PUBLICATIONS MATHÉMATIQUES ET INFORMATIQUES DE RENNES

QUANSHENG LIU

On the Integrability of the Limit of a Supercritical Branching Process

Publications de l'Institut de recherche mathématiques de Rennes, 1994, fascicule 2
« Fascicule de probabilités », , p. 1-4
<http://www.numdam.org/item?id=PSMIR_1994___2_A7_0>

© Département de mathématiques et informatique, université de Rennes, 1994, tous droits réservés.

L'accès aux archives de la série « Publications mathématiques et informatiques de Rennes » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

ON THE INTEGRABILITY OF THE LIMIT OF A SUPERCRITICAL BRANCHING PROCESS

Quansheng LIU
IRMAR, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes, France

Let Z_n be a Galton-Watson process with $Z_0=1$ and $1<m=\mathbb{E}Z_1<\infty$ and put $W=\lim_{n\to\infty} Z_n/m$. It is well-known that for all $\beta\geq 0$,

$$0<\mathbb{E}W(\log^+W)^\beta<\infty \text{ if and only if } \mathbb{E}Z_1(\log^+Z_1)^{\beta+1}<\infty.$$

Inspired by this, Athreya and Ney (1972,p.63) conjectured that for all $\alpha>1$ and $\beta>0$, $0<\mathbb{E}W^\alpha(\log^+W)^\beta<\infty$ if and only if $\mathbb{E}Z_1{}^\alpha(\log^+Z_1)^{\beta+1}<\infty$. Quite curiously, this is not the case, as is shown in the following

Theorem. *(a) If $\alpha>1$ is not an integer and $\beta\geq 0$, then*

$$0<\mathbb{E}W^\alpha(\log^+W)^\beta<\infty \text{ if and only if } \mathbb{E}Z_1{}^\alpha(\log^+Z_1)^\beta<\infty.$$

(b) If $\alpha\geq 1$ is an integer and $\beta\geq 0$, then we have the following implications:

$$\mathbb{E}Z_1{}^\alpha(\log^+Z_1)^{\beta+1}<\infty \Rightarrow 0<\mathbb{E}W^\alpha(\log^+W)^\beta<\infty \Rightarrow \mathbb{E}Z_1{}^\alpha(\log^+Z_1)^\beta<\infty.$$

We can prove this theorem in a similar way as Asmussen and Hering (1983) for a proof of the Kesten-Stigum theorem. Simiar arguments were used in Wen (1986, section 4). We shall need the following generization of a result of Asmussen and Hering (1983,P.41), whose proof is postponed to the end of the note.

Lemma. *Let $S_n=\gamma_1+\gamma_2+...+\gamma_n$ be the sum of independent and identically distributed random variables $\gamma_i\geq 0$. If $\phi(x)\geq 0$ is a non-decreasing concave function on $[0,\infty)$ with $\phi(0)=0$, then for all $k\geq 1$,*

$$\mathbb{E}S_n^k\phi(S_n) \leq (\mathbb{E}S_n)^k\phi(S_n) + c_k n^{k-1}(\mathbb{E}\gamma_1)\phi(\mathbb{E}S_n) + 2k\, n^k\mathbb{E}\gamma_1^k\phi(\gamma_1), \tag{1}$$

where $c_k=\frac{1}{2}(k-1)k$.

Proof of Theorem. Using the Lemma above for the sum

$$\frac{Z_{n+1}}{m^{n+1}} = \sum_{i=1}^{Z_n} \frac{X_{n,i}}{m^{n+1}}, \tag{2}$$

where given $\mathbb{F}_n=\sigma(Z_1,...Z_n)$, $\{X_{n,i}\}$ are independent copies of Z_1, we obtain, for all $k\geq 1$,

$$\mathbb{E}[W^k_{n+1}\phi(W_{n+1})\,|\,\mathbb{F}_n] \leq W^k_n\phi(W_n)+ c_k W_n^{k-1}\phi(W_n)\,\frac{\mathbb{E}Z_1^k}{m^{n+k}} + 2k\; W^k_n\; \mathbb{E}\left[\frac{Z_1^k}{m^k}\,\phi(\frac{Z_1}{m^{n+1}})\right],$$

where $W_n=Z_n/m^n$, and c_k and ϕ are as in the lemma. Therefore

$$\sup_{n\geq 0} \mathbb{E}W_{n+1}^k\phi(W_{n+1}) \leq \phi(0) + c_k \sum_{n=0}^{\infty} \frac{\mathbb{E}Z_1^k}{m^{n+k}} \sup_{n\geq 0} \mathbb{E}W_n^{k-1}\phi(W_n)$$
$$+ 2km^{-k}\mathbb{E}\Big[Z_1^k \sum_{n=0}^{\infty} \phi(\frac{Z_1}{m^{n+1}})\Big] \sup_{n\geq 0} \mathbb{E}W_n^k . \tag{3}$$

If we choose $\phi(x)=x$, an induction on k shows the well-known result that $\mathbb{E}Z_1^k<\infty$ implies $\mathbb{E}W^k<\infty$. In the general case, let us take $k=[\alpha]$ (the integral part of α) and define

$$\phi(x)= \begin{cases} Ax & \text{if } 0\leq x\leq x_0 \\ x^{\alpha-[\alpha]}\log^\beta x + B & \text{if } x>x_0, \end{cases} \tag{4}$$

where $x_0>1$ is choosen so large that for all $x\geq x_0$, $\frac{d^2}{dx^2}\{x^{\alpha-[\alpha]}\log^\beta x\}<0$, $A=\frac{d}{dx}\{x^{\alpha-[\alpha]}\log^\beta x\}\big|_{x=x_0}>0$ and $B=c_1x_0-x_0^{\alpha-[\alpha]}\log^\beta x_0 \geq 0$.

Then ϕ satisfies the conditions of the Lemma. We claim that

$$\sum_{n=0}^{\infty} \phi(\frac{x}{m^{n+1}}) = O(x^{\alpha-[\alpha]}(\log^+ x)^{\beta+1}), \tag{5}$$

which can be improved as

$$\sum_{n=0}^{\infty} \phi(\frac{x}{m^{n+1}}) = O(x^{\alpha-[\alpha]}(\log^+ x)^{\beta}) \tag{6}$$

if $\alpha-[\alpha]>0$. [Here $f(x)=O(g(x))$ means that for some constant $c>0$ and all large $x>0$, $|f(x)|\leq cg(x)$.] In fact, for all $x>x_0$, choosing an integer $k\geq 0$ such that $x/m^{k+1} \leq x_0 < x/m^k$, we have

$$\sum_{n=0}^{\infty} \phi(\frac{x}{m^{n+1}}) = c_1 \sum_{n=k+1}^{\infty} \frac{x}{m^{n+1}} + \sum_{n=0}^{k}\Big[(\frac{x}{m^{n+1}})^{\alpha-[\alpha]}\log^\beta \frac{x}{m^{n+1}} + c_2\Big]$$
$$\leq cx_0 + x^{\alpha-[\alpha]} \sum_{n=0}^{k} m^{-(n+1)(\alpha-[\alpha])}\log^\beta \frac{x}{m^{n+1}} + c_2k,$$

for some canstant $c>0$. Since $k< \log(x/x_0)/\log m$, this gives (5) and (6). By (3), (5) and (6), an induction argument on $k\geq 1$ shows immedeately the sufficient parts of both (a) and (b). It remains to prove that for all $\alpha\geq 1$ and $\beta\geq 0$, if $0<\mathbb{E}W^\alpha(\log^+ W)^\beta<\infty$ then $\mathbb{E}Z_1^\alpha(\log^+ Z_1)^\beta <\infty$. Since the function $f(x)=x^\alpha\log^\beta x$ $+c$ is nonnegative and convex on $(0,\infty)$ if $c>0$ is sufficiently large, Jensen's inequality gives

$$\mathbb{E}(W^\alpha\log^\beta W + c) = \mathbb{E}[\mathbb{E}(W^\alpha\log^\beta W + c\,|\,\mathbb{F}_1)]$$
$$\geq \mathbb{E}[(\mathbb{E}W\,|\,\mathbb{F}_1)^\alpha\log^\beta \mathbb{E}(W\,|\,\mathbb{F}_1)+c] = \mathbb{E}(Z_1/m)^\alpha\log^\beta(Z_1/m) + c.$$

Therefore $\mathbb{E}W^\alpha(\log^+ W)^\beta<\infty$ implies $\mathbb{E}Z_1^\alpha(\log^+ Z_1)^\beta <\infty$. ■

It remains to prove the Lemma.

Proof of Lemma. Since ϕ is concave with $\phi(0)=0$, ϕ is subadditive. Thus

$$\mathbb{E}S_n^k\phi(S_n) = \mathbb{E}\sum_{i_1,\ldots,i_k=1}^{n} \gamma_{i_1}\ldots\gamma_{i_k}\phi(S_n)$$

$$\leq \mathbb{E}\sum_{i_1,\ldots,i_k=1}^{n} \gamma_{i_1}\ldots\gamma_{i_k}\left[\phi\Big(\sum_{j\notin\{i_1,\ldots,i_k\}}\gamma_j\Big)+\phi\Big(\sum_{j\in\{i_1,\ldots,i_k\}}\gamma_j\Big)\right]$$

$$= \sum_{i_1,\ldots,i_k=1}^{n} \mathbb{E}\gamma_{i_1}\ldots\gamma_{i_k}\mathbb{E}\phi\Big(\sum_{j\notin\{i_1,\ldots,i_k\}}\gamma_j\Big)$$

$$+ \mathbb{E}\sum_{i_1,\ldots,i_k=1}^{n} \gamma_{i_1}\ldots\gamma_{i_k}\left[\phi\Big(\sum_{j\in\{i_1,\ldots,i_k\}}\gamma_j\Big)\right]$$

$$\leq \sum_{i_1,\ldots,i_k=1}^{n} \mathbb{E}\gamma_{i_1}\ldots\gamma_{i_k}\mathbb{E}\phi(S_n)$$

$$+ \mathbb{E}\sum_{i_1,\ldots,i_k=1}^{n} \gamma_{i_1}\ldots\gamma_{i_k}\left[\sum_{j\in\{i_1,\ldots,i_k\}}\phi(\gamma_j)\right]. \tag{7}$$

We now estimate the last two sums. For the first, we write

$$\sum_{i_1,\ldots,i_k=1}^{n} \mathbb{E}\gamma_{i_1}\ldots\gamma_{i_k} = \sum_{(i_1,\ldots,i_k)\in D_k} + \sum_{(i_1,\ldots,i_k)\in D_k^c}, \tag{8}$$

where $D_k:=\{(i_1,\ldots,i_k):\ i_m\neq i_n \text{ if } m\neq n\}\subseteq\{1,\ldots,n\}^k$, and D_k^c is the complement of D in $\{1,\ldots,n\}^k$. We claim that

$$\text{card } D_k^c \leq \frac{1}{2}(k-1)kn^{k-1}. \tag{9}$$

To see this, put $D_{k,1}^c := D_k^c\cap\{i_1=1\}$ and devide it into two parts. The first part is of the form $(1,i_2,\ldots,i_k)$ with $i_m=1$ for some $2\leq m\leq k$, whose cardinality $\leq(k-1)n^{k-2}$. The second part contains the elements $(1,i_2,\ldots,i_k)$ with $i_m\neq i_n$ if $m\neq n$; this part has ardinality $\leq$ card D_{k-1}^c. Hence

$$\text{card } D_{k,1}^c \leq (k-1)n^{k-2} + \text{card } D_{k-1}^c,$$

and consequently

$$\text{card } D_k^c \leq (k-1)n^{k-1} + n\ \text{card } D_{k-1}^c.$$

This gives (9) by induction on k. Since

$$\mathbb{E}\gamma_{i_1}\ldots\gamma_{i_k} \leq (\mathbb{E}\gamma_{i_1}^k)^{1/k}\ldots(\mathbb{E}\gamma_{i_k}^k)^{1/k} = \mathbb{E}\gamma_1^k,$$

by the generalized Höld's inequality, and $\mathbb{E}\gamma_{i_1}\ldots\gamma_{i_k}=\mathbb{E}\gamma_{i_1}\ldots\mathbb{E}\gamma_{i_k}$ if

$(i_1,\dots,i_k)\in D_k$, (8) and (9) give

$$\sum_{1\le i_1,\dots,i_k\le n} \mathbb{E}\gamma_{i_1}\dots\gamma_{i_k} \le \sum_{(i_1,\dots,i_k)\in D_k} \mathbb{E}\gamma_{i_1}\dots\mathbb{E}\gamma_{i_k} + \frac{1}{2}(k-1)k\, n^{k-1}\mathbb{E}\gamma_1^k$$

$$\le (\mathbb{E}S_n)^k + \frac{1}{2}(k-1)k\, n^{k-1}\mathbb{E}\gamma_1^k \qquad (10)$$

We now estimate the second sum in (7). Since ϕ is increasing,

$$(\gamma_i^k-\gamma_j^k)[\phi(\gamma_i^k)-\phi(\gamma_j^k)]\ge 0;$$

so

$$\gamma_i^k\phi(\gamma_j^k) + \gamma_j^k\phi(\gamma_i^k) \le \gamma_i^k\phi(\gamma_i^k)+ \gamma_j^k\phi(\gamma_j^k),$$

and consequently

$$\mathbb{E}\gamma_i^k\phi(\gamma_j^k)\le 2\mathbb{E}\gamma_1^k\phi(\gamma_1^k).$$

Therefore,

$$\mathbb{E}\gamma_{i_1}\dots\gamma_{i_k}\phi(\gamma_j)= \mathbb{E}\gamma_{i_1}\phi^{1/k}(\gamma_j)\dots\gamma_{i_k}\phi^{1/k}(\gamma_j)$$

$$\le [\mathbb{E}\gamma_{i_1}^k\phi(\gamma_j)]^{1/k}\dots[\mathbb{E}\gamma_{i_k}^k\phi(\gamma_j)]^{1/k} \le 2[\mathbb{E}\gamma_1^k\phi(\gamma_1)]. \qquad (11)$$

The conclusion then follows from (7),(10) and (11). ∎

Remark. The result and the proof of the Theorem can obviously be generalized. For example, if $\alpha>1$ is not an integer and

$$\ell(x)= c(\log^+ x)^{\alpha_1}(\log_2^+ x)^{\alpha_2}\dots(\log_k^+ x)^{\alpha_k}$$

($\alpha_i\in\mathbb{R}$), where $\log_1^+ x:=\log^+ x$, $\log_k^+ x:=\log^+\log_{k-1}^+ x$ if $k>1$, and the first non-vanishing α is positive, then

$$\mathbb{E}W^{\alpha}\ell(W)<\infty \textit{ if and only if } \mathbb{E}Z_1^{\alpha}\ell(Z_1)<\infty.$$

Note sur épreuves. -Depuis l'achèvement de cette note, l'auteur s'est rendu compte que N.H.Bingham et R.A.Doney [1974: Asympotonic properties of supercritical branching processes. Adv.Appl.Prob., 6, 711-731] ont obtenu des résultats plus fins par des théorèmes Tauberiens et avec des calculs relativement compliqués; en particulier ils ont montré l'equivalence entre $0<\mathbb{E}W^{\alpha}(\log^+W)^{\beta}<\infty$ et $\mathbb{E}Z_1^{\alpha}(\log^+Z_1)^{\beta}<\infty$ pour tout $\alpha>1$ (entier ou non) et $\beta\ge 0$. Cependant, la simplicité de notre nouvelle preuve nous a paru garder un intérêt.

References

Athreya,K.B.and Ney,P.E. (1972): Branching processes. Springer-Verlag.

Asmussen,S.and Hering,H.(1983): Branching processes. Birkhäuser.

Wen,Z.Y.(1986): Des etudes de certaines processus de naissance. Publication Mathématiques d'Orsay, 86-03.