PUBLICATIONS MATHÉMATIQUES ET INFORMATIQUES DE RENNES

KARINE CHEMLA

Equations with General Coefficients in the [U+6D4B] [U+5706] [U+6D77] [U+955C]

Publications de l'Institut de recherche mathématiques de Rennes, 1985, fascicule 2 « Science, histoire et société », , p. 23-30

<http://www.numdam.org/item?id=PSMIR_1985___2_23_0>

EQUATIONS WITH GENERAL COEFFICIENTS IN THE 測圓海鏡

Karine CHEMLA (CNRS, France)
3, Square Simon Bolivar
F 75019 PARIS

INTRODUCTION

Li Ye's first mathematical book[1], Ce Yuan Hai Jing, (Sea-mirror of the circle measurements) begins with a well-known diagram, reproduced in figure 1, on which it entirely relies. The purpose of the 170 problems it contains is to find the length of the diameter of the circle, when given a couple (sometimes a triplet) of data, that are the lengths of some segments of that diagram or simple combinations of such lengths. Once the data are given in the outline of a problem, the text splits up into 法 /fa/ and 草 / cao/. I would like to stress the possible interest of a study on the part played by the /fa/ in the virtual unity the book perhaps alludes to. As for the / cao/, it has been already analysed because our text is the earliest extant document of the so-called Song polynomial algebra : throughout the book, Li Ye uses, at the same time as he presents them, arithmetical operations on polynomials with numerical coefficients. These polynomials receive a positional notation : they are denoted here as the sequence of their coefficients, in a decreasing order with respect to the powers of the unknown[2]. They are involved in any kind of arithmetical operations, just similary as the numbers.

So, for each problem, an unknown is chosen, which is linked with the wanted diameter in the course of the text when it happens to be different from this diameter ; the main purpose is to find a numerical equation which has this unknown as positive root. This is specific of the parts /cao/ (草 , detailed methods) which are generally built according to the following pattern : they give two sequences of geometrical relations which, starting with the data and the unknwon, lead both to some equivalent to the same geometrical quantity, that I shall call the "go-between" quantity. The geometrical properties that are used come from a formulary given in the chapter I of the book or are sometimes displayed just before the beginning of the computation.

Let us show an instance of these two sequences (cf. problem VII-1). Here the data are LS and BE, the chosen unknwon is the diameter D.
We have :

1 - (a) D-2(LS + BE) = d, diameter of the circle inscribed in the triangle LUB.
(b) D.d = 2 UL. UB, go-between quantity

2 - (a') 2 LS . 2 BE = 2 UL . UB, go-between quantity.

These sequences are built under the constraint that one must be able to numerically compute the polynomial expression of each intermediate geometrical step. These polynomials are actually computed step by step and as a final step we get two different polynomial expressions of the go-between quantity ; then by elimination the numerical equation, which is looked for and whose solution can give the value of the unknwon.

However, just after the outline of each problem and before its actual computation via the /cao/, Li Ye gives in the /fa/ (法 , method) a rather abstract and complex mathematical description of the equation just mentionned, coefficient by coefficient. I mean, instead of using the numerical values of the data he calls them in terms reminding us the way they were given in the outline : it is either a narrative of how far men (Jia, Yi, Bing...) walked in certain directions or the simple data of numerical values for lengths of segments called by their geometrical names -all the segments are given names in the first chapter[3]. With these terms, he describes how each coefficient is reached. This description is made step by step, using some intermediate quantities. Let us give an idea with an instance transposed from problem III-9 :

A minus twice the difference of the two data ;
what is left multiplies A ;
then the double of this is the constant coefficient

Translated in modern symbolism, the sequence of intermediate quantities is :

A-2 (A-B),
(A-2(A-B))A, and eventually
(A-2(A-B))A.2, constant coefficient.

The description is rather intricate because Li Ye did not actually choose the simplest expression of the intermediaries. For instance, why did he use such a term as A-2(A-B) ? Besides, how did he find this description of each coefficient, whereas, in the text, he only got them numerically ?

With the intention of investigating these problems, we took the sequences of geometrical relations Li Ye wrote in each / cao/, and, using the tools of modern algebra, we computed the successive polynomial expressions in terms of the data given in the outline, thus not numerically as he did in the third section of each problem. I will refer to this as "symbolic computation". One is tempted to notice striking correlations between the sequences of the intermediaries used in the description (/fa/) and some sequences of expressions in A and B appearing in the symbolic computation.

I - CORRELATIONS :

Let us turn back to the problem III-9, for which the constant coefficient of the equation as described in the /fa/ section has been given previously : we find that, in the course of the symbolic computation which we derive from the /cao/, we have to take the difference of the two polynomials[4].

2(A-B) x	and	-1 A x

; or in modern notations : $(X^2+AX) - 2(A-B)X$

so, we get, as coefficient of least degree, A - 2(A-B). In the following operation, this term has to be multiplied by A and eventually by 2. Thus, the sequence of intermediaries happens to correspond with the succesive stages of the term of least degree in the course of the computation; furthermore the shape of the first expression in the description, A-2(A-B), is closely related to the way this quantity has been actually computed.

For the solution of another problem (III-13), Li Ye gives two different /cao/ which end at the same numerical equation ; the /fa/ which corresponds to the second /cao/ says that the coefficients of the second equation are the same as those of the first equation, except for the constant term. Therefore this /fa/ contains another description of the constant term. We can notice that, although these coefficients are, here, numerically the same in the /cao/, they are described in a different way in the /fa/.

4

Let us compare the sequences of expressions that are used for the description of these constant terms in each case :

first description	second description
A-B 2(A-B)-A A(2(A-B)-A B^2.A(2(A-B)-A)	 A-2B A(A-2B) B^2.A(A-2B)

So, actually, these descriptions are equivalent, the only difference lies in the way of giving the two expressions of the second line.

On the other hand, if we compare the two geometrical procedures of the two /cao/, we find that difference between them is the way both reach their own polynomial expression of the quantity BE. After that computation, they are identical. Let us focus on the step where the constant coefficient of the polynomial expression for this quantity BE is built in each case. We notice that, in the first method, this appears as :

2(A-B)-A in the operation	-2x 2(A-B)	minus	A.	;

in the second, it comes as :

A-B-B in the operation	-1x A-B	minus	B.

Thus the "artificial" distinction which has been made between the two terms in the descriptions of the equation in the /fa/, actually, bears the mark of the points where the procedures were different. It reminds us of the different successive stages of the constant coefficient through the two methods of computation. Perhaps, here, Li Ye points out the close relationship that exists between the abstract description of the equation and the procedure of numerical computations.

We found out that most examples follow this rule and from now on we will take it as granted : the description fits the procedure. From this hypothesis, we will answer the objection which could be formulatd about the previous example : how to prove that Li Ye knows 2(A-B)-A and A-2B are identical ? Let us, for that purpose, introduce another notion, i.e symmetry.

II - SYMMETRY :

The two sides TQ and CQ of the big triangle play symmetrical parts in the whole theory ; hence triangles are grouped into couples of symmetrical triangles which play the same part. For instance, the triangles CHA and FTZ or the triangles COF and HNT. Through these symmetries, the height of the first triangle corresponds to the basis of the second triangle and the converse.

Li Ye uses these correspondances in the composition of the formulary in the first chapter where each formula is coupled with a symmetrical formula. As for the problems, each of them is also usually associated with a symmetrical problem, which means a problem which has symmetrical data. For example, the problems with same numbers in the chapters III and IV on one hand, in the chapters V and VI on the other hand, are symmetrical.

One can notice that in the case where the data are symmetrical, the case where these outlines are in form of a narrative, men with same names walk along the symmetrical segments of road. Therefore, the terms by which the data are denoted in the description of the corresponding equations are the same with respect to the directions or to the geometric denominations.

Furthermore, for symmetrical problems, the description of the respective equations and the procedures of the respective computations are usually symmetrical. Thus a kind of analogy can be established between equations which are numerically different.

******

The lacks of symmetry between symmetrical problems deserve attention. If we look, for instance, at what happens for the problems V-1 and VI-1, we can see that, if the procedures are exactly the same, there is a difference in the descriptions of the equations. But, if these problems follow the principle of fitness of the description to the procedure, then these equations are to be said formally identical. If we now turn back to the difference in the descriptions of the equations, we find that, for the problem V-1, there is, as an intermediary in the description of the constant term, the quantity 2(A-B)-A, which is one of the successive stages of the constant coefficient. The symmetrical term for the problem VI-1 is described as A-2B. Thus the identity that the symmetry established, gives the equivalence between these two formulations.

CONCLUSION

We have seen the way /fa/and/cao/ refer each to the other, as well as the formal analogies that were expressed between different numerical equations. We would now like to sketch some of the problems that can arise from this mathematical situation and from the mathematical object it defines : equation in terms of data.

Li Ye seems to have met the situation of looking for some /cao/ that would fit a given "formal" equation : for instance in the problem VIII-2, we could see the question of finding a procedure for another solution to the problem, solution that would obey the following constraint: its equation must be the same as the previous one when all the coefficients have been divided by the factor B. Besides, he may have thought of the transformations that could affect an equation expressed in terms of the data when one chooses, in the /cao/, another unknown to solve the problem (changes of x into x/2, into x+a).

There are experiments of these changements inside the same problem ; in this case, the changes of the equation may be also experimented on the numerical values. Or in symmetric problems ; then the changes can be only seen on the formal equations and not on the numerical values that are different.

One would have also to try to answer the question : how did he manage to get such a description of the equations if he only computed them numerically ? One could think that he carefully paid attention to the different stages of each coefficient through the computation and in some way recorded them. This has actually been a method and some rare descriptions bear the mark of it. But their expression is very different from that of other descriptions in the book so we could also imagine that Li Ye got another more powerful method, perhaps that he got these equations at the end of a more formal computation, where he could have used caracters instead of numbers.

NOTES

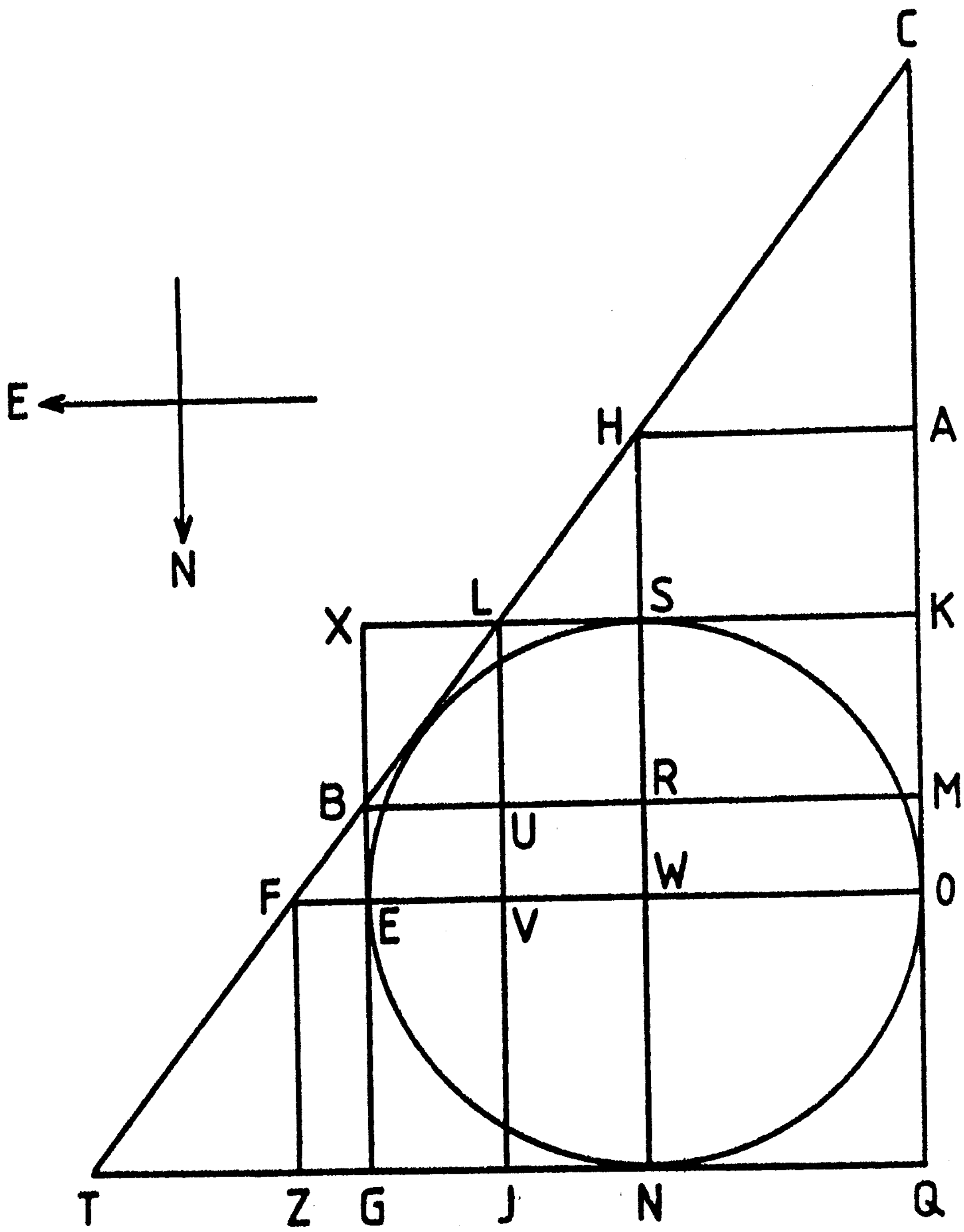
(1) Li Ye wrote that book in 1248, as is said in the annals of the Yuan dynasty. In 1259, he wrote another book about the method of the unknown 天 tian, sky, the name of which is 益古演段 Yi Gu-Yan--Duan. The level of that book seems inferior to the level of the Ce-Yuan-Hai-Jing. It has been thought that the latter was written towards reseachers whereas the former was written to introduce Li Ye's students to this method.

(2) Li Ye's notation is different from other mathematicians' notations and changed when he wrote the Yi Gu Yan Duan. We'll also note the polynomials as the sequence of their coefficients in a frame. The point (resp. x) on the left of one of them denotes it is the constant term (resp. the term of first degree).

(3) in the book, each triangle is given a name related to one of its geometrical properties or characteristic feature. Afterwards, each segment is called by the name of the triangle it belongs to, followed by the name of the kind of side it is in that triangle. Here, wherever could be the origins of the data, we'll use A and B to denote them.

(4) To make our computations, we kept the positionnal notation for polynomial to make clear the differences between different terms.

LEUVEN, augustus 1982


C
E
H
A
N
X
L
S
K
R
M
B
U
W
F
O
E
V
T
Z
G
J
N
Q