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On regression models with non-square integrable martingale-like errors

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The paper is concerned with two types of estimators of an unknown parameter θ of the drift of an observed semimartingale X. A martingale part M of the semimartingale X is not a local square integrable martingale in general. As a rule we suppose only that M has a r-th moment, $r \in [1,2]$.

The first part of the paper is devoted to an investigation of strong consistency of the least-square estimators (LS-estimators). Our approach is based on a multidimensional large numbers law for local martingales (see [1] , where the results were announced particularly, see also [2] - [3]).

In the second part of the paper another type estimators of θ are studied. They are so-called sequential estimators (SQ-estimators), and were systematically investigated in [4] for regression models with local square integrable martingales and quasi-left-continuous local martingales as errors. It was proved there that these estimators have a very important property-a guaranted accuracy. Here we get rid of from these assumptions proved a generalisation of Novikov's [2] inequality and Metivier-Pellaumail's one [5] for general local martingales and using the approach of the paper [4].

Let $(\Omega , \mathcal{F} , (\mathcal{F}_t)_{t \geq o} , P)$ be a standard stochastic basis on which we consider all stochastic processes whose paths are regular.

Let us denote (see, for references [2]) $\mathcal{M}_{loc} (R^d)$ the set of local martingales, which values in R^d $d \geq 1$;

$\mathcal{A}^+{}_{loc} (R^d)$ the set of predicatble processes, whose values are positive definite operators (matrix) from R^d into R^d such that $A_t - A_s \geq o$, $t \geq s$.

Let $\lambda_1 (A)$, $\lambda_2 (A)$ and $tr (A)$ be the minimal, maximal egenvalues and the trace of the operator (matrix) A. Let us denote A^* a transpose matrix of A.

For a random process X with values in R^d , $d \geq 1$, let $\{w : X_t \to\}$ be the set of $\omega \in \Omega$ such that $\lim\limits_{t \to \infty} X_t (\omega) = X_\infty(\omega)$ exists for the norm $\|.\|$ of the space R^d .

If A , $B \in \mathcal{F}$ and $P\{A \cap \{\Omega \setminus B\}\} = o$, then we write $A \leq B$ (a.s.).

Let $M \in \mathcal{M}_{loc}(R^d)$ and

$$M_t = M_t^c + \int_o^t \int_{R_o^d} x \, d(\mu - \nu) , \qquad (1)$$

be the canonical decompostion of M , where $R_o^d = R^d \setminus \{o\}$, M^c be a continuous part of M (and $< M^c >$ be its (matrix) quadratic characteristic), μ be a random measure of jumps of M and ν be its compensator (see [2]).

Theorem 1 : Assume the following conditions : (a.s.)

1) $\lim_{t \to \infty} \lambda_1(A_t) = \infty$;

2) $\limsup_{t \to \infty} \frac{\lambda_1(A_t)}{\lambda_2(A_t)} < \infty$;

3) $\int_o^\infty \lambda_1^{-2}(A_s) \, d < M^c >_s + \int_o^{+\infty} \int_{R_o^d} \lambda_1^{-r}(A_s) \, \|x\|^r \, d\nu < +\infty$

for some $r \in [1 , 2]$.

Then $A_t^{-1} M_t \to o$ (a.s.) as $t \uparrow \infty$.

Particularly, if V is predictable increasing process such that (a.s.)

$$\frac{d < M^c >_t}{d V_t} + \frac{d}{d V_t} \int_o^t \int_{R_o^d} ||x||^r \, d\nu \leq \xi < \infty$$

and (a.s.)

$$3') \int_o^\infty \lambda_1^{-r}(A_s) \, d V_s < \infty,$$

then 1), 2), 3') $\Rightarrow A_t^{-1} M_t \to o$ (a.s.) as $t \uparrow \infty$.

Proof : Denote $\tilde{B}$ a compensator of an increasing process B. Then as in one-dimensional case (see [2]) it is proved that (a.s.).

$$\{\omega : tr < M^c >_\infty + \widetilde{\sum_s \frac{\|\Delta M_s\|^2}{1 + \|\Delta M_s\|}} < \infty\} \leq \{\omega : M_t \to\}.$$

Particularly, for some $r \in [1, 2]$ (a.s.)

$$\{\omega : tr < M^c >_\infty + \sum_s \|A M_s\|^r < \infty\} \leq \{\omega : M_t \to\} \qquad (2)$$

The last statement follows from

$$\frac{\|x\|^2}{1 + \|x\|} \leq \|x\|^r \text{ for all } x \in R^d, r \in [1, 2].$$

Now define as in [2] - [3] the process

$$Y_t = \int_o^t A_s^{-1} \, d\, M_s.$$

Using the same arguments we have that (a.s.)

$$\{\lambda_1 (A_t) \to \infty\} \cap \left\{ \lim_{t \to \infty} \frac{\lambda_2 (A_t)}{\lambda_1 (A_t)} < \infty \right\} \subset \{A_t^{-1} M_t \to o\}.$$

To complete the proof note that the condition 3) 3') implies (a.s.)

$$tr < Y^c >_\infty + \widetilde{\sum_s \|\Delta Y_s\|^r} < \infty$$

(in the case of 3'))

and in view of (2) we get the statement of the theorem 1.

This theorem gives us a possibility to prove the strong consistency of the LS-estimators in regression models with non-square integrable martingale errors.

Consider the following regression model

$$X_t = \int_o^t f s \, d\, V_s \theta + m_t, \qquad (3)$$

where m is a pure discontinuous (for simplicity) local martingale from $\mathcal{M}_{loc}(R^d)$, a predictable process $V \in \mathcal{A}^+_{loc}(R^1)$, f is a predictable $(d \times k)$ – matrix, $\theta \in R^k$, $k \geq 1$, is an unknown parameter.

Let $F_t = \int_o^t f_s^* f_s \, d V_s, F_t > o, t \geq t_o$.

In this case we can define the estimator of θ :

$$\theta_t = F_t^{-1} \int_o^t f_s^* \, d X_s = \theta + F_t^{-1} \int_o^t f_s^* \, d m_s .$$

Theorem 2 : Suppose for the model (3) the following conditions hold (a.s.)

1) $\lim_{t \to \infty} \lambda_1 (F_t) = \infty$;

2) $\limsup_{t \to \infty} \frac{\lambda_2 (F_t)}{\lambda_1 (F_t)} < \infty$;

3) $\int_o^\infty \int_{R_o^d} \lambda_1^{-r} (F_s) \|f_s\|^r \|x\|^r \, d v < \infty$

where $r \in [1, 2]$, v is a compensator of a measure μ of jumps of M.

Then $\theta_t \to \theta$ (a.s.) as $t \uparrow \infty$.

It is possible to unify the conditions of the theorem 2 , if we suppose that (a.s.)

$$\frac{d}{d V_t} \int_o^t \int_{R_o^d} \|x\|^r \, d v \leq \xi < \infty,$$

and (a.s.)

3') $\int_o^\infty \lambda_1^{-r} (F_s) \|f_s\|^r \, d V_s < \infty$.

Then 1) - 2) - 3') $\Rightarrow \theta_t \to \theta$ (a.s.) as $t \uparrow \infty$.

Proof : It is sufficient to note that

$$\theta_t - \theta = A_t^{-1} M_t ,$$

where $A_t = F_t$, $M_t = \int_o^t \int_{R_o^d} f^* x \, d(\mu - \nu)$.

Using the theorem 1 we get immediatly the statement of the theorem 2.

Remark : Note that the consistency of LS-estimators for the model (3) with non-random regressors was proved by Novikov [6]. The strong consistency of the LS-estimators for this model with non-random regressors was studied also in [7]-[8].

Now consider another type of estimators of θ in the one-dimensional model (3). These are SQ-estimators, which systematically were studied in [4]. But the case of non-square integrable errors was handed there for the quasi-left continuous martingale errors m only. Here we prove an estimate for pure-discontinuous martingales and apply it to give an upper estimate for the r–th moment of the difference between the SQ-estimator and θ . This result gives us (in some sense) a guaranted accuracy of these estimators.

Denote $\mathcal{B}(R)$– Borel σ-algebra of the space R. Let

$$M_t = \int_o^t \int_{R_o} x \, d(\mu - \nu)$$

be a purely discontinuous local martingale of the classe $\mathcal{M}_{loc}(R^1)$ (see decomposition (1)).

Let U be a $\mathcal{B}(R_+^1) \otimes \mathcal{F} \otimes \mathcal{B}(R_0^1)$–measurable function such that for some $r \in [1 , 2]$

$$\int_o^t \int_{R_o} |U|^r \, d\nu \in \mathcal{A}_{loc}^+ (R^1)$$

and

$$\int_{R_o} U(t, x, \omega)\, \nu(\{t\}, dx) = o \qquad (4)$$

Denote $Y_t(U) = \int_o^t \int_{R_o} U\, d(\mu - \nu)$ and $Y_t^*(U) = \sup_{s \le t} | Y_s(U) |$.

Theorem 3 : Suppose the function U satisfies to the condition (4) and τ is a prédictable stopping time (s.t.). Then

$$E\, |Y^*_{\tau-}(U)|^r \le A_r E \int_o^{\tau-} \int_{R_o} |U|^r\, d\nu, \qquad (5)$$

where $A_r \le 3 \left(\frac{r}{r-1}\right)^r$, $r \in (1, 2]$, $A_1 = Z$ and Y_{t-} is left limit of Y_t .

Proof. We shall use Novikov's method [5]. Let us involve the $s.t.$ $(a > o)$

$$\tau_a = inf\,(t \le \tau : \int_o^t \int_{R_o} |U|^r\, d\nu \ge a),$$

$inf\ (\emptyset) = \tau$.

Of course, τ_a is a predictable $s.t.$

Therefore there is a sequence of $s.t.$'s $(\tau_a^n)_{n \ge 1}$ such that

$\tau_a^n \uparrow \tau_a$ $(a.s.)$ as $n \uparrow \infty$,

$\tau_a^n < \tau_a$ on the set $(\omega : \tau_a < \infty\}$.

It follows from here that

$$\int_o^{\tau_a^n} \int_{R_o} |U|^r\, d\nu < a .$$

Let us show that $E\, Y^*_{\tau_a^n}(U) < \infty$, we have (as usually, I_c is an indicator of c)

$$E\,Y^*_{\tau_a^n}(U) \le E \sup_{t\le \tau_a^n} \left| \int_o^t \int_{R_o} U \,.\, I_{|U|>1}\, d(\mu - \nu) \right| +$$

$$+ E \sup_{t\le \tau_a^n} \left| \int_o^t \int_{R_o} U \,.\, I_{|U|\le 1}\, d(\mu - \nu) \right| \le$$

$$\le const\,.\,E \int_o^{\tau_a^n} \int_{R_o} |U|\,.\,I_{|U|>1}\, d\nu + const\,.\,E \left[\int_o^{\tau_a^n} \int_{R_o} |U|^2 \,.\, I_{|U|\le 1}\, d\nu \right]^{1/2} \le$$

$$\le const\,.\,E \int_o^{\tau_a^n} \int_{R_o} |U|^r \,.\,I_{|U|>1}\, d\nu + const\,.\,E \left[\int_o^{\tau_a^n} \int_{R_o} |U|^r\, d\nu \right]^{1/2} \le$$

$$\le const\,(a + a^{1/2}) < \infty.$$

Using this fact and the elementary inequality

$$\left|\, |x+y|^r - |x|^r \,\right| \le C_r \left(|x|^{r-1}\, |y| + |y|^r \right)$$

we get that

$$E \int_o^{\tau_a^n} \int_{R_o} I_{|U|>1} \left|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\right| d\nu < \infty,$$

$$E \left[\int_o^{\tau_a^n} \int_{R_o} I_{|U|\le 1} \left|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\right|^2 d\nu \right]^{1/2} < \infty \qquad (6)$$

This first inequality of (6) follows from

$$E \int_o^{\tau_a^n} \int_{R_o} I_{|U|>1} \left|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\right| d\nu \le$$

$$\leq const\,(r)\, E\,(1 + |Y_{\tau_a^n}|^{r-1}) \int_o^{\tau_a^n} \int_{R_o} |U|^r\, d\, v \leq$$

$$\leq a\,.\, const\,(r)\,.\, E\,(1 + |Y^*_{\tau_a^n}|^{r-1}) < \infty.$$

The second one follows from

$$E\left[\int_o^{\tau_a^n} \int_{R_o} I_{|U|\leq 1} \left|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\right| d\, v\right]^{1/2} \leq$$

$$\leq const\,(r)\;\; E\left[\int_o^{\tau_a^n} \int_{R_o} I_{|U|\leq 1} (|Y_{s-}|^{r-1}\, |U| + |U|^r)^2\, d\, v\right]^{1/2} \leq$$

$$\leq const\,(r)\, E\,(1 + (Y^*_{\tau_a^n})^{2(r-1)})^{1/2} \left(\int_o^{\tau_a^n} \int_{R_o} I_{|U|\leq 1}\, |U|^r\, d\, v\right)^{1/2}$$

and

$$E\,(Y^*_{\tau_a^n}\,(U))^{r-1} \leq (E\, Y^*_{\tau_a}\,(U\, I_{|U|>1}))^{r-1} + (E\, Y^*_{\tau_a^n}\,(U\,.\, I_{|U|\leq 1}))^{r-1}$$

Now using the Ito's formula (see [2] ; p. 150-151) we get

$$|Y_{\tau_a^n}|^r = \int_o^{\tau_a^n} \int_{R_o} (|Y_{s-} + U|^r - |Y_{s-}|^r)\, d\,(\mu - \nu) +$$

$$+ \int_o^{\tau_a^n} \int_{R_o} \{\, |Y_{s-} + U|^r - |Y_{s-}|^r - r\, |Y_{s-}|^{r-2}\, Y_{s-} U\,\}\, d\, v \qquad (7)$$

It follows from (6) that

E (martingale part of (7)) = 0

Applying the elementary inequality

$$\left|x+y\right|^{r} - \left|x\right|^{r} - r\,x\,y\left|x\right|^{r-2} \le B_r \left|y\right|^{r},$$

where $B_r \le 3$, $r \in [1,2]$ and $B_1 = 2$,

to the second part of (7), we have

$$E\left|Y_{\tau_a^n}\right|^{r} \le B_r E \int_o^{\tau_a^n}\int_{R_o} \left|U\right|^{r} d\,\nu . \qquad (8)$$

Using the Doob's inequality [2] , we get

$$E\,(Y^{*}_{\tau_a^n})^{r} \le 3\left(\frac{r}{r-1}\right)^{r} E\int_o^{\tau_a^n}\int_{R_o} \left|U\right|^{r} d\,\nu .$$

To tend $n \uparrow \infty$ and $a \uparrow \infty$ we complete the proof.

We note that the inequality (8) for $r=1$ is true with $B_1 = 2$ and therefore $A_1 = 2$.

Now consider the one-dimensional regression model (3) and suppose that

$$\frac{d}{d\,V_t}\int_o^{t}\int_{R_o} \left|x\right|^{r} d\,\nu \le \gamma_t , \qquad (9)$$

where $r \in [1,2]$, γ is a predictable process such that

$$K_t = \int_o^{t} \gamma_s^{1-r} \left|f_s\right|^{r} d\,V_s \in \mathcal{A}^{+}_{loc}\,(R^1).$$

we define the following SQ-estimator

$$\theta_H = H^{-1}\int_o^{\tau_H^-} \gamma_s^{-1} f_s\, d\,X_s + H^{-1}\,\beta_H\,\gamma_{\tau_H}^{-1} f_{\tau_H}\,\Delta X_{\tau_H},$$

where $H > o$, $\tau_H = inf\,(t : K_t \geq H)$,
β_H – $\mathcal{F}_{\tau_{H^-}}$ -measurable random variable such that
$\beta_H \in [o\,, 1]$,

$$\int_o^{\tau_{H^-}} \gamma_s^{-1}\ |f_s|^r\ d\,V_s + \beta_H\ \gamma_{\tau_H}^{-1}\ |f_{\tau_H}|^r\ \Delta\ V_{\tau_H} = H. \qquad (10)$$

Theorem 4 : Let the conditions (9) - (10) are fulfield.

Then $\int_o^{\infty} \gamma_s^{1-r}\ |f_s|^r d\ V_s = \infty (a.s) \ \Rightarrow E\ \theta_H = \theta$ and $E\ \left|\theta_H - \theta\right|^r \leq const\ (r).\ H^{-r}\ (H{+}\Delta)$,
where $\Delta = E\ \sup_t\ \Delta\,K_t$, $r \in [1,2]$.

Proof : The first statement is the direct consequence of (10). Now we have, using the theorem 3, that

$$E\ \left|\theta_H - \theta\right|^r = E\ \Big|H^{-1} \int_o^{\tau_{H^-}} \int_{R_o} \gamma_s^{-1}\ f_s\,x\ d\,(\mu - \nu) +$$

$$+ \beta_H H^{-1}\ \gamma_{\tau_H}^{-1} f_{\tau_H} \Delta\,M_{\tau_H}\ \Big|^r \leq A_r\,2^{\,r-1}\ E\,H^{-r} \int_o^{\tau_{H^-}} \int_{R_o} \gamma_s^{-r}\ |f_s|^r |x|^r\,d\,\nu +$$

$$+ 2^{\,r-1}\,H^{-r}\ E\,\beta_H\ \gamma_{\tau_H}^{-r}\ |f_{\tau_H}|^r \cdot \int_{R_o} |x|^r\,\nu\,(\{\tau_H\}\,, d\,x)\ \leq$$

$$\leq H^{-r}\ 2^{\,r-1} \left[A_r\ E \int_o^{\tau_{H^-}} \gamma_s^{1-r}\ |f_s|^r\ d\,V_s + E\,\beta_H\ \gamma_{\tau_H}^{1-r}\ |f_{\tau_H}|^r \Delta\,V_{\tau_H} \right] \leq$$

$$\leq 2^{\,r-1}\ H^{-r}\ [A_r H + \Delta] \leq const\ (r)\ .\ H^{-r}\,(H + \Delta)$$

The theorem is proved.

REFERENCES

1. A.V. Mel'nikov , On a large numbers law for multidimensional local martingales, IV International Vilnius Conference on Probab. and Math. Stat. , Summary of papers, Vilnius, 1985.

2. R. Sh. Liptser, A. N. Shiryaev, Theory of martingales, Moscow, Nauka, 1986.

3. A.V. Mel'nikov, The law of large numbers for multidimensional martingales, Soviet Math. Dokl. , V. 33 (1986), N° 1, p. 131-135.

4. A.V. Mel'nikov, A. A. Novikov, On sequential inferences with guaranted accuracy for semimartingales, Theory Probab. and Appl. , XXXIII , 3 (1988).

5. A. A. Novikov, On discontinuous martingales, Theory Probab. and Appl. , XX , 1 , 1975, p. 13-28.

6. A. A. Novikov, Consistency of LS-estimates in regression models with martingale errors, In Statistics and Control of Stochastic Processes, Optimization software, Springer-Verlag, Berlin and etc, 1985.

7. A. Lebreton, M. Musiela, Strong consistency of LS-estimates in linear regression models driven by semimartingales, J. Multiv. Anal.,1987, 23 , 1 , p. 77-92.

8. A. V. Mel'nikov, On Strong consistency of LS-estimates in regression models with depending errors, Proceedings of IV European Young Statisticians Meeting, Pliska, Sofia, 1988.