\ \forall q \in H$).

The proof can be found in CROUZEIX [11].

Remark 3.4 : The discrete forms of $\mathcal{A}$ are in general full matrices.

From Theorem 3.1 we shall derive a family of gradient methods (steepest descent) for the solution of Stokes problem.

3.2.2. Gradient methods and variant.

Let $\{\vec{u},p\} \in (H^1_0(\Omega))^N \times L^2(\Omega)$ be the solution of Stokes' problem 2, and let $\vec{u}_0$ be the solution of

$$(11)\quad \begin{cases} -\nu\Delta\vec{u}_0 = \vec{f} \text{ in } \Omega, \\ \vec{u}_0 \in (H^1_0(\Omega))^N. \end{cases}$$

By substracting 2 and 11 we have

$$\begin{cases} \nu\Delta(\vec{u}-\vec{u}_0) = \nabla p \text{ in } \Omega, \\ \vec{u}-\vec{u}_0 \in (H^1_0(\Omega))^N. \end{cases}$$

Hence $\mathcal{A}p = \nu\nabla\cdot(\vec{u}-\vec{u}_0) = -\nu\nabla\cdot\vec{u}_0$. In other words the pressure is the unique solution in $L^2(\Omega)/\mathbb{R}$ of

$$(12)\quad \mathcal{A}p = -\nu\nabla\cdot\vec{u}_0.$$

Owing to the properties of $\mathcal{A}$ (see Theorem 3.-1) it is natural to solve 12 (and therefore 2) by iterative methods such as the method of steepest descent.

Gradient method with fixed step size : For a given $\rho > 0$ consider the following algorithm :

$$(13)\quad p^0 \in L^2(\Omega) \text{ given arbitrarily,}$$

for $n \geq 0$, p^n given compute ,

14 $p^{n+1} = p^n - \rho(\mathcal{A}\, p^n + \nu\nabla\cdot\vec{u}_o)$.

In practice one has to replace (13.14) by

14.1 $$\begin{cases} -\nu\Delta\vec{u}^n = \vec{f} - \nabla p^n \text{ in } \Omega, \\ \vec{u}^n \in (H_o^1(\Omega))^N, \end{cases}$$

14.2 $p^{n+1} = p^n - \rho\nu\nabla\cdot\vec{u}^n$.

Remark 3.5 : To solve 14.1 one has to solve N independent Dirichlet problems for $-\Delta$ (in practice N=2 or 3).

Remark 3.6 : The previous method is close to the artificial compressibility methods of CHORIN and YANENKO.

We recall the following result :

Theorem 3.2 : If in 13, 14 we have

15 $0 < \rho < \frac{2}{N}$,

then $\forall p^o \in L^2(\Omega)$ we have

16 $$\lim_{n\to+\infty} \{\vec{u}^n, p^n\} = \{\vec{u}, p\} \text{ in } (H_o^1(\Omega))^N \times L^2(\Omega) \text{ , strongly}$$

where $\{\vec{u},p\}$ is the solution of Stokes' problem 2 with $\int_\Omega p\, dx = \int_\Omega p^o\, dx$. Moreover the rate of convergence is linear.

We remind the reader that the $(H_o^1(\Omega))^N$-norm is

$$||\vec{v}|| = \left(\int_\Omega |\nabla\vec{v}|^2 dx\right)^{\frac{1}{2}} = \sum_{i=1}^{N} \left(\int_\Omega |\nabla v_i|^2 dx\right)^{\frac{1}{2}} .$$

Variants of 13 14

One can find in [11] variants of 13 , 14 where a sequence of parameters $\{\rho_n\}_{n\geq 0}$ (cyclic in particular) is used instead of a fixed ρ. Accelerating methods of Tchebycheff type can also be found in [11] for 13 , 14 .

Steepest descent and minimal residual procedures for 13 14 can also be found in FORTIN-GLOWINSKI [12] and FORTIN-THOMASSET [13]. Each of these methods requires N uncoupled Dirichlet problems for $-\Delta$ to be solved at each iteration . However these variants of 13 , 14 seem less efficient than the conjugate gradient method of Sec. 3.2.3 which, by the way, is only slightly costlier to implement.

3.2.3. A conjugate gradient method.

It follows from DANIEL [14] that one may solve 2 via 12 by a conjugate gradient method. Sending back to [12], [13] for more details, we shall limit ourselves to the description of the algorithm. For the sake of clarity, but without loss of generality we set $\nu=1$. Then the conjugate gradient algorithm is as follows :

17 $\quad p^o \in L^2(\Omega)$, given arbitrarily,

18 $$\begin{cases} -\Delta\vec{u}^o = \vec{f}-\nabla p^o, \\ \vec{u}^o \in (H_o^1(\Omega))^N, \end{cases}$$

19 $$g^o = \nabla\cdot\vec{u}^o,$$

20 $$z^o = g^o,$$

then for $n \geq 0$,

21 $$\rho_n = \frac{(z^n,g^n)_{L^2(\Omega)}}{(\mathcal{A}\, z^n,z^n)_{L^2(\Omega)}} = \frac{\|g^n\|^2_{L^2(\Omega)}}{(\mathcal{A}z^n,z^n)_{L^2(\Omega)}},$$

(22) $$p^{n+1} = p^n - \rho_n z^n ,$$

(23) $$g^{n+1} = g^n - \rho_n \mathcal{A} z^n ,$$

(24) $$\gamma_n = \frac{\|g^{n+1}\|^2_{L^2(\Omega)}}{\|g^n\|^2_{L^2(\Omega)}} ,$$

(25) $$z^{n+1} = g^{n+1} + \gamma_n z^n ,$$

then n = n+1 and go to (21).

To implement (17)-(25) it is necessary to know $\mathcal{A} z^n$.
From Theorem 3.1, $\mathcal{A} z^n$ can be obtained by

(26) $$\begin{cases} \Delta \vec{\chi}^n = \nabla z^n, \\ \vec{\chi}^n \in (H^1_o(\Omega))^N, \end{cases}$$

(27) $$\mathcal{A} z^n = \nabla \cdot \vec{\chi}^n .$$

Thus each iteration costs N uncoupled Dirichlet problem for $-\Delta$. The strong convergence of p^n to p can be shown as in Theorem 3.2.

Remark 3.7 : Owing to the H-ellipticity of $\mathcal{A}$ it is not necessary to precondition (i.e. to scale) the conjugate gradient algorithm above.

3.3. Penalty-duality methods

It is shown in [12], [13] for example (see also [1]) that Stokes problem can be solved by a penalty-duality method (in the sense of HESTENES [15], POWELL [16]).

Therefore let $r > 0$. We note that Stokes' problem (2) is equivalent to

(28) $$\begin{cases} -\Delta \vec{u} - r\nabla(\nabla \cdot \vec{u}) + \nabla p = \vec{f} \text{ in } \Omega, \\ \nabla \cdot \vec{u} = 0 \text{ in } \Omega, \\ \vec{u}|_\Gamma = 0. \end{cases}$$

It is then natural to generalise algorithm (3.13), (3.14) by

(3.29) $p^o \in L^2(\Omega)$ arbitrarily given,

and for $n \geq 0$, p^n being known :

(3.30)
$$\begin{cases} -\Delta\vec{u}^n - r\nabla(\nabla\cdot\vec{u}^n) = \vec{f}-\nabla p^n \text{ in } \Omega, \\ \vec{u}^n \in (H_o^1(\Omega))^N \quad (\Longrightarrow \vec{u}^n|_\Gamma = \vec{0}) , \end{cases}$$

(3.31) $p^{n+1} = p^n - \rho\nabla\cdot\vec{u}^n \,, \rho > 0.$

For the convergence of (3.29)-(3.31) one shows the following

<u>Theorem 3.3</u> : <u>If in</u> (3.29)-(3.31), ρ <u>satisfies</u>

(3.32) $0 < \rho < 2(r + \frac{1}{N})$,

<u>then</u> $\forall p^o \in L^2(\Omega)$ <u>one has</u>

(3.33) $\lim_{n\to\infty} \{\vec{u}^n,p^n\} = \{\vec{u},p\}$ <u>in</u> $(H_o^1(\Omega))^N \times L^2(\Omega)$ <u>strongly</u>

<u>where</u> $\{\vec{u},p\}$ <u>is the solution of the Stokes problem</u> (2.2) <u>with</u> $\int_\Omega p\,dx = \int p^o\,dx$. <u>Moreover the convergence is linear.</u> ■

The above results can be made more precise by observing that

$$p^{n+1}-p = (I-\rho(rI+\mathcal{A}^{-1})^{-1}(p^n-p)$$

(where $\mathcal{A}$ is as in Theorem 3.1). Each operator being in $\mathcal{L}(L^2(\Omega),L^2(\Omega))$ we have

(3.34) $\|p^{n+1}-p\|_{L^2(\Omega)} \leq \|I-\rho(rI+\mathcal{A}^{-1})^{-1}\| \; \|p^n-p\|_{L^2(\Omega)}$.

And

$$I-\rho(rI+\mathcal{A}^{-1})^{-1} = (rI+\mathcal{A}^{-1})^{-1}((r-\rho)I+\mathcal{A}^{-1})$$

yields

$$\|I-\rho(rI+\mathcal{A}^{-1})^{-1}\| \le \frac{1}{r}(|r-\rho|+\|\mathcal{A}^{-1}\|).$$

It follows from 34, 35 that for the classical choice (see [12]) $\rho=r$, we have

$$(36)\quad \|p^{n+1}-p\|_{L^2} \le \frac{\|\mathcal{A}^{-1}\|}{r}\|p^n-p\|_{L^2}.$$

Therefore if r is large enough the convergence ratio of algorithm 29)-31 is of order $\frac{1}{r}$.

Remarks on algorithm 29)-31) :

Remark 3.8 : The system 30) is closely related to the linear elasticity system. Once it is discretized by finite differences or finite elements, it can be solved using a Cholesky's factorization LL^t or LDL^t, done once and for all (this remark holds also for the algorithms of Sec. 3.2 above).

Remark 3.9 : The method of 29 - 31 has the drawback of requiring the solution of a system of N partial differential equations coupled (if $r>0$) by $r\nabla(\nabla\cdot)$, while this is not so for algorithms of Sec. 3.2. Hence much more computer storage is required.

Remark 3.10 : By inspection of 3.6) it seems that one should take $\rho=r$, and r as large as possible. However 30 and its discrete forms will be ill-conditioned when r is large. In practice if 36 is solved by a direct method (Gauss, Cholesky) one should take r in the range of 10^2 to 10^5. In such cases and if $\rho=r$ the convergence of 29), 31 is extremely fast (about 3 iterations). Under such conditions it is not necessary to use a conjugate gradient accelerating scheme.

Remark 3.11 : In fact, 29 - 31 is a UZAWA algorithm (see for example [12], GLOWINSKI-LIONS-TREMOLIERES [17, Ch. 2]) applied to the computation of the saddle-points of the augmented Lagrangian $\mathcal{L}_r : (H_0^1(\Omega))^N \times L^2(\Omega) \to \mathbb{R}$ defined by

37 $$\mathcal{L}_r(\vec{v},q) = \frac{1}{2}\int_\Omega |\nabla\vec{v}|^2\, dx + \frac{r}{2}\int_\Omega (\nabla\cdot\vec{v})^2 dx - \int_\Omega f\cdot\vec{v}\, dx - \int_\Omega q\nabla\cdot\vec{v}\, dx .$$

This remark holds also for algorithms of Sec. 3.2 with $r=0$ in 37. Formula 37 is directly related to the fact that the pressure p is a Lagrange multiplier to the condition of incompressibility $\nabla\cdot\vec{v} = 0$ in the equivalent formulation of Stokes problem :

38 $$\begin{cases} \min\limits_{\vec{v}\in V} \{ \frac{1}{2}\int_\Omega |\nabla\vec{v}|^2\, dx - \int_\Omega \vec{f}\cdot\vec{v}\, dx\} , \\ V = \{v \in (H_0^1(\Omega))^N \ ;\ \nabla\cdot\vec{v} = 0\} . \end{cases}$$

4. ON A NEW METHOD FOR THE SOLUTION OF STOKES PROBLEM

In this section we shall describe a new class of methods, due to GLOWINSKI-PIRONNEAU [18], [19], for the numerical solution of the Stokes problem. Unlike the previous methods, the trace of the pressure on $\partial\Omega$ will play an important role. It leads also to the construction of a Stokes solver easy to implement, once in possession of a subroutine for the numerical solution of the Dirichlet problem for $-\Delta$. This method is closely related to the ideas used by the authors in [20] for the biharmonic equation.

4.1. The continuous case : motivation.

As before Ω is bounded and $\nu=1$. Let

$$\mathcal{H}^{1/2}(\Gamma) = \{\mu \in H^{1/2}(\Gamma),\ \int_\Gamma \mu d\Gamma = 0\} .$$

The methods below are based on the following result :

Theorem 4.1 : Let $\lambda \in H^{-1/2}(\Gamma)$; let $A : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ be defined by

39 $$\begin{cases} \Delta p_\lambda = 0 \ \text{in}\ \Omega, \\ p_\lambda \in H(\Omega;\Delta) = \{q \mid q \in L^2(\Omega),\ \Delta q \in L^2(\Omega)\} , \\ p_\lambda = \lambda \ \text{on}\ \Gamma, \end{cases}$$

$$\text{(40)}\quad \begin{cases} \Delta\vec{u}_\lambda = \nabla p_\lambda \ \text{in}\ \Omega, \\ u_\lambda \in (H_o^1(\Omega))^N, \end{cases}$$

$$\text{(41)}\quad \begin{cases} -\Delta\psi_\lambda = \nabla\cdot u_\lambda \ \text{in}\ \Omega, \\ \psi_\lambda \in H_o^1(\Omega)\ , \end{cases}$$

$$\text{(42)}\quad A\lambda = -\frac{\partial\psi_\lambda}{\partial n}\Big|_\Gamma \ .$$

Then A *is an isomorphism from* $H^{-1/2}(\Gamma)/R$ *onto* $H^{1/2}(\Gamma)$. *Moreover the bilinear form* $a(\cdot,\cdot)$ *defined by*

$$a(\lambda,\mu) = \langle A\lambda,\mu\rangle \ ,$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H^{1/2}(\Gamma)$ *and* $H^{-1/2}(\Gamma)$, *is continuous, symmetric and* $H^{-1/2}(\Gamma)/\mathbb{R}$-*elliptic*.

The reader is sent to [21] for the proof.

Application of Theorem 4.1 to the solution of Stokes problem

Assume that $\vec{f} \in (L^2(\Omega))^N$, and define $p_o,\vec{u}_o,\psi_o$ by

$$\text{(43)}\quad \begin{cases} \Delta p_o = \nabla\cdot\vec{f} \ \text{in}\ \Omega, \\ p_o \in H_o^1(\Omega), \end{cases}$$

$$\text{(44)}\quad \begin{cases} -\Delta\vec{u}_o = \vec{f} - \nabla p_o \ \text{in}\ \Omega, \\ \vec{u}_o \in (H_o^1(\Omega))^N, \end{cases}$$

$$\text{(45)}\quad \begin{cases} -\Delta\psi_o = \nabla\cdot\vec{u}_o \ \text{in}\ \Omega, \\ \psi_o \in H_o^1(\Omega). \end{cases}$$

The following is easy to prove :

Theorem 4.2 : If $\{\vec{u},p\}$ *is the solution of Stokes' problem* (2), *then the trace* λ *of* p *on* Γ *is the unique solution of the linear variational equation* :

$$\text{(E)}\quad \begin{cases} \lambda \in H^{-1/2}(\Gamma)/\mathbb{R}, \\ \langle A\lambda,\mu\rangle = \langle \frac{\partial\psi_o}{\partial n},\mu\rangle \quad \forall \mu \in H^{-1/2}(\Gamma)/\mathbb{R} . \blacksquare \end{cases}$$

Theorem 4.2 implies that Stokes' problem 2 can be broken down to a <u>finite number</u> of Dirichlet problems for $-\Delta$ (N+2 for ψ_o, N+1 for $\{\vec{u},p\}$ once λ is known) plus the problem (E) on $\partial\Omega$; the main difficulty being that A <u>is not known explicitly</u>.

<u>Remark 4.1</u> : If μ is sufficiently regular, Green's formula yields

$$46 \quad \langle \frac{\partial\psi_o}{\partial n},\mu\rangle = \int_\Omega \nabla\psi_o\cdot\nabla\tilde{\mu}\, dx - \int_\Omega \nabla\cdot\vec{u}_o\tilde{\mu}\, dx = \int_\Omega (\nabla\psi_o+\vec{u}_o)\cdot\nabla\tilde{\mu}\, dx$$

where $\tilde{\mu}$ is a regular extention of μ in Ω. Note that in 46 $\frac{\partial\psi_o}{\partial n}$ does not appear explicitly. We shall use this remark to approximate (E). ■

To approximate (E) will require to introduce a new variational formulation of Stokes' problem,discretized in turn by mixed finite elements (see Sec. 4.3).

4.2 <u>A new variational formulation of Stokes' problem</u>

Let

$$W_o = \{\{\vec{v},\phi\} \in (H_o^1(\Omega))^{N+1} , \int_\Omega \nabla\phi\cdot\nabla w\, dx = \int_\Omega \nabla\cdot\vec{v}\, w\, dx \quad \forall w \in H^1(\Omega)\}.$$

<u>Proposition 4.1</u> : <u>If</u> $\{\vec{u},\phi\} \in W_o$ <u>then</u> $-\Delta\phi = \nabla\cdot\vec{v}$ <u>in</u> Ω <u>and</u> $\phi = \frac{\partial\phi}{\partial n} = 0$ <u>on</u> Γ.

As above for the sake of clarity we assume that $\vec{f} \in (L^2(\Omega))^N$. Consider the following problem

$$\text{(P)}\quad \begin{cases} \underline{\text{Find}}\ \{\vec{u},\psi\} \in W_o\ \underline{\text{such that}} \\ \int_\Omega \nabla\vec{u}\cdot\nabla\vec{v}\, dx = \int_\Omega \vec{f}\cdot(\vec{v}+\nabla\phi)dx \quad \forall \{\vec{v},\phi\} \in W_o. \end{cases}$$

Then we have

<u>Theorem 4.3</u> : (P) <u>has a unique solution</u> $\{\vec{u},\psi\}$ <u>where</u> $\psi=0$ <u>and</u> $\vec{u}$ <u>is the solution of the Stokes problem</u> 2 .

Remark 4.3 : The formulation (P) can be interpreted as follows : if $\vec{v} \in (H_o^1(\Omega))^N$ and $\partial\Omega$ is sufficiently smooth, there exists $\phi \in H^2(\Omega) \cap H_o^1(\Omega)$ and $\vec{\omega} \in (H^1(\Omega))^N$ with $\nabla\cdot\vec{\omega} = 0$, such that

(4.47) $$\vec{v} = -\nabla\phi + \vec{\omega},$$

and the decomposition (4.47) is unique.

In the formulation (P), instead of directly imposing $\nabla\cdot\vec{v} = 0$, we try to impose $\phi=0$; these precedures are equivalent in the continuous case but not in the discrete case.

4.3 A mixed finite element approximation

In this section we proceed to define a mixed finite element approximation to the Stokes problem. We limit ourselves to the case where Ω is polygonal and bounded in $\mathbb{R}^2$, but the following extends to $\Omega \subset \mathbb{R}^3$ (see [22] for computational results).

4.3.1. Triangulation of Ω. Fundamental discrete spaces.

Let $\{\mathcal{T}_h\}_h$ be a family of regular triangulations of Ω such that $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$. We set $h(T)$ = length of the greatest side of T, $h = \max_{T \in \mathcal{T}_h} h(T)$ and we assume that

(4.48) $$\frac{h}{\min_{T \in \mathcal{T}_h} h(T)} \leq \beta \quad \forall \mathcal{T}_h .$$

Then we define the following finite dimensional spaces :

(4.49) $$\begin{cases} H_h^1 = \{\phi_h \in C^o(\overline{\Omega}) \ , \ \phi_h|_T \in P_1 \quad \forall T \in \mathcal{T}_h\}, \\ H_{oh}^1 = H_h^1 \cap H_o^1(\Omega) = \{\phi_h \in H_h^1, \ \phi_h|_{\partial\Omega} = 0\}, \end{cases}$$

(4.50) $$\begin{cases} V_h = \{\vec{v}_h \in (C^o(\overline{\Omega}))^2 \ , \ \vec{v}_h|_T \in (P_2)^2 \ \forall T \in \mathcal{T}_h\}, \\ V_{oh} = V_h \cap (H_o^1(\Omega))^2. \end{cases}$$

We will also consider V_h defined by

50 bis $$V_h = \{\vec{v}_h \in C^o(\bar{\Omega})^2 \ , \ \vec{v}_h|_T \in (P_1)^2 \ \forall T \in \tilde{\mathcal{T}}_h\}$$

where $\tilde{\mathcal{T}}_h$ is the triangulation deduced from $\mathcal{T}_h$ by dividing each triangle $T \in \mathcal{T}_h$ into 4 equal triangles (by joining the mid-sides). We record that P_k denote the space of polynomial of degree $\leq k$. Finally we define

$$\overset{\circ}{W}_{oh} = \{\{\vec{v}_h,\phi_h\} \in V_{oh} \times H^1_{oh} \ , \ \int_\Omega \nabla\phi_h \cdot \nabla w_h \ dx = \int_\Omega \nabla\cdot\vec{v}_h w_h \ dx \ \ \forall w_h \in H^1_h\}.$$

4.3.2. Definition of the approximate problem; characterization of the approximate solution

We approximate (P) (i.e. the Stokes problem) by

(P_h) $\begin{cases} \underline{\text{Find}} \ \{\vec{u}_h,\psi_h\} \in W_{oh} \ \underline{\text{such that}} \\ \int_\Omega \nabla\vec{u}_h \cdot \nabla\vec{v}_h \ dx = \int_\Omega f\cdot(\vec{v}_h + \nabla\phi_h) dx \ \ \forall \{\vec{v}_h,\phi_h\} \in W_{oh}. \end{cases}$

Then the following is shown in [6] :

Theorem 4.4 : (P_h) has a unique solution and it satisfies

51 $$\int_\Omega \nabla p_h \cdot \nabla w_h \ dx = \int_\Omega \vec{f}\cdot\nabla w_h \ dx \ \ \forall w_h \in H^1_{oh} \ , \ p_h \in H^1_h \ ,$$

52 $$\int_\Omega \nabla\vec{u}_h \cdot \nabla\vec{v}_h \ dx = \int_\Omega (-\nabla p_h + \vec{f})\cdot\vec{v}_h \, dx \ \forall \vec{v}_h \in V_{oh} \ , \ \vec{u}_h \in V_{oh} \ ,$$

53 $$\{\vec{u}_h,\psi_h\} \in W_{oh}.$$

Remark 4.4 : The discrete pressure p_h is the Lagrange multiplier of condition 53.

Remark 4.5 : If in W_{oh} and (P_h) we impose $\phi_h = \psi_h = 0$ (which may be, since $\psi=0$), then the scheme is identical to the one in TAYLOR-HOOD [23] for the Stokes problem whose convergence was established by BERCOVIER-PIRONNEAU [24].

4.3.3. Error estimates.

In the sequel C will denote various constants. The following lemma, proved in [6], [24], plays a fundamental part.

Lemma 4.1 : It is assumed that no $T \in \mathcal{T}_h$ has two sides or more belonging to $\partial\Omega$. Then, provided that (4.48) - (4.50) (resp. (4.50)$_{bis}$) holds, there exists C independant of h such that

$$(4.54)\qquad \|\nabla q_h\|_{L^2(\Omega)} \leq C \max_{\vec{v}_h \in V_{oh} - \{\vec{0}\}} \frac{\int_\Omega \vec{v}_h \cdot \nabla q_h \, dx}{\|\vec{v}_h\|_{L^2(\Omega)}} \quad \forall q_h \in H_h^1 . \blacksquare$$

It is easy to show that (4.54) implies the uniqueness of p_h in H_h^1.
From Lemma 4.1 and following THOMAS [25], one can show the following :

Theorem 4.5 : Assume that (4.48), (4.50), (4.54) hold and that Ω is a convex polygonal. Then if $\{\vec{u},p\}$, solution of Stokes' problem, belongs to $(H^3(\Omega))^2 \times H^2(\Omega)$:

$$(4.55)\qquad \|\vec{u}_h - \vec{u}\|_{(H^1(\Omega))^2} \leq C\, h^2 (\|\vec{u}\|_{(H^3(\Omega))^2} + \|p\|_{H^2(\Omega)/\mathbb{R}}) ,$$

$$(4.56)\qquad \|p_h - p\|_{H^1(\Omega)/\mathbb{R}} \leq Ch(\|\vec{u}\|_{(H^3(\Omega))^2} + \|p\|_{H^2(\Omega)/\mathbb{R}}).$$

Remark 4.6 : If we use V_h defined by (4.50)$_{bis}$ and if $\{\vec{u},p\} \in (H^2(\Omega))^2 \times H^1(\Omega)/\mathbb{R}$ then,

$$(4.57)\qquad \|\vec{u}_h - \vec{u}\|_{(H^1(\Omega))^2} \leq Ch(\|\vec{u}\|_{(H^2(\Omega))^2} + \|p\|_{H^1(\Omega)/\mathbb{R}}) .$$

Remark 4.7 : The above error estimates have an optimal order.

4.3.4. Comments.

The above methods, based on Lagrangian finite triangular elements, conforming in $H^1(\Omega)$, are easier to implement that the non conforming methods (cf. [1], [26], [27]). They generalize naturally to the 3-D case, to quadrilateral elements as well as curved boundaries (with curved elements (see ZIENKIEWICZ [28]) isoparametric for the velocity, superparametric for the pressure).

Finally let us mention that LE TALLEC [29] has extended the error estimate theorems to the stationary Navier-Stokes equations.

4.4. Approximation of Problem (E).

We shall now use the finite elements of Sec. 4.3. to approximate (E) defined in Sec. 4.1.

4.4.1. The space $\mathcal{M}_h$. Approximation of $a(\cdot,\cdot)$.

Let $\mathcal{M}_h$ be a complementary space of H^1_{oh} in H^1_h ; i.e. $H^1_h = \mathcal{M}_h \oplus H^1_{oh}$.
In practice $\mathcal{M}_h$ is defined by

58 $$\begin{cases} \mathcal{M}_h \oplus H^1_{oh} = H^1_h \, , \\ \phi_h \in \mathcal{M}_h \Longrightarrow \phi_h|_T = 0 \quad \forall T \in \mathcal{T}_h \text{ such that } \partial T \cap \partial\Omega = \emptyset \, . \end{cases}$$

Let $N_h = \dim \mathcal{M}_h$; if H^1_h , H^1_{oh} are defined by 49, then N_h equals the number of nodes of $\mathcal{T}_h$ which belong to $\partial\Omega$. Notice that if $\phi_h \in \mathcal{M}_h$, $\text{supp}(\phi_h) \subset \bar{\Omega}_{\gamma_h} = \bigcup_{T \cap \partial\Omega \neq \emptyset} T$ and that, $\lim_{h\to 0} \text{meas}\,(\bar{\Omega}_{\gamma_h}) = 0$.

Approximation of $a(\cdot,\cdot)$

With the notation of Section 4.1, if μ is sufficiently regular, Green's formula yields

59 $$\begin{cases} a(\lambda,\mu) = -\displaystyle\int_\Gamma \frac{\partial\psi_\lambda}{\partial n}\,\mu d\Gamma = -\int_\Omega \nabla\psi_\lambda\cdot\tilde{\mu}\,dx - \int_\Omega \Delta\psi_\lambda\tilde{\mu}dx \\ \quad = -\displaystyle\int_\Omega \nabla\psi_\lambda\cdot\nabla\tilde{\mu}dx + \int_\Omega \nabla\cdot\vec{u}_\lambda\,\tilde{\mu}\,dx = -\int_\Omega (\nabla\psi_\lambda+\vec{u}_\lambda)\cdot\nabla\tilde{\mu}\,dx \, , \end{cases}$$

where $\tilde{\mu}$ is a regular extention of μ in Ω. Now let $\lambda_h,\mu_h \in \mathcal{M}_h$ and define $a_h(\cdot,\cdot) : \mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$ by

60 $$\begin{cases} \displaystyle\int_\Omega \nabla p_h\cdot\nabla q_h\,dx = 0 \quad \forall q_h \in H^1_{oh} \, , \\ p_h - \lambda_h \in H^1_{oh} \, , \end{cases}$$

$$(61)\quad \begin{cases} \int_\Omega \nabla\vec{u}_h \cdot \nabla\vec{v}_h \, dx = - \int_\Omega \nabla p_h \cdot \vec{v}_h \, dx \quad \forall \vec{v}_h \in V_{oh} , \\ \vec{u}_h \in V_{oh} , \end{cases}$$

$$(62)\quad \begin{cases} \int_\Omega \nabla\psi_h \cdot \nabla\phi_h \, dx = \int_\Omega \nabla\cdot\vec{u}_h \, \phi_h \, dx \quad \forall \phi_h \in H^1_{oh} , \\ \psi_h \in H^1_{oh} , \end{cases}$$

$$(63)\quad a_h(\lambda_h,\mu_h) = - \int_\Omega (\nabla\psi_h + \vec{u}_h)\cdot\nabla\mu_h \, dx .$$

Then the following holds (see [21]) :

Lemma 4.2 : If 54 holds, the bilinear form $a_h(\cdot,\cdot)$ is symmetric, positive definite on $(\mathcal{M}_h/\mathbb{R}_h)^2$ where $\mathbb{R}_h = \{\mu_h \in \mathcal{M}_h,\ \mu_h = \text{constant on } \partial\Omega\}$.

4.4.2. Transformation of (P_h) into a variational problem in $\mathcal{M}_h$.

In 51 - 53 of Section 4.3, an approximate pressure p_h was found unique in $H^1_h/\mathbb{R}$ once 54 holds. Therefore we can now state the discrete analogue of Theorem 4.2 (see [21]) :

Theorem 4.6 : Let p_h be the discrete pressure. If 54 holds the component λ_h of p_h in $\mathcal{M}_h$ is the unique solution of

$$(E_h)\quad \begin{cases} \lambda_h \in \mathcal{M}_h/\mathbb{R}_h , \\ a_h(\lambda_h,\mu_h) = \int_\Omega (\nabla\psi_{oh} + \vec{u}_{oh})\cdot\nabla\mu_h \, dx \quad \forall\mu_h \in \mathcal{M}_h/\mathbb{R}_h \end{cases}$$

where p_{oh}, $\vec{u}_{oh}$, ψ_{oh} are respectively the solutions of

$$(64)\quad \int_\Omega \nabla p_{oh}\cdot\nabla q_h \, dx = \int_\Omega \vec{f}\cdot\nabla q_h \, dx \ \forall q_h \in H^1_{oh} , \ p_{oh} \in H^1_{oh} ,$$

$$(65)\quad \begin{cases} \int_\Omega \nabla\vec{u}_{oh}\cdot\nabla\vec{v}_h \, dx = \int_\Omega (\vec{f} - \nabla p_{oh})\cdot\vec{v}_h \, dx \quad \forall\vec{v}_h \in V_{oh}, \\ \vec{u}_{oh} \in V_{oh} , \end{cases}$$

$$(66)\quad \int_\Omega \nabla\psi_{oh}\cdot\nabla\phi_h dx = \int_\Omega \nabla\cdot\vec{u}_{oh} \, \phi_h \, dx \quad \forall\phi_h \in H^1_{oh} , \ \psi_{oh} \in H^1_{oh} . \ \blacksquare$$

Remark 4.8 : The reader will recognize that (13.64)-(13.66) are the discrete analogue of (43)-(45).

Remark 4.9 : To compute the right hand side of (E_h) it is necessary to solve the 4 (5 if $\Omega \subset \mathbb{R}^3$) approximate Dirichlet problems (64), (65). Similarly if λ_h is known, to compute the approximate solution $\{\vec{u}_h, p_h\}$ of the Stokes problem (2) it is necessary to solve

$$(67) \quad \begin{cases} \int_\Omega \nabla p_h \cdot \nabla q_h \, dx = \int_\Omega \vec{f} \cdot \nabla q_h \, dx \quad \forall q_h \in H^1_{oh} , \\ p_h - \lambda_h \in H^1_{oh} , \end{cases}$$

and (52) ; i.e. 3 approximate Dirichlet problems (4 in $\mathbb{R}^3$).

Remark 4.10 : On account of the choice (58) for the space $\mathcal{M}_h$, the integrals in the definition of $a_h(\cdot,\cdot)$ (see (63)) and of the right hand side of (E_h), involve functions whose supports are in the neighborhood of $\partial\Omega$ only.

4.5 . Solution of (E_h) by a direct method.

4.5.1. Construction of the linear system equivalent to (E_h).

Generalities : As before $\mathcal{M}_h$ is defined by (58) ; let $\mathcal{B}_h = \{w_i\}_{i=1}^{N_h}$ be a basis of $\mathcal{M}_h$. Then $\forall \mu_h \in \mathcal{M}_h$

$$(68) \quad \mu_h = \sum_{i=1}^{N_h} \mu_i w_i ,$$

and from now on we shall write

$$(69) \quad r_h \mu_h = \{\mu_1, \ldots, \mu_{N_h}\} \in \mathbb{R}^{N_h} .$$

In practice $\mathcal{B}_h$ is defined by

$$(70) \quad \mathcal{B}_h = \{w_i\}_{i=1}^{N_h}$$

and (see Figure 2)

$$(71)\quad \begin{cases} \forall i=1,\ldots,N_h \\ w_i(P_i) = 1, \\ w_i(Q) = 0 \quad \forall Q \text{ vertex of } \mathcal{C}_h,\ Q \neq P_i\ , \end{cases}$$

where we assumed implicitly (but in practice it is not necessary) that the boundary nodes are numbered first.
With this choice for $\mathcal{B}_h$, $\mu_i = \mu_h(P_i)$ in 68 .

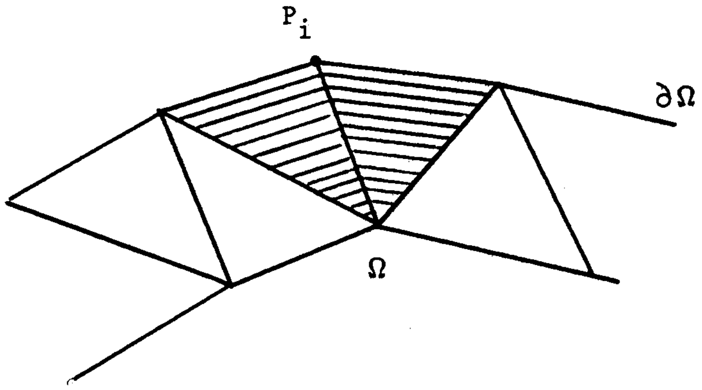

Figure 1

(The support of w_i is shown).

Then problem (E_h) is equivalent to the linear system

$$(72)\quad \begin{cases} \displaystyle\sum_{i=1}^{N_h} a_h(w_j,w_i)\lambda_j = \int_\Omega (\nabla\psi_{oh}+\vec{u}_{oh})\cdot\nabla w_i \, dx\ , \\ 1 \le i \le N_h . \end{cases}$$

Let $a_{ij} = a_h(w_j,w_i), A_h = (a_{ij})_{1\le i,j\le N_h}$, $b_i = \int_\Omega (\nabla\psi_{oh}+\vec{u}_{oh})\cdot\nabla w_i \, dx$, $b_h = \{b_i\}_{i=1}^{N_h}$. The matrix A_h is full and symmetric, positive , semi definite. If 54 is verified, then 0 is a single eigenvalue of A_h ; furthermore if $\mathcal{B}_h$ is defined by 71 then

$$(73)\quad \mathrm{Ker}(A_h) = \{y \in \mathbb{R}^{N_h}\ ,\ y_1=y_2=\ldots=y_{N_h}\}.$$

As to the conditioning of A_h restricted to $R(A_h)$ $(= \mathbb{R}^{N_h} - Ker(A_h))$, it can be shown that the ratio $\nu(A_h)$ of the largest eigenvalue to the smallest is of order h^{-2}, if 54 holds. In fact, by analogy with [20, Sec. 4] it is reasonable to conjecture that $\nu(A_h) = 0(\frac{1}{h})$ but we were not able to obtain this estimate.

Construction of A_h : A_h is constructed column by column according to the relation $a_{ij} = a_h(w_j, w_i)$. To compute the j^{th} column of A_h we solve 60 - 62 with $\lambda_h = w_j$ and compute a_{ij} from 63. Thus 4 Dirichlet problems must be solved for each column (5 in $\mathbb{R}^3$). The matrix A_h being symmetric one may restrict i to be greater or equal to j. By the way Remark 13.4.10 applies for the computation of the b_i and a_{ij}'s.

4.5.2 Solution of 72 by the Cholesky method

Assume that 54 , 71 hold. Then one shows from 73 (see [21]) that the submatrix $\tilde{A}_h = (a_{ij})_{1 \le i,j \le N_h - 1}$ is symmetric and positive definite. Therefore one may proceed as follows :

Take $\lambda_{N_h} = 0$ and solve

74 $$\tilde{A}_h \tilde{r}_h \lambda_h = \tilde{b}_h$$

(where $\tilde{r}_h \lambda_h = \{\lambda_1 \ldots \lambda_{N_h - 1}\}$, $\tilde{b}_h = \{b_1, \ldots, b_{N_h - 1}\}$) by the Cholesky method via a factorization :

75 $$\tilde{A}_h = \tilde{L}_h \tilde{L}_h^t \quad (\text{or } \tilde{A}_h = \tilde{L}_h \tilde{D}_h \tilde{L}_h^t)$$

where $\tilde{L}_h$ is lower triangular non singular (and $\tilde{D}_h$ is diagonal).

Let us review the sub-problems arising in the computation of $\{\vec{u}_h, p_h\}$ via (E_h) if the Cholesky method is used :

- The 4 approximate Dirichlet problems 64 - 66 to compute p_{oh}, $\vec{u}_{oh}$, ψ_{oh} and $\tilde{b}_h$ (5 if $\Omega \subset \mathbb{R}^3$),
- $4(N_h - 1)$ approximate Dirichlet problems to construct $\tilde{A}_h$ ($5(N_h - 1)$ if $\Omega \subset \mathbb{R}^3$),
- 2 triangular systems to compute λ_h : $\tilde{L}_h \tilde{y}_h = \tilde{b}_h$, $\tilde{L}_h^t \tilde{r}_h \lambda_h = \tilde{y}_h$,

. 3 approximate Dirichlet problems to obtain p_h and $\vec{u}_h$ from λ_h (4 if $\Omega \subset \mathbb{R}^3$).

<u>Hence if</u> $\Omega \subset \mathbb{R}^N$ (N=2,3) <u>it is necessary to solve</u> $(N+2)(N_h+1)-1$ <u>approximate Dirichlet problems</u>.

In practice the matrices of the approximate Dirichlet problem should be factorized once and for all (there are two symmetric positive matrices, one for the affine elements, one for the quadratic elements (or affine on $\tilde{\mathcal{T}}_h$ if 50 $_{bis}$ is used)).

4.6. <u>Solution of</u> (E_h) <u>by the conjugate gradient method</u>.

We may also solve (E_h) (and therefore (P_h)) by a <u>conjugate gradient method, which does not require the knowledge of</u> A_h but requires 4 approximate Dirichlet problems to be solved at each iteration (5 if $\Omega \subset \mathbb{R}^3$) :

76 $\lambda_h^o \in \mathcal{M}_h$, arbitrarily given,

77 $g_h^o = A_h r_h \lambda_h^o - b_h$

78 $z_h^o = g_h^o$,

and for $n \geq 0$

79 $$\rho_n = \frac{(z_h, g_h^n)_h}{(A_h z_h^n, z_h^n)_h} \quad \left(\text{or } \frac{\|g_h^n\|_h^2}{(A_h z_h^n, z_h^n)_h} \right),$$

80 $r_h \lambda_h^{n+1} = r_h \lambda_h^n - \rho_n z_h^n$,

81 $g_h^{n+1} = g_h^n - \rho_n A_h z_h^n$,

82 $$\gamma_n = \frac{\|g_h^{n+1}\|_h^2}{\|g_h^n\|_h^2} ,$$

83 $z_h^{n+1} = z_h^n + \gamma_n z_h^n$.

In 76) - 83), $(\cdot,\cdot)_h$ stands for the standard euclidian scalar product of $\mathbb{R}^{N_h}$ (but one could use a conjugate gradient method with preconditioning in the sense of [30]).

The matrix A_h being symmetric, positive semi-definite, one can show that $\{\lambda_h^n\}_{n\geq 0}$ converges to λ_h, solution of (E_h) ; the componant of λ_h in $\mathbb{R}_h$ is that of λ_h^o. Implementing 76) - 83) requires the solution of 4 Dirichlet problems at each iteration (5 if $\Omega\subset\mathbb{R}^3$) to compute $A_h\, z_h^n$ from

$$a_h(\lambda_h,\mu_h) = (A_h r_h\lambda_h, r_h\mu_h)_h \quad \forall\, \lambda_h,\mu_h \in \mathcal{M}_h \;. \tag{84}$$

Here also one should factorize the matrices of the approximate Dirichlet problem.

4.7 Comments.

In Section 13.4 a new mixed finite element method was described for Stokes problem 2). The direct method described in Sec. 4.4 has been used in 2-D and 3-D cases for the computation of unsteady incompressible viscous flows. We recommend the method if the Stokes problem has to be solved many times on a given domain. On the other hand if the Stokes problem is to be solved once only or if N_h, the number of boundary nodes, is large, we recommend the conjugate gradient method of Section 4.6. The ideas of Sec. 4 will be developed in [6], [21] where the proofs will be included together with most of the results shown here.

5. FURTHER REFERENCES AND CONCLUSION

To conclude with we would like to mention the works of BERCOVIER [31], ARGYRIS-DUNNE [32], JOHNSON [33] on Stokes and Navier-Stokes equations, and incompressible media. These appear to us connected with some of the ideas developed in this chapter.

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