LAPLACE ASYMPTOTICS FOR GENERALIZED K.P.P. EQUATION

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Abstract. Consider a one dimensional non linear reaction-diffusion equation (KPP equation) with non-homogeneous second order term, discontinuous initial condition and small parameter. For points ahead of the Freidlin-KPP front, the solution tends to 0, and we obtain sharp asymptotics (i.e. non logarithmic). Our study follows the work of Ben Arous and Rouault who solved this problem in the homogeneous case. Our proof is probabilistic, and is based on the Feynman-Kac formula and the large deviation principle satisfied by the related diffusions. We use the Laplace method on Wiener space. The main difficulties come from the non-linearity and the possibility for the endpoint of the optimal path to lie on the boundary of the support of the initial condition.

1. Introduction

The purpose of this paper is to obtain precise (i.e. non logarithmic) asymptotics of $u^\varepsilon(T,x)$ for certain values of (T,x), where $u^\varepsilon(T,x)$ is the solution of generalized KPP equation

$$\begin{cases} \partial_t u^\varepsilon = \dfrac{\varepsilon^2}{2}\,\sigma^2(x)\,\partial^2_{xx}u^\varepsilon + \dfrac{c(x)}{\varepsilon^2}u^\varepsilon(1-r(u^\varepsilon)) \\ u^\varepsilon(0,x) = 1_{\{x\le 0\}} \end{cases} \tag{1.1}$$

where c is a non-negative C^3 function such that there exists $k>0$ satisfying

$$c(x)\le k\;(1+|x|),\quad |c'(x)|\le k\;(1+x^2), \tag{1.2}$$

σ is C^2, with bounded derivatives, such that there exists $M>m>0$ satisfying

$$m\le\sigma(x)\le M, \tag{1.3}$$

r is a one-to-one C^1 increasing function from $[0,1]$ to $[0,1]$.

Our study follows the work of Ben Arous and Rouault (1993) who solved this problem when $\sigma\equiv 1$ and $r(u)=u$.

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The Feynman-Kac formula gives an equivalent form of (1.1), that is

$$u^\varepsilon(T,x) = E1_{\{X_T^\varepsilon \leq 0\}} \exp \frac{1}{\varepsilon^2} \int_0^T c(X_s^\varepsilon)[1 - r(u^\varepsilon(T-s, X_s^\varepsilon))]\, ds \quad (1.4)$$

where X^ε is the diffusion solution of the stochastic differential equation $X_t^\varepsilon = x + \varepsilon \int_0^t \sigma(X_s^\varepsilon) dW_s$. According to the large deviation principle satisfied by the laws of $(X^\varepsilon)_{\varepsilon>0}$, the limit of $\varepsilon^2 \ln E[f(X^\varepsilon) \exp \varepsilon^{-2} F(X^\varepsilon)]$ exists for many continuous functionals f and F. By adapting this technique, Freidlin (1985 and 1990) derived the asymptotics of $\ln u^\varepsilon(T,x)$ (the difficulty coming from the u^ε in the expectation of (1.4)). He proved the existence of a non-positive function $V^*(T,x)$ such that

i) if $V^*(T,x) < 0$, then $\varepsilon^2 \ln u^\varepsilon(T,x) \to V^*(T,x)$

ii) if (T,x) is in the interior of $\{V^* = 0\}$, then $u^\varepsilon(T,x) \to 1$.

This makes clear the propagation of a wave front: (t,x) is ahead (resp. behind) of the front if $u^\varepsilon(t,x) \to 0$ (resp. $u^\varepsilon(t,x) \to 1$).

In order to get precise asymptotics of $E[f(X^\varepsilon) \exp \varepsilon^{-2} F(X^\varepsilon)]$, one can use the Laplace method on Wiener space (see Doss (1980), Azencott (1980-81), Ben Arous (1988)) under the standard hypothesis: the maximum of $F - I$ on $\{\psi \mid f(\psi) \neq 0\}$, where I is the action functional of $(X^\varepsilon)_{\varepsilon>0}$, is attained at a unique path φ, and this maximum is non-degenerate. When f is not continuous at φ, new techniques are required (see Azencott (1985)). We will use them since the initial condition $1_{]-\infty,0]}$ is discontinuous at 0.

However, precise asymptotics of $u^\varepsilon(t,x)$ cannot be obtained by using directly Laplace method because of the presence of u^ε in the expectation of (1.4), presence related to the non-linearity of (1.1). Nevertheless, $u^\varepsilon(t,x)$ approaches 0 with exponential speed ahead of the front. Thus, if the path $s \to X_{T-s}^\varepsilon(\omega)$ stays ahead of the front, we can neglect most of its contribution; i.e. for all $t < T$ and $\alpha > 0$,

$$\exp -\frac{1}{\varepsilon^2} \int_0^t c(X_s^\varepsilon(\omega)) r(u^\varepsilon(T-s, X_s^\varepsilon(\omega)))\, ds = O(\exp -\frac{\alpha}{\varepsilon^2}).$$

More precisely, only $\exp -\frac{1}{\varepsilon^2} \int_{T-\varepsilon^a}^T c(X_s^\varepsilon(\omega)) r(u^\varepsilon(T-s, X_s^\varepsilon(\omega)))\, ds$ contributes ($a \in]0,1[$; $T - \varepsilon^a$ and T define a boundary layer, see Ben Arous and Rouault (1993)).

But what happens for other ω? Under the Laplace method usual hypothesis, only paths $X^\varepsilon(\omega)$ close to the optimal path φ really contribute. So, if we want to approximate the case where the paths $s \to X_{T-s}^\varepsilon(\omega)$ stay ahead of the front, we will assume that φ stays, in reversed time, ahead of the front (i.e. $\forall s \in [0,T[\;\; V^*(T-s, \varphi_s) < 0$). This hypothesis is close to condition (N) of Freidlin (see Freidlin (1985 p. 408), and Freidlin (1990) where one can find several examples where this condition is satisfied). A recent result of Barles and Souganidis (1994) concerning the asymptotics of $\varepsilon^2 \ln\big(1 - u^\varepsilon(t,x)\big)$ behind the front might enable us to carry the proof to the end without this hypothesis.

In section 2, we state our main results, give connections with branching diffusions and summarize the proof, which starts in section 3 where we carry out the Laplace method. We prove in section 4 that only a boundary layer can contribute. In section 5, we construct diffusion bridges using the Skorokhod integral. In section 6, we study the contribution of the non-linear part and end the proof in section 7.

2. Results

Let $T \in]0, +\infty[$ and $x \in \mathbb{R}$. H_x stands for the Cameron-Martin space

$$\{\psi : [0,T] \to \mathbb{R} \mid \psi_0 = x, \psi \text{ absolutely continuous}, \int_0^T \dot{\psi}_s^2 \, ds < \infty\}.$$

If f is a continuous function on $[0,T]$, we let $\|f\| = \sup_{0 \leq t \leq T} |f(t)|$.
If f is continuous on $[a,b]$, we let $\|f\|_a^b = \sup_{a \leq t \leq b} |f(t)|$.
$o(1), O(\varepsilon) \ldots$ and positive constants denoted "cst" are all universal, i.e. may only depend on σ, c, r, T and x.

2.1. The linear problem

We introduce the linear problem related to (1.1)

$$\begin{cases} \partial_t v^\varepsilon = \dfrac{\varepsilon^2}{2}\, \sigma^2(x)\, \partial_{xx}^2 v^\varepsilon + \dfrac{c(x)}{\varepsilon^2}\, v^\varepsilon \\ v^\varepsilon(0,x) = 1_{\{x \leq 0\}} \end{cases} \tag{2.1}$$

Precise asymptotics of v^ε will help computing the ones of u^ε since $u^\varepsilon \leq v^\varepsilon$ (consequence of the maximum principle or of the Feynman-Kac formula). Let

$$F(\psi) = \int_0^T c(\psi_s)\, ds.$$

According to the Feynman-Kac formula,

$$v^\varepsilon(T,x) = E 1_{\{X_T^\varepsilon \leq 0\}} \exp \varepsilon^{-2} F(X^\varepsilon). \tag{2.2}$$

The laws of $(X_s^\varepsilon)_{s \leq T}$ satisfy a large deviation principle with action functional

$$\begin{aligned} I(\psi) &= \frac{1}{2} \int_0^T \dot{\psi}_s^2 S(\psi_s)\, ds \ \text{ if } \psi \in H_x \\ &= +\infty \ \text{ otherwise,} \end{aligned}$$

where $S = \sigma^{-2}$. Therefore, $\lim_{\varepsilon \to 0} \varepsilon^2 \ln v^\varepsilon(T,x) = V(T,x)$, where

$$V(T,x) = \sup\{F(\psi) - I(\psi) \mid \psi \in H_x,\ \psi_T \leq 0\}.$$

We will obtain asymptotics of $v^\varepsilon(T,x)$ under the Laplace method usual hypothesis:

(H1) the maximum in $V(T,x)$ is attained at a unique path φ;
(H2) φ is a non-degenerate maximum.

Let $R = F - I$. For $\psi \in H_x$ and $h \in H_0$,

$$R'(\psi).h = \int_0^T [c'(\psi_s) + \frac{1}{2}\dot{\psi}_s^2 S'(\psi_s) + \ddot{\psi}_s S(\psi_s)]h_s ds - \dot{\psi}_T S(\psi_T) h_T. \quad (2.3)$$

Therefore, (H1) yields:

$$\begin{aligned} & c'(\varphi) + \frac{1}{2}\dot{\varphi}^2 S'(\varphi) + \ddot{\varphi} S(\varphi) = 0 \quad \text{(Euler equation)} \\ & -\dot{\varphi}_T \geq 0 \\ & \varphi_T\, \dot{\varphi}_T = 0 \quad \text{(complementary slackness)} \\ & R'(\varphi).h = -\dot{\varphi}_T S(\varphi_T) h_T \qquad \text{for } h \in H_0. \end{aligned} \quad (2.4)$$

The case $\varphi_T < 0$ can be reduced to a problem without constraint and the result is known (see Azencott (1980-81) and Ben Arous (1988)). We will hence study the case $\varphi_T = 0$.

Remark 2.1. (2.4) has a geometrical interpretation: since R attains its maximum on $E = \{\psi \mid \psi_T \leq 0\}$ on the boundary of E, its gradient at φ and the outwardly normal of E in φ are positively linked.

We will now focus on the meaning of (H2). We have:

$$R''(\varphi).h^2 = \int_0^T [c''(\varphi_s) + \frac{1}{2}\dot{\varphi}_s^2 S''(\varphi_s) + \ddot{\varphi}_s S'(\varphi_s)]h_s^2 ds - \dot{\varphi}_T S'(0) h_T^2 - \langle h, h \rangle$$

where $\langle \ , \ \rangle$ is the scalar product defined on H_0 by $\langle h, h \rangle = \int_0^T \dot{h}_s^2 S(\varphi_s) ds$.

We will say that φ is a non-degenerate maximum if there exists $\lambda > 0$ such that

$$R''(\varphi).h^2 \leq -\lambda \langle h, h \rangle \qquad \text{for all } h \in H_0. \quad (2.5)$$

Introduce now A, the self-adjoint Hilbert-Schmidt operator on H_0 defined by

$$\langle Ah, h \rangle = \int_0^T [c''(\varphi_s) + \frac{1}{2}\dot{\varphi}_s^2 S''(\varphi_s) + \ddot{\varphi}_s S'(\varphi_s)]h_s^2 ds - \dot{\varphi}_T S'(0) h_T^2. \quad (2.6)$$

Since A is self-adjoint and compact, there exists a basis $(f_n)_{n\geq 1}$ of eigenvectors of A, orthonormal with respect to $\langle \ , \ \rangle$. Define $(\lambda_n)_{n\geq 1}$ the corresponding eigenvalues, λ_1 being the largest one. Condition (2.5) is thus equivalent to

$$\lambda_1 < 1. \quad (2.7)$$

We now define a gaussian process and its corresponding bridge by

$$g_1(t) = \int_0^t \sigma(\varphi_s)\, dW_s, \quad g_1^0(t) = g_1(t) - \frac{\int_0^t \sigma^2(\varphi_s)\, ds}{\int_0^T \sigma^2(\varphi_s)\, ds} g_1(T) \ t \leq T. \quad (2.8)$$

By extending definition (2.6) to continuous functions h on $[0, T]$, we define

$$K(h) = \frac{1}{2}\langle Ah, h \rangle \qquad \text{for all } h \in C[0, T]. \quad (2.9)$$

Finally, let $p = -\dot{\varphi}_T \sigma^{-2}(0)$.

THEOREM 2.2. *Assume (H1), (H2) (i.e. (2.5) or (2.7)), $\varphi_T = 0$ and $-\dot{\varphi}_T > 0$. Then*

$$v^\varepsilon(T,x) = [\, A_1 \varepsilon + o(\varepsilon)]\, \exp[\varepsilon^{-2}\, V(T,x)]$$

where $A_1 = p^{-1}(2\pi \int_0^T \sigma^2(\varphi_s) ds)^{-\frac{1}{2}}\, E \exp K(g_1^0)$.

REMARKS 2.3. i) If $\varphi_T < 0$, by directly using the Laplace method, we get

$$v^\varepsilon(T,x) = [E \exp K(g_1) + o(1)]\, \exp[\varepsilon^{-2}\, V(T,x)].$$

ii) If $\varphi_T = 0$ and $-\dot{\varphi}_T = 0$, it is easy to get (see section 3)

$$v^\varepsilon(T,x) = [E 1_{\{g_1(T) \le 0\}} \exp K(g_1) + o(1)]\, \exp[\varepsilon^{-2}\, V(T,x)].$$

The previous constants are finite, as a consequence of

LEMMA 2.4. *There exists $\beta > 0$ such that, for all $\alpha \le \beta$,*

$$E[\exp{(1+\alpha)}\, K(g_1)] < \infty \quad \textit{and} \quad E[\exp{(1+\alpha)}\, K(g_1^0)] < \infty.$$

Proof. We adapt here a computation of Ben Arous (1988). The equality $\langle Ah, h\rangle = \sum_{n=1}^{\infty} \lambda_n \langle h, f_n\rangle^2$ can be extended by density to continuous martingales h such that $h(0) = 0$ if we let $\langle h, f_n\rangle = \int_0^T \dot{f}_n(s) S(\varphi_s) dh_s$. Moreover,

$$E[\langle g_1, f_n\rangle \langle g_1, f_m\rangle] = \int_0^T \dot{f}_n(s) S(\varphi_s)\sigma(\varphi_s) \dot{f}_m(s) S(\varphi_s)\sigma(\varphi_s)\, ds = \langle f_n, f_m\rangle.$$

Hence, $\langle g_1, f_n\rangle$ are independent, gaussian, centered, reduced. Since A is defined by restriction to H_0 of a continuous quadratic form on $C[0,T]$, A is a trace class operator H_0 (see Kuo (1975) th. 4.6 p.83), i.e. $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. Hence, for $(1+\alpha)\lambda_1 < 1$, we have

$$\begin{aligned} E[\exp{(1+\alpha)}\, K(g_1)] &= \prod_n E[\exp \frac{1+\alpha}{2} \lambda_n \langle g_1, f_n\rangle^2] \\ &= \prod_n [1 - (1+\alpha)\lambda_n]^{-\frac{1}{2}} \\ &= \big(\det[I - (1+\alpha)A]\big)^{-\frac{1}{2}} < \infty. \end{aligned}$$

A similar computation with the restriction of A to $\{h \in H_0 / h(T) = 0\}$ yields $E \exp(1+\alpha) K(g_1^0) < \infty$.

□

2.2. The nonlinear problem

Let

$$V^*(T,x) = \sup\{\inf_{0\le t\le T} \int_0^t [c(\psi_s) - \frac{1}{2}\dot\psi_s^2 S(\psi_s)]\, ds \ \mid \psi \in \ H_x,\ \psi_T \le 0\}.$$

Our first hypothesis is

(H3) i) The maximum in $V^*(T,x)$ is attained at a unique path φ,
ii) $V^*(s,\varphi_{T-s}) < 0$ for all $s \in]0,T]$,
iii) $V^*(T,x) = V(T,x)$.

ii) means that the optimal path φ runs always ahead of the front. ii) and iii) are satisfied under condition (N) of Freidlin.

(H3) is equivalent to $V^*(T,x) < 0$ and the set of (ψ,t) realizing the equality in $V^*(T,x)$ is a singleton of the form (φ,T). This was proved by Ben Arous and Rouault (1993) in the case $\sigma \equiv 1$, and it can be extended easily.

(H3) yields $x > 0,\ \varphi_T = 0,\ -\dot\varphi_T \ge 0$, and $c(0) - \frac{1}{2}\dot\varphi_T^2 S(0) \le 0$.

We need more than this last inequality to analyse the boundary layer, i.e.

(H4) $-\dot\varphi_T > \sqrt{2c(0)\sigma^2(0)}$.

(H4) means that φ, in reversed time, moves quickly away from the front.

Since (H3) implies $V(T,x) = R(\varphi)$, we can select, as a non-degeneracy hypothesis for the nonlinear problem, that of the linear problem (H2).

Finally, let

$$g(y) = E\exp -c(0)\int_0^{+\infty} r(\tilde u(s, -\dot\varphi_T s + y + \sigma(0)W_s))ds \quad \text{where}$$

$$\begin{cases} \partial_t \tilde u = \dfrac{1}{2}\,\sigma^2(0)\,\partial^2_{xx}\tilde u + c(0)\tilde u(1-r(\tilde u)) \\ \tilde u(0,x) = 1_{\{x\,\le\,0\}} \end{cases}$$

Theorem 2.5. *Under hypothesis (H2), (H3) and (H4),*

$$u^\varepsilon(T,x) = [\,A_2\,\varepsilon + o(\varepsilon)]\,\exp[\varepsilon^{-2}\,V^*(T,x)]$$

where $A_2 = \displaystyle\int_{-\infty}^0 g(y)\exp py\ dy\ (2\pi\int_0^T \sigma^2(\varphi_s)ds)^{-\frac{1}{2}}\,E\exp\ K(g_1^0)$.

Remarks 2.6. i) If $\sigma \equiv 1$, we can weaken the hypothesis on r: r is a one-to-one increasing continuous function from $[0,1]$ to $[0,1]$, C^1 on $]0,1]$, $\lim_{u\to 0} ur'(u) = 0$ and

$$\exists\,\theta > 0 \quad \int_0^1 \frac{r(u)}{u^{1+\theta}}\,du < +\infty. \tag{2.10}$$

ii) By using transformations $x \to x-a$ or $x \to -x+a$, we get similar results with initial condition $f(x) = 1_{\{x\,\le\,a\}}$ or $f(x) = 1_{\{x\,\ge\,a\}}$.

We can also treat the case $f(x) = 0$ if $x \geq 0$, $f(x) \in]0,1]$ if $x < 0$ and f smooth.

Let $k = \inf\{n \geq 1 \mid f^{(n)}(0) \neq 0\}$. We have

$$u^\varepsilon(T,x) = [\, A_3\, \varepsilon^{k+1} + o(\varepsilon^{k+1})]\, \exp[\varepsilon^{-2}\, V^*(T,x)]$$

where $A_3 = \dfrac{f^{(k)}(0)}{k!} \displaystyle\int_{-\infty}^0 y^k g(y) \exp py\; dy\; (2\pi \int_0^T \sigma^2(\varphi_s) ds)^{-\frac{1}{2}}\, E \exp\, K(g_1^0)$.

iii) Let us prove that $g(y) > 0$ for all y, and therefore $A_2 > 0$.

Let $\alpha \in]\sqrt{2c(0)\sigma^2(0)}, -\dot{\varphi}_T[$. We have a.s.

$$\exists s_0 \geq 0\ \forall s \geq s_0 \quad -\dot{\varphi}_T s + y + \sigma(0) W_s \geq \alpha s.$$

Since $\tilde{u}(s,.)$ is non increasing (Kolmogorov, Petrovskii and Piscunov(1937)), we get a.s.

$$\forall s \geq s_0 \quad \tilde{u}(s, -\dot{\varphi}_T s + y + \sigma(0) W_s) \leq \tilde{u}(s, \alpha s).$$

Since $\alpha > \sqrt{2c(0)\sigma^2(0)}$, we have (see Freidlin (1985)),

$$\lim_{s\to+\infty} \frac{\ln \tilde{u}(s,\alpha s)}{s} = -\frac{1}{2}\Big(\frac{\alpha^2}{\sigma^2(0)} - 2c(0)\Big).$$

Therefore, $\displaystyle\int_0^{+\infty} r(\tilde{u}(s, -\dot{\varphi}_T s + y + \sigma(0) W_s)) ds < \infty$ a.s. and $g(y) > 0$.

2.3. Connection with branching diffusions

Let λ be a non-negative function on $\mathbb{R}$. Consider the following branching diffusion:
i) a particle starts from $x \in \mathbb{R}$, and executes a small diffusion

$$dX_s^\varepsilon = \varepsilon \sigma(X_s^\varepsilon) dW_s$$

ii) its lifetime τ is given by

$$P(\tau \in\ [t, t+dt[\ \mid X_t^\varepsilon = y,\ \tau \geq t) = \varepsilon^{-2} \lambda(y) dt + o(dt)$$

iii) when it dies, it is replaced by a random number of descendants N
iv) each descendant, starting from where its parent died, repeats i), ii), iii).
All diffusions, lifetimes and number of descendants are independent of one another.

Let N_t^ε be the number of particles in $]-\infty, 0]$ at time t.

Then, $P_x(N_t^\varepsilon \neq 0)$ is the solution of

$$\begin{cases} \partial_t u^\varepsilon = \dfrac{\varepsilon^2}{2}\, \sigma^2(x)\, \partial^2_{xx} u^\varepsilon + \dfrac{c(x)}{\varepsilon^2} f(u^\varepsilon) \\ u^\varepsilon(0,x) = 1_{\{x \leq 0\}} \end{cases}$$

where $c(x) = (EN - 1)\lambda(x)$ and $f(u) = (EN-1)^{-1}\big(1 - u - E(1-u)^N\big)$. Assume $P(N=0) = 0$ and $EN > 1$. Then f is a KPP type non-linearity, i.e.

$$f(0) = f(1) = 0, \quad f(u) > 0 \ \text{ for } u \in]0,1[$$
$$f \in C^1[0,1], \quad f'(u) < f'(0) \ \text{ for } u \in]0,1].$$

Define r such that $f(u) = u(1 - r(u))$. Then r is increasing continuous one-to-one from $[0,1]$ to $[0,1]$, C^1 on $]0,1]$, and $\lim_{u\to 0} ur'(u) = 0$.

The additional hypothesis made on r can be expressed in terms of N:
- r is C^1 on $[0,1]$ if and only if $E(N^2) < \infty$ (easy check).
- (2.10) holds if and only if $E(N(\log N)^{1+\theta}) < \infty$ (see Athreya and Ney (1972) p. 26).

In the homogeneous case (σ and c constant), Chauvin and Rouault (1988, th.1) obtained asymptotics of $P_x(N_t^\varepsilon \neq 0)$ for (t,x) in the sub-critical area (i.e. ahead of the front) under the weaker condition $E(N \log N) < \infty$.

We can translate our results into branching diffusions language: since $E_x N_t^\varepsilon$ is solution of linear equation (2.1), we get, under hypothesis of th. 2.1 and 2.3, asymptotics for $E_x N_T^\varepsilon$ and $P_x(N_T^\varepsilon \neq 0)$. For instance, $E_x[N_T^\varepsilon \mid N_T^\varepsilon \neq 0]$ goes to a finite limit. This means that, when $]-\infty, 0]$ is visited at time T (a rare event), the average number of particles in this area is finite.

2.4. Summary of the proof

Starting with (2.2) and (1.4), we implement the Laplace method (section 3) which consists in localizing around φ, applying the Girsanov formula, then performing a stochastic Taylor expansion of the diffusion $Z_t^\varepsilon = \varepsilon \int_0^t \sigma(\varphi_s + Z_s^\varepsilon) dW_s$ of the form $Z^\varepsilon = \varepsilon g_1 + \varepsilon^2 g_2 +$ remainder (g_1 is gaussian). We get $v^\varepsilon(T,x) \sim v_1^\varepsilon \exp[\varepsilon^{-2} V(T,x)]$ and $u^\varepsilon(T,x) \sim u_1^\varepsilon \exp[\varepsilon^{-2} V^*(T,x)]$ where

$$v_1^\varepsilon = E 1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \exp[\, p\,(\varepsilon^{-1} g_1(T) + g_2(T)) + K(g_1)]$$
$$u_1^\varepsilon = E 1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}}$$
$$\exp[\, p\,(\varepsilon^{-1} g_1(T) + g_2(T)) + K(g_1) - \varepsilon^{-2} F_\varepsilon(0, T, \varphi + Z^\varepsilon)]$$
$$F_\varepsilon(t, t', \psi) = \int_t^{t'} c(\psi_s) r(u^\varepsilon(T-s, \psi_s))\, ds.$$

For v_1^ε, we now use the following strategy.

i) We prove $v_1^\varepsilon = E\Psi_1^\varepsilon(g_1, g_2) + o(\varepsilon)$ for some Ψ_1^ε (lemma 7.1).
ii) We construct a process Δ independent of $g_1(T)$ such that g_1 and g_2 can be expressed in terms of Δ and $g_1(T)$ (lemma 5.1).
iii) Therefore $v_1^\varepsilon = E\Psi_2^\varepsilon(\Delta, g_1(T)) + o(\varepsilon)$ for some Ψ_2^ε. We condition on Δ and prove that the gaussian integral $\varepsilon^{-1} E\Psi_2^\varepsilon(\delta, g_1(T))$ goes to a non zero finite limit (for fixed δ). This implies $v_1^\varepsilon \sim cst\ \varepsilon$ (section 7.1).

Concerning the nonlinear problem, we prove in section 4 that we can neglect $F_\varepsilon(0, T - \varepsilon^a, \varphi + Z^\varepsilon)$ for $a \in]0,1[$, i.e. $u_1^\varepsilon = u_2^\varepsilon + o(\varepsilon)$ where

$$u_2^\varepsilon = E 1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \tag{2.11}$$
$$\exp[\, p\,(\varepsilon^{-1} g_1(T) + g_2(T)) + K(g_1) - \varepsilon^{-2} F_\varepsilon(T - \varepsilon^a, T, \varphi + Z^\varepsilon)].$$

For u_2^ε, the strategy is close to that used for v_1^ε.

i) We condition on $\sigma(Z_s^\varepsilon, s \le T-\varepsilon^d;\ Z_T^\varepsilon)$ for $d \in]0,a[$.
From $F_\varepsilon(T-\varepsilon^a, T, \varphi + Z^\varepsilon)$ arises a functional of Z^ε conditioned on its final position Z_T^ε studied in section 6.
ii) We prove $u_2^\varepsilon = E\Psi_3^\varepsilon(g_1, g_2) + o(\varepsilon)$ for some Ψ_3^ε (lemma 7.4).
iii) We construct a process Δ_ε independent of $(g_1(T-\varepsilon^d), g_1(T))$ such that g_1 and g_2 can be expressed in terms of Δ_ε and $(g_1(T-\varepsilon^d), g_1(T))$ (lemma 5.3).
iv) Therefore $u_2^\varepsilon = E\Psi_4^\varepsilon(\Delta_\varepsilon, g_1(T-\varepsilon^d), g_1(T)) + o(\varepsilon)$ for some Ψ_4^ε. We condition on Δ_ε and prove that the gaussian double integral $\varepsilon^{-1}E\Psi_4^\varepsilon(\delta, g_1(T-\varepsilon^d), g_1(T))$ goes to a non zero finite limit (for fixed δ). This implies $u_2^\varepsilon \sim cst\ \varepsilon$ (section 7.2).

3. The Laplace method

Let $\rho > 0$. By (H1), (1.2) and large deviation arguments, there exists $\tau > 0$ such that

$$v^\varepsilon(T,x) = E[1_{\{X_T^\varepsilon \le 0,\, \|X^\varepsilon - \varphi\| \le \rho\}} \exp \frac{F(X^\varepsilon)}{\varepsilon^2}] + O[\exp \frac{(V(T,x)-\tau)}{\varepsilon^2}].$$

Then, we apply the Girsanov formula and get (see Azencott (1980-81) pp. 265-266)

$$\begin{aligned} v^\varepsilon(T,x) &= E1_{\{Z_T^\varepsilon \le 0,\ \|Z^\varepsilon\| \le \rho\}} \exp \varepsilon^{-2}[F(\varphi+Z^\varepsilon) - G(Z^\varepsilon)] \quad (3.1)\\ &\qquad + O[\exp \varepsilon^{-2}\,(V(T,x)-\tau)] \\ Z_t^\varepsilon &= \varepsilon \int_0^t \sigma(\varphi_s + Z_s^\varepsilon) dW_s \\ G(Z^\varepsilon) &= \int_0^T \dot\varphi_s\, S(\varphi_s + Z_s^\varepsilon)[dZ_s^\varepsilon + \frac{1}{2}\,\dot\varphi_s\, ds]. \end{aligned}$$

Z^ε is known to have the following stochastic Taylor expansion (see Azencott (1980-81) p. 251)

$$Z^\varepsilon = \varepsilon\, g_1 + \varepsilon^2\, g_2 + \varepsilon^3 \Gamma^\varepsilon \quad \text{where} \tag{3.2}$$

$$g_1(t) = \int_0^t \sigma_0(s) dW_s, \quad g_2(t) = \int_0^t \sigma_1(s) g_1(s) dW_s, \quad \text{and} \quad \sigma_i(s) = \sigma^{(i)}(\varphi_s).$$

The remainder Γ^ε is such that: $\exists\, c_1, c_2 > 0 \quad \forall \rho \ge \varepsilon > 0 \quad \forall r \ge c_1\, \rho^{-1}$

$$P(\|Z^\varepsilon\| \le \rho\,,\, \varepsilon\,\|\Gamma^\varepsilon\| \ge r) \le \exp - c_2 r\, \rho^{-1}. \tag{3.3}$$

(3.3) yields, for fixed $\alpha > 0$ and $\rho < c_2\, \alpha^{-1}$,

$$\sup_{\varepsilon \in\ ]0,\rho]} E1_{\{\|Z^\varepsilon\| \le \rho\}} \exp\ \alpha\, \varepsilon\, \|\Gamma^\varepsilon\| \le cst(\alpha, \rho, c_1, c_2).$$

Remark 3.1. Azencott inequalities can sometimes be improved since their left-hand side members are non-decreasing function of ρ.

Thus, the previous inequality becomes: $\forall \alpha > 0 \quad \exists \rho_1(\alpha) > 0$

$$\sup_{\varepsilon \leq \rho \leq \rho_1(\alpha)} E 1_{\{\|Z^\varepsilon\| \leq \rho\}} \exp \alpha \varepsilon \, \|\Gamma^\varepsilon\| < \infty \tag{3.4}$$

Let $\overline{Z}^\varepsilon = \varepsilon g_1 + \varepsilon^2 g_2$. By composition of Taylor expansions, we get

$$F(\varphi + Z^\varepsilon) - G(Z^\varepsilon) = F(\varphi) + F'(\varphi).\overline{Z}^\varepsilon + \frac{\varepsilon^2}{2} F''(\varphi).g_1^2 - \overline{G}_\varepsilon + \varepsilon^3 \Lambda_\varepsilon \tag{3.5}$$

$$\text{where} \quad \overline{G}_\varepsilon = \int_0^T \dot{\varphi}_s [S(\varphi_s) + S'(\varphi_s).\overline{Z}_s^\varepsilon + \frac{\varepsilon^2}{2} S''(\varphi_s).g_1^2(s)][d\overline{Z}_s^\varepsilon + \frac{1}{2}\dot{\varphi}_s ds].$$

The remainder $\varepsilon^3 \Lambda_\varepsilon$ satisfies (see Azencott (1980-81) p. 271 and remark 3.1)

$$\forall \alpha > 0 \quad \exists \rho_2(\alpha) > 0 \qquad \sup_{\varepsilon \leq \rho \leq \rho_2(\alpha)} E 1_{\{\|Z^\varepsilon\| \leq \rho\}} \exp \alpha \varepsilon |\Lambda_\varepsilon| < \infty \tag{3.6}$$

$$\forall \rho > 0 \, \exists c_3(\rho) > 0 \, \forall r \geq c_3(\rho) \;\; P(\|Z^\varepsilon\| \leq \rho \, , |\Lambda_\varepsilon| \geq r) \leq \exp - \frac{r^{2/3}}{c_3(\rho)}. \tag{3.7}$$

We transform (3.5) by using (2.9) and (2.3) (which is still meaningful when h is a continuous martingale on $[0, T]$ such that $h(0) = 0$). We get

$$F(\varphi + Z^\varepsilon) - G(Z^\varepsilon) = V(T, x) + R'(\varphi).\overline{Z}^\varepsilon + \varepsilon^2 K(g_1) + \varepsilon^3 \Lambda_\varepsilon. \tag{3.8}$$

We can extend the identity $R'(\varphi).h = ph_T$ by density to continuous martingales h such that $h(0) = 0$. Hence, according to (3.1) and (3.8)

$$v^\varepsilon(T, x) = \exp[\frac{V(T, x)}{\varepsilon^{-2}}] \, E 1_{\{Z_T^\varepsilon \leq 0, \; \|Z^\varepsilon\| \leq \rho\}} \exp[\frac{p\overline{Z}_T^\varepsilon}{\varepsilon^2} + K(g_1) + \varepsilon \Lambda_\varepsilon] \\ + O[\exp \varepsilon^{-2} \, (V(T, x) - \tau)].$$

Let $v_1^\varepsilon = E 1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \exp[p \, \overline{Z}_T^\varepsilon \, \varepsilon^{-2} + K(g_1)]$.

Lemma 3.2. *If ρ is small enough, then*

$$v^\varepsilon(T, x) = [v_1^\varepsilon + o(v_1^\varepsilon) + o(\varepsilon)] \exp[\varepsilon^{-2} \, V(T, x)\,].$$

Proof. Let $b \in]0, 1[$.

It is enough to prove $w_1^\varepsilon = o(\varepsilon)$ and $w_2^\varepsilon = o(v_1^\varepsilon) + o(\varepsilon)$ where

$$w_1^\varepsilon = E 1_{\{|\Lambda_\varepsilon| \geq \varepsilon^{b-1}, \, Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \exp[\frac{p\overline{Z}_T^\varepsilon}{\varepsilon^2} + K(g_1)][\exp(\varepsilon \Lambda_\varepsilon) - 1]$$

$$w_2^\varepsilon = E 1_{\{|\Lambda_\varepsilon| \leq \varepsilon^{b-1}, \, Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \exp[\frac{p\overline{Z}_T^\varepsilon}{\varepsilon^2} + K(g_1)][\exp(\varepsilon \Lambda_\varepsilon) - 1]$$

Lemma 2.2, (3.4), (3.6) and Hölder inequality yield: $\forall \alpha \in]0, \beta[\;\; \exists \rho_3(\alpha) > 0$

$$\sup_{\varepsilon \leq \rho \leq \rho_3(\alpha)} E 1_{\{\|Z^\varepsilon\| \leq \rho\}} \exp \, (1 + \alpha)[-p\varepsilon \Gamma_T^\varepsilon + K(g_1) + \varepsilon |\Lambda_\varepsilon|] < \infty, \tag{3.9}$$

(see remark 3.1), which implies the very useful result

$$\sup_{\varepsilon\le\rho\le\rho_3(\alpha)} E1_{\{\|Z^\varepsilon\|\le\rho\}} \exp{(1+\alpha)[-p\varepsilon\Gamma_T^\varepsilon + K(g_1)]} < \infty. \tag{3.10}$$

(3.7) and (3.9) yield: for $\varepsilon\le\rho\le\rho_3(\frac{\beta}{2})$ and $\varepsilon^{b-1}\ge c_2(\rho)$,

$$\begin{aligned}|w_1^\varepsilon| &\le E1_{\{\varepsilon|\Lambda_\varepsilon|\ge\varepsilon^b,\, \|Z^\varepsilon\|\le\rho\}}\exp[-p\varepsilon\Gamma_T^\varepsilon + K(g_1)+\varepsilon|\Lambda_\varepsilon|]\\ &= O(P(|\Lambda_\varepsilon|\ge\varepsilon^{b-1}, \|Z^\varepsilon\|\le\rho)^{1-2(2+\beta)^{-1}}) = O(\exp - cst\ \varepsilon^{\frac{2}{3}(b-1)}).\end{aligned}$$

Let $w_3^\varepsilon = E1_{\{\varepsilon|\Lambda_\varepsilon|\le\varepsilon^b,\, Z_T^\varepsilon\le 0,\, \|Z^\varepsilon\|\le\rho\}}\exp[p\overline{Z}_T^\varepsilon\varepsilon^{-2}+K(g_1)+\varepsilon\Lambda_\varepsilon]$.

Then, according to (3.7) and (3.9)

$$|w_2^\varepsilon|\le w_3^\varepsilon(\exp\varepsilon^b - 1) \text{ and } v_1^\varepsilon - w_3^\varepsilon = O(\exp - cst\ \varepsilon^{\frac{2}{3}(b-1)}),$$

and therefore $w_2^\varepsilon = o(v_1^\varepsilon)+o(\varepsilon)$.

□

Let us now deal with the nonlinear problem. Let

$$u_1^\varepsilon = E1_{\{Z_T^\varepsilon\le 0,\, \|Z^\varepsilon\|\le\rho\}}\exp[\frac{p\bar{Z}_T^\varepsilon}{\varepsilon^2}+K(g_1)-\frac{1}{\varepsilon^2}F_\varepsilon(0,T,\varphi+Z^\varepsilon)]. \tag{3.11}$$

Lemma 3.3. *If ρ is small enough, then*

$$u^\varepsilon(T,x) = [u_1^\varepsilon + o(u_1^\varepsilon)+o(\varepsilon)]\exp[\varepsilon^{-2}V^*(T,x)].$$

Proof. If (H1) holds, the proof for the localization is the same. If not, the large deviation argument is not sufficient any more. Ben Arous and Rouault (1993, appendix 3) proved that we can neglect the contribution of the other maxima if $\sigma\equiv 1$. This proof can be extended easily.

The other computations carried out for $v^\varepsilon(T,x)$ are still valid for $u^\varepsilon(T,x)$.

□

4. The boundary layer

This section concerns the nonlinear problem. Let $a\in\,]0,1[$, and $T(\varepsilon) = T-\varepsilon^a$. Ben Arous and Rouault (1993) proved that, when $\sigma\equiv 1$ and $r(u)=u$, (H3ii) and (H4) allow us to neglect $F_\varepsilon(0,T(\varepsilon),\varphi+Z^\varepsilon)$ in (3.11). The proof can be extended easily. We will not deal with it in detail. Let $\overline{f}(\eta) = \sup_{|y|\le\eta}|f(y)|$ for any function f.

Lemma 4.1. *Let $\eta>0$. Then,*

$$\overline{\sigma}(\eta)\sqrt{2(\overline{c}(2\eta)+\eta)}s \le y\le \overline{\sigma}(\eta)\eta M^{-1} \Longrightarrow u^\varepsilon(s,y)\le 4\exp -\eta s\,\varepsilon^{-2}.$$

Proof. We apply the strong Markov property in (1.4) with $\tau=\inf\{u\le s \mid |X_u^\varepsilon|>\eta\}$. □

LEMMA 4.2. *For ρ small enough,* $u_1^\varepsilon = u_2^\varepsilon + o(\varepsilon)$ *where*

$$u_2^\varepsilon = E1_{\{Z_T^\varepsilon \le 0, \|Z^\varepsilon\| \le \rho\}} \exp[p\overline{Z}_T^\varepsilon \varepsilon^{-2} + K(g_1) - \varepsilon^{-2} F_\varepsilon(T(\varepsilon), T, \varphi + Z^\varepsilon)]. \tag{4.1}$$

Proof . For $\delta > 0$, let us define the event

$$G = \{\overline{\sigma}(\eta)\sqrt{2(\overline{c}(2\eta)+\eta)}(T-s) \le \varphi_s + Z_s^\varepsilon \le \frac{\overline{\sigma}(\eta)\eta}{M},\ \forall s \in\ [T-\delta, T(\varepsilon)]\ \}.$$

(H4) yields the existence of ρ, δ, η such that, for ε small enough,

$$P\big((\Omega \setminus G) \cap \{\|X^\varepsilon\| \le \rho\}\big) = O(\exp -cst\, \varepsilon^{a-2}).$$

Thus, it is sufficient to find an upper bound for

$$\begin{aligned} w^\varepsilon = E1_{G \cap \{\|X^\varepsilon\| \le \rho\}} & 1_{\{Z_T^\varepsilon \le 0, \|Z^\varepsilon\| \le \rho\}} \\ & [1 - \exp -\varepsilon^{-2} F_\varepsilon(0, T(\varepsilon), \varphi + Z^\varepsilon)] \exp[p\, \overline{Z}_T^\varepsilon \varepsilon^{-2} + K(g_1)]. \end{aligned}$$

According to lemma 4.1, on G,

$$\begin{aligned} F_\varepsilon(T-\delta, T(\varepsilon), \varphi + Z^\varepsilon) &\le \overline{c}(\eta) \int_{\varepsilon^a}^{\delta} r(4\exp -\eta s \varepsilon^{-2})\ ds \\ &= \varepsilon^2 \overline{c}(\eta)\eta^{-1} \int_{4\exp -\eta\delta\varepsilon^{-2}}^{4\exp -\eta\varepsilon^{-2+a}} \frac{r(u)}{u}\ du \\ &\le \begin{cases} cst\ \varepsilon^2 \overline{c}(\eta)\eta^{-1} \exp -\eta\varepsilon^{-2+a} & \text{if } r \text{ is } C^1 \text{ on } [0,1] \\ cst\ \varepsilon^2 \overline{c}(\eta)\eta^{-1} \exp -\theta\eta\varepsilon^{-2+a} & \text{if (2.10) holds.} \end{cases} \end{aligned} \tag{4.1}$$

(H3ii) yields, on $\{\|X^\varepsilon\| \le \rho\}$ and for $s \in\ [0, T-\delta]$,

$$\begin{aligned} u^\varepsilon(s, \varphi_s + Z_s^\varepsilon) &\le \exp -cst\, \varepsilon^{-2} \\ F_\varepsilon(0, T-\delta, \varphi + Z^\varepsilon) &\le cst\ r(\exp -cst\, \varepsilon^{-2}) \\ &\le \begin{cases} cst\ \exp -cst\, \varepsilon^{-2} & \text{if } r \text{ is } C^1 \text{ on } [0,1] \\ cst\ \varepsilon^4 & \text{if (2.13) holds.} \end{cases} \end{aligned} \tag{4.2}$$

Actually, the fact that r is non-decreasing and (2.13) imply: $r(\exp -\varepsilon^{-1}) = O(\varepsilon^\mu)$ for all $\mu > 0$. Therefore, according to (4.1) and (4.2), $F_\varepsilon(0, T(\varepsilon), \varphi + Z^\varepsilon) = o(\varepsilon^3)$, and

$$w^\varepsilon \le o(\varepsilon)\ E1_{\{\|Z^\varepsilon\| \le \rho\}} \exp\big(-p\varepsilon\Gamma_T^\varepsilon + K(g_1)\big).$$

We conclude by using (3.10). □

5. Construction of diffusion bridges

To understand the asymptotics of v_1^ε and u_2^ε, Ben Arous and Rouault ((1993) pp. 272, 274) introduced the brownian bridge $W^0 = \left(W_s - s\,T^{-1}\,W_T\right)_{0\le s\le T}$, and used independence between W^0 and W_T. In this section, we construct "bridges" associated to the non gaussian diffusion (g_1, g_2).

For $\alpha, \lambda, \mu > 0$, define $\mathcal{E}(\alpha, \lambda, \mu)$ to be the set of continuous processes X such that

$$\forall r \ge 0 \quad P(\|X\| \ge r) \le \lambda\ \exp -\mu r^\alpha.$$

Let $\mathcal{E} = \cup_{\alpha,\,\lambda,\,\mu\,>\,0}\,\mathcal{E}(\alpha, \lambda, \mu)$ and $Y = -g_1(T)$.

Lemma 5.1. Construction of a single bridge Δ

There exists a process $\Delta = \{G_{10}, G_{20}, G_{21}\}$, independent of Y, whose components are in $\mathcal{E}$, and there exist G_{11} and G_{22} in $C^1[0,T]$ such that $G_{10} = g_1^0$ and

$$g_1 = G_{10} + G_{11}Y, \quad g_2 = G_{20} + G_{21}Y + G_{22}Y^2.$$

Proof . Let $a_t = -(\int_0^T \sigma_0^2(s)ds)^{-1} \int_0^t \sigma_0(s)ds$ and $\widetilde{W}_t = W_t - a_tY$ for $t \in [0,T]$. The gaussian process $\widetilde{W}$ is independent of Y.

Then, define $\int_0^t X_s d\widetilde{W}_s = \int_0^t X_s dW_s - Y \int_0^t X_s da_s$ for X a continuous and adapted process of $L^2(\Omega \times [0,T])$. We have,

$$g_1 = G_{10} + G_{11}Y, \qquad G_{10}(.) = \int_0^\cdot \sigma_0(s)d\widetilde{W}_s, \qquad G_{11}(.) = \int_0^\cdot \sigma_0(s)da_s$$

$$g_2(.) = \int_0^\cdot \sigma_1(s)[G_{10}(s) + G_{11}(s)Y]dW_s.$$

Since G_{10} and $G_{11}Y$ are not adapted to the filtration of W, we cannot develop this previous expression of g_2 by linearity. So we use the Skorokhod integral which extends Itô integral and accepts non-adapted integrands. $\int_0^t X_s \delta W_s$ stands for the Skorokhod integral of $X \in \mathrm{dom}\ \delta$. Define $\delta\widetilde{W}_s$ in the natural way. We have,

$$\begin{aligned} g_2(.) &= \int_0^\cdot \sigma_1(s)G_{10}(s)\delta W_s + \int_0^\cdot \sigma_1(s)G_{11}(s)Y\delta W_s \\ &= \int_0^\cdot \sigma_1(s)G_{10}(s)\delta\widetilde{W}_s + Y\int_0^\cdot \sigma_1(s)G_{10}(s)da_s \\ &\qquad + \int_0^\cdot Y\sigma_1(s)G_{11}(s)\delta\widetilde{W}_s + Y^2\int_0^\cdot \sigma_1(s)G_{11}(s)da_s. \end{aligned}$$

We know that (see for instance Nualart and Pardoux (1988)),

$$\begin{aligned} &\int_0^t Y\sigma_1(s)G_{11}(s)\delta W_s = \\ &\qquad Y\int_0^t \sigma_1(s)G_{11}(s)dW_s - \int_0^t \sigma_1(s)G_{11}(s)D_sY\,ds \end{aligned} \tag{5.1}$$

where D_s stands for the Malliavin derivative. Since $D_sY = -\sigma_0(s)$, we get

$$g_2 = G_{22}Y^2 + G_{21}Y + G_{20}, \qquad G_{22}(.) = \int_0^{\cdot} \sigma_1(s)G_{11}(s)da_s$$

$$G_{20}(.) = \int_0^{\cdot} \sigma_1(s)G_{10}(s)\delta\widetilde{W}_s + \int_0^{\cdot} \sigma_0(s)\sigma_1(s)G_{11}(s)ds$$

$$G_{21}(.) = \int_0^{\cdot} \sigma_1(s)G_{10}(s)da_s + \int_0^{\cdot} \sigma_1(s)G_{11}(s)\delta\widetilde{W}_s.$$

Let $\mathcal{H}_m$ be the m$^{\text{th}}$ Wiener chaos.

LEMMA 5.2. *Let $X = (X^1, \ldots, X^n)$ be a continuous $\mathbb{R}^n$-valued process. Assume*

i) $(X, \widetilde{W})$ and Y are independent

ii) $\forall i \in \{1, \ldots, n\}$ $\forall s \in [0,T]$ $X_s^i \in \mathcal{H}_0 \oplus \mathcal{H}_1$.

Then, $(\int_0^{\cdot} X_s\delta\widetilde{W}_s,\ \widetilde{W})$ and Y are independent.

Lemma 5.2. is proved below. Lemma 5.2 with $X = \sigma_0$ yields independence between $(G_{10}, \widetilde{W})$ and Y. Then, with $X = (\sigma_0,\ \sigma_1 G_{10},\ \sigma_1 G_{11})$, it yields that (G_{10}, G_{20}, G_{21}) and Y are independent.

It remains to prove that G_{10}, G_{20} and G_{21} belong to $\mathcal{E}$. For G_{10} and G_{21}, it is a consequence of the stability of $\mathcal{E}$ under sum, product, Riemann and Itô integration (see Azencott (1980-81) p. 252). For G_{20}, we just need to prove $\int_0^{\cdot} \sigma_1(s)G_{10}(s)\delta W_s \in \mathcal{E}$. The space $\mathcal{E}$ is not stable under Skorokhod integration, but identity $G_{10} = g_1 - YG_{11}$ and (5.1) allow us to conclude. □

Proof of lemma 5.2. Let us prove that

$$Ef(\int_0^t X_s\delta\widetilde{W}_s,\ \widetilde{W}_t)h(Y) = Ef(\int_0^t X_s\delta\widetilde{W}_s,\ \widetilde{W}_t)\ Eh(Y)$$

for $t \in [0,T]$ and f, h bounded continuous functions (we only deal with one-dimensional marginals not to overload the notations).

Let $\Pi^p:\ 0 = t_{0,p} < \cdots < t_{p,p} = t$ be a sequence of partitions of $[0,t]$ whose meshes go to 0, and let

$$X_p = \sum_{k=0}^{p-1} \overline{X}_{k,p} 1_{[t_{k,p},\, t_{k+1,p}[}, \qquad \overline{X}_{k,p} = (t_{k+1,p} - t_{k,p})^{-1} \int_{t_{k,p}}^{t_{k+1,p}} X_u\, du.$$

We have (see Nualart and Pardoux (1988), prop. 4.3 and remark p. 546)

$$\int_0^t X_p(s)\delta W_s \longrightarrow \int_0^t X_s\delta W_s \qquad \text{in } L^2(\Omega).$$

Therefore, there exists a subsequence of Π^p such that this convergence holds a.s. It yields

$$Z_p := \int_0^t X_p(s)\, \delta\widetilde{W}_s \longrightarrow \int_0^t X_s\delta\widetilde{W}_s \quad \text{a.s.}$$

By dominated convergence, it is enough to prove that Z_p is independent of Y.

Since the Malliavin derivative and Riemann integral commute,

$$\begin{aligned}Z_p =& \sum_{k=0}^{p-1} \overline{X}_{k,p}(\widetilde{W}_{t_{k+1,p}} - \widetilde{W}_{t_{k,p}}) - \int_{t_{k,p}}^{t_{k+1,p}} D_s \bar{X}_{k,p} ds \\ =& \sum_{k=0}^{p-1} \overline{X}_{k,p}(\widetilde{W}_{t_{k+1,p}} - \widetilde{W}_{t_{k,p}}) \\ & \qquad - (t_{k+1,p} - t_{k,p})^{-1} \int_{[t_{k,p};\, t_{k+1,p}]^2} D_s X_u \, du \, ds.\end{aligned}$$

But $X_u^i \in \mathcal{H}_0 \oplus \mathcal{H}_1$ yields that $D_s X_u$ is deterministic. Therefore, Z_p can be expressed in terms of $(X, \widetilde{W})$ which is independent of Y. □

Let $V_\varepsilon = (Y_\varepsilon, Y) = (-g_1(T'(\varepsilon)), -g_1(T))$ where $T'(\varepsilon) = T - \varepsilon^d$ and $d \in]0, a[$.

Lemma 5.3. Construction of a double bridge Δ_ε. *There exists a process $\Delta_\varepsilon = \{G^\varepsilon_{ijk},\ 0 \le j+k \le i \le 2\}$, independent of V_ε, such that:*

i) $g_i = \sum_{0 \le j+k \le i} G^\varepsilon_{ijk} Y_\varepsilon^j Y^k, \quad i = 1, 2$

ii) If $j + k = i$, G^ε_{ijk} is a deterministic C^1 function on $[0, T]$.

iii) $\exists\, \alpha, \lambda, \mu > 0\ \forall \varepsilon > 0\ \forall\ 0 \le j+k \le i \le 2 \quad G^\varepsilon_{ijk} \in \mathcal{E}(\alpha, \lambda, \mu)$.

Proof. Define $\widetilde{W}^\varepsilon$, a gaussian process independent of V_ε:

$$\begin{aligned}\widetilde{W}^\varepsilon_t &= W_t - b^\varepsilon(t) Y_\varepsilon - a^\varepsilon(t) Y, \qquad 0 \le t \le T \\ a^\varepsilon_t &= -1_{\{t \ge T'(\varepsilon)\}} \Big(\int_{T'(\varepsilon)}^{T} \sigma_0^2(s) ds\Big)^{-1} \int_{T'(\varepsilon)}^{t} \sigma_0(s) ds \\ b^\varepsilon_t &= a^\varepsilon_t - \Big(\int_0^{T'(\varepsilon)} \sigma_0^2(s) ds\Big)^{-1} \int_0^{\inf(t, T'(\varepsilon))} \sigma_0(s) ds.\end{aligned}$$

Then, let $d\widetilde{W}^\varepsilon_s = dW_s - Y_\varepsilon db^\varepsilon(s) - Y da^\varepsilon(s)$. The proof is now similar to that of lemma 5.1. The only new point is to prove that the processes G^ε_{ijk} belong to the same $\mathcal{E}(\alpha, \lambda, \mu)$ for all ε. If $j + k = i$, it is clear because G^ε_{ijk} is deterministic and uniformly bounded in ε. If $j + k < i$, it comes from Azencott results (see Azencott (1980-81) p. 252). □

Exact expressions of G^ε_{ijk} do not matter, except for

$$G^\varepsilon_{100}(.) = \int_0^{\cdot} \sigma_0(s) dW^\varepsilon_s. \tag{5.2}$$

6. The non-linear part contribution

6.1. Results

Let $\alpha \in]0, 1-a[$ and $g^\varepsilon(y,z)$ given by

$$E[1_{\{\|Z^\varepsilon\|_{T(\varepsilon)}^{T} \leq \varepsilon^{1-\alpha}\}} \exp -\frac{1}{\varepsilon^2} F_\varepsilon(T(\varepsilon), T, \varphi + Z^\varepsilon) \mid Z^\varepsilon_{T'(\varepsilon)} = \varepsilon z, Z^\varepsilon_T = \varepsilon^2 y]. \tag{6.1}$$

In this section, we prove $g^\varepsilon(y,z) \to g(y)$, thus understanding the contribution of u^ε in the expectation of (2.11). The key result is lemma 6.1 that establishes the convergence of $u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x)$.

If $\sigma \equiv 1$, the equality between the laws of a brownian motion knowing its final position and the related bridge makes the computation more simple (see Ben Arous and Rouault 1993). Our method is based upon the following classical result (see Fitzsimmons, Pitman and Yor (1993)).

Let $(X_t)_{t\leq T}$ be a real Markov process with transition density $p_{s,t}(x,y)$, and $(\mathcal{F}_t)_{t\leq T}$ be its natural filtration. Let $x \in \mathbb{R}$, $t < T$ and Z a r.v. $\mathcal{F}_t$−measurable. Then, for almost every $y \in \mathbb{R}$,

$$E_x[Z \mid X_T = y]\; p_{0,T}(x,y) = E_x[Z\, p_{t,T}(X_t, y)]. \tag{6.2}$$

In order to use (6.2) and make $u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x)$ appear, let $Y^\varepsilon_s = \varepsilon^{-2} Z^\varepsilon_{T-\varepsilon^2 s}$.

We have

$$g^\varepsilon(y,z) = E[1_{B_\varepsilon} \exp -I_\varepsilon \mid \; Y^\varepsilon_{\varepsilon^{d-2}} = \varepsilon^{-1} z] \tag{6.3}$$

$$Y^\varepsilon_t = y + \int_0^t \sigma(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s) dW_s$$

$$B_\varepsilon = \{\|Y^\varepsilon\|_0^{\varepsilon^{a-2}} \leq \varepsilon^{-1-\alpha}\}$$

$$I_\varepsilon = \int_0^{\varepsilon^{a-2}} c(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s) r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s))\, ds.$$

We will prove that, if $z\varepsilon^{-1}$ is not too large, the conditioning on $Y^\varepsilon_{\varepsilon^{d-2}} = z\varepsilon^{-1}$ does not alter the asymptotics. Actually, "tying up" a diffusion Y at time t on point $y_t \in R$ does not alter $E[\Psi(Y_s,\ s \leq t')]$ when t and t' go to $+\infty$, provided that $t' = o(t)$ and that y_t is not too large. Let

$$g^\varepsilon(y) = E[1_{B_\varepsilon} \exp -I_\varepsilon]. \tag{6.4}$$

Proposition 6.1. *Assume $2/3 < d < a < 1$. Then*
i) For all y, $g^\varepsilon(y) \to g(y)$.
ii) Assume $z = O(\varepsilon^{1-\nu})$ where $\nu \in]0, 1 + \frac{a}{2} - d[$. Then, $g^\varepsilon(y,z) - g^\varepsilon(y) \to 0$.
We need, in section 7, the following result on the modulus of continuity of $g^\varepsilon(y,z)$.

Proposition 6.2. *Assume $r \in C^1[0,1]$ (case σ non constant).*
Assume $2/3 < d < a < 1$. Let $\eta > 0$ small enough, and $b > 0$. Then
i) If $|y-y'| \leq \varepsilon^b$ and $|y|, |y'| \leq \varepsilon^{-\eta}$, then $g^\varepsilon(y) - g^\varepsilon(y') = o(1)$.
ii) If moreover $|z-z'| \leq \varepsilon^{d+\eta}$ and $|z|, |z'| \leq \varepsilon^{\frac{1}{2}d-\eta}$, then $g^\varepsilon(y,z) - g^\varepsilon(y',z') = o(1)$.

Remark 6.3. We state propositions 6.1.i and 6.2.i to show that the conditioning has no influence and to make the proofs easier to read.

6.2. Proof of proposition 6.1.i.

Lemma 6.4. *Let $t_0 > 0$. Then $u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x) \to \tilde{u}(t,x)$ uniformly for $t \in]0, t_0[$ and for x such that $\varepsilon^2 x \to 0$.*

Proof. Since $r \in C^1]0,1]$ and $\lim_{u\to 0} ur'(u) = 0$, there exists a continuous function ρ on $[0,1]^2$ such that $\rho(u,v) = \dfrac{ur(u) - vr(v)}{u - v}$ if $u \neq v$.

Let $h^\varepsilon(t,x) = u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x) - \tilde{u}(t,x)$. Then $h^\varepsilon(0,x) = 0$ and

$$\partial_t h^\varepsilon = \frac{\sigma^2(0)}{2}\partial^2_{xx} h^\varepsilon + \alpha_\varepsilon + \beta_\varepsilon h^\varepsilon$$
$$\alpha_\varepsilon(t,x) = \frac{1}{2}\big(\sigma^2(\varepsilon^2 x) - \sigma^2(0)\big)\partial^2_{xx} u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x) + \big(c(\varepsilon^2 x) - c(0)\big)\tilde{u}(1 - r(\tilde{u}))$$
$$\beta_\varepsilon(t,x) = c(\varepsilon^2 x)[1 - \rho(\tilde{u}^\varepsilon(t,x), \tilde{u}(t,x))].$$

Therefore, the Feynman-Kac formula yields

$$h^\varepsilon(t,x) = E\int_0^t \alpha_\varepsilon(t-s, x+\sigma(0)W_s)\exp\Big(\int_0^s \beta_\varepsilon(t-u, x+\sigma(0)W_u)du\Big)ds. \tag{6.5}$$

(1.2) implies $|\beta_\varepsilon(t,x)| \leq k\sup_{[0,1]^2}|1-\rho|\ (1+\varepsilon^2|x|)$.

Appendix 8.1 implies: $\exists C > 0\quad |\partial^2_{xx}u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x)| \leq C\ (\sqrt{t} + t^2 x^2 + t^{\frac{5}{2}})$.

(1.2), (1.3) and the previous two bounds allow us to end the proof by applying to (6.5) a joint dominated convergence in ω and s. □

We come back to the proof of proposition 6.1.i.

Define R^ε such that $Y^\varepsilon_s = y + \sigma(0)W_s + \varepsilon R^\varepsilon_s$. Then $dR^\varepsilon_s = A^\varepsilon_s dW_s$ where

$$A^\varepsilon_s = \varepsilon^{-1}\big(\sigma(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s) - \sigma(0)\big) = O(\varepsilon^{-\alpha}) \qquad \text{on } B_\varepsilon \text{ and if } s \leq \varepsilon^{-1}.$$

Therefore $P(\|R^\varepsilon\|_0^{\varepsilon^{-1}} \geq \varepsilon^{-\delta};\ B_\varepsilon) \leq \exp -cst\ \varepsilon^{1+2\alpha-2\delta}$.

Since $\alpha < 1 - a < \frac{1}{2}$, there exists $\delta \in]\frac{1}{2}+\alpha, 1[$. Hence,

$$g^\varepsilon(y) = E[1_{\{\|R^\varepsilon\|_0^{\varepsilon^{-1}} \leq \varepsilon^{-\delta}\}\cap B_\varepsilon}\exp -I_\varepsilon] + o(1).$$

Let $L = c(0)\displaystyle\int_0^{+\infty} r(\tilde{u}(s, -\dot{\varphi}_T s + y + \sigma(0)W_s))\ ds$.

Since $P(\{\|R^\varepsilon\|_0^{\varepsilon^{-1}} \leq \varepsilon^{-\delta}\}\cap B_\varepsilon) \to 1$, proposition 6.1.i is implied by

$$E1_{\{\|R^\varepsilon\|_0^{\varepsilon^{-1}} \leq \varepsilon^{-\delta}\}\cap B_\varepsilon}[\exp(-I_\varepsilon) - \exp(-L)] \to 0.$$

By dominated convergence, it is enough to prove

$$1_{\{\|R^\varepsilon\|_0^{\varepsilon^{-1}} \leq \varepsilon^{-\delta}\}\cap B_\varepsilon}(I_\varepsilon - L) \to 0 \ \text{ a.s.} \tag{6.6}$$

Remark 6.5. ε stands for the general term of a sequence going to 0, thus avoiding non-countability problems. Th. 2.1 and 2.3 can easily be reduced to this case.

Therefore, we can assume $\omega \in \{\|R^\varepsilon\|_0^{\varepsilon^{-1}} \leq \varepsilon^{-\delta}\} \cap B_\varepsilon$ for all ε.

For fixed s and $\varepsilon \leq s^{-1}$,

$$|Y_s^\varepsilon(\omega) - y - \sigma(0)W_s(\omega)| \leq \varepsilon^{1-\delta}.$$

Hence, by lemma 6.3, $c(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y_s^\varepsilon(\omega))\ r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y_s^\varepsilon(\omega)))$ goes to $c(0)r(\tilde{u}(s, -\dot{\varphi}_T s + y + \sigma(0)W_s(\omega)))$ for all s.

(6.6) will be deduced, provided that we have a dominated convergence in s. Let

$$\Omega_1 = \{\omega \mid \forall\delta > 0\ \ \exists s_0(\omega)\ \ \forall\varepsilon\ \ \forall s \geq s_0(\omega)\ \ |Y_s^\varepsilon(\omega)| \leq \delta\, s\}.$$

(8.8) implies $P(\Omega_1) = 1$. Hence we assume $\omega \in \Omega_1$.

By lemma 4.1, for all $\eta > 0$,

$$\bar{\sigma}(\eta)\sqrt{2(\bar{c}(2\eta) + \eta)}\ s \leq x \leq \bar{\sigma}(\eta)\eta M^{-1}\varepsilon^{-2} \Longrightarrow \tilde{u}^\varepsilon(s, x) \leq 4\exp -\eta s. \quad (6.7)$$

(H4) yields the existence of $\eta > 0$ such that $-\dot{\varphi}_T > \bar{\sigma}(\eta)\sqrt{2(\bar{c}(2\eta) + \eta)} + \eta$.

For $s \in [0, \varepsilon^{a-2}]$ and ε small enough, $|e^{-2}s^{-1}\varphi_{T-\varepsilon^2 s} + \dot{\varphi}_T| \leq \frac{\eta}{2}$.

$\omega \in \Omega_1$ implies: $\exists s_0(\omega)\ \ \forall\varepsilon\ \ \forall s \geq s_0(\omega)\ \ |Y_s^\varepsilon(\omega)| \leq \frac{\eta}{2}\, s$.

Hence, according to (6.7), for $s_0(\omega) \leq s \leq \varepsilon^{a-2}$ and ε small enough,

$$\begin{aligned}
&\bar{\sigma}(\eta)\sqrt{2(\bar{c}(2\eta) + \eta)}\ s \leq \varphi_{T-\varepsilon^2 s}\varepsilon^{-2} + Y_s^\varepsilon(\omega) \leq \bar{\sigma}(\eta)\eta M^{-1}\varepsilon^{-2} \\
&u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y_s^\varepsilon(\omega)) \leq 4\exp -\eta s \\
&r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s}\varepsilon^{-2} + \varepsilon^2 Y_s^\varepsilon(\omega))) \leq r(4\exp -\eta s)
\end{aligned}$$

and $r \in C^1[0,1]$ or (2.10) yields dominated convergence in s.

6.3. Proof of proposition 6.2.i.

We can define on the same sample space (see remark 6.1.) the diffusions

$$Y_t^\varepsilon = y + \int_0^t \sigma(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y_s^\varepsilon)dW_s, \quad Y_t^{'\varepsilon} = y' + \int_0^t \sigma(\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y_s^{'\varepsilon})dW_s.$$

Define B'_ε and I'_ε from $Y^{'\varepsilon}$ as B_ε and I_ε were defined from Y^ε (see (6.3)).

Since $|y| \leq \varepsilon^{-\eta}$, (8.7) yields

$$\begin{aligned}
P(\Omega\backslash B_\varepsilon), P(\Omega\backslash B'_\varepsilon) &\leq 2\exp -\frac{(\varepsilon^{-1-\alpha} + |\varepsilon^{-\eta}|)^2}{2\varepsilon^{a-2}M^2} \leq \exp -cst\,\varepsilon^{-2\alpha-a} \\
g^\varepsilon(y) - g^\varepsilon(y') &= E[1_{B_\varepsilon}\exp(-I_\varepsilon) - 1_{B'_\varepsilon}\exp(-I'_\varepsilon)] \\
&= E1_{B_\varepsilon \cap B'_\varepsilon}[\exp(-I_\varepsilon) - \exp(-I'_\varepsilon)] + o(1).
\end{aligned}$$

Lemma 6.6. *Let $\mu \in]0, \inf(b, 3a/2 - 1)[$. Then*

$$P(\{\|Y^\varepsilon - Y'^\varepsilon\|_0^{\varepsilon^{a-2}} \geq \varepsilon^\mu\} \cap B_\varepsilon \cap B'_\varepsilon) = o(1).$$

Proof. Let $X_t = y+\sigma(0)W_t$, $X'_t = y'+\sigma(0)W_t$ and $\mu' \in]\mu, \inf(b, 3a/2-1)[$. We get

$$P(\{\|Y^\varepsilon - Y'^\varepsilon\|_0^{\varepsilon^{a-2}} \geq \varepsilon^\mu\} \cap B_\varepsilon \cap B'_\varepsilon) \leq P(\{\|Y^\varepsilon - X\|_0^{\varepsilon^{a-2}} \geq \varepsilon^{\mu'}\} \cap B_\varepsilon) + P(\{\|Y'^\varepsilon - X'\|_0^{\varepsilon^{a-2}} \geq \varepsilon^{\mu'}\} \cap B'_\varepsilon).$$

We have $Y^\varepsilon_t - X_t = \int_0^t A^\varepsilon_s dW_s$ where $A^\varepsilon_s = \sigma(\varphi_{T-\varepsilon^2 s}+\varepsilon^2 Y^\varepsilon_s) - \sigma(0) = O(\varepsilon^a)$ on B_ε and if $s \leq \varepsilon^{a-2}$. Hence, (8.7) yields

$$P(\{\|Y^\varepsilon - X\|_0^{\varepsilon^{a-2}} \geq \varepsilon^{\mu'}\} \cap B_\varepsilon) \leq \exp - cst\, \varepsilon^{2+2\mu'-3a}.$$

□

Let $D_\varepsilon = \{\|Y^\varepsilon - Y'^\varepsilon\|_0^{\varepsilon^{a-2}} \leq \varepsilon^\mu\} \cap B_\varepsilon \cap B'_\varepsilon$. Lemma 6.4 yields

$$|g^\varepsilon(y) - g^\varepsilon(y')| \leq E1_{D_\varepsilon}|I_\varepsilon - I'_\varepsilon| + o(1).$$

On the event D_ε and if $s \leq \varepsilon^{a-2}$, then $\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s$ and $\varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y'^\varepsilon_s$ stay in a compact K. Let $k_1 = \sup_K |c|$ and $k_2 = \sup_K |c'|$. We get

$$1_{D_\varepsilon}|I_\varepsilon - I'_\varepsilon| \leq k_2\, \varepsilon^{a-2}\varepsilon^{2+\mu} + k_1 \int_0^{\varepsilon^{a-2}} |r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s)) - r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y'^\varepsilon_s))|\, ds.$$

Therefore $|g^\varepsilon(y) - g^\varepsilon(y')| \leq k_1 l^\varepsilon(0, \varepsilon^{a-2}) + o(1)$ where $l^\varepsilon(a_1, a_2) =$

$$E1_{D_\varepsilon}\int_{a_1}^{a_2} |r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s)) - r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y'^\varepsilon_s))|\, ds.$$

Lemma 6.7. *If* $0 < \gamma_1 < \gamma_2 < 2\gamma_1 \leq 2 - a$, *then* $l^\varepsilon(\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}) = o(1)$, *where* $o(1)$ *depends only on* γ_1 *and* γ_2, *not on* y, y'.

Proof. According to (H4), there exists $\delta > 0$ such that

$$l = -\dot\varphi_T - \overline{\sigma}(\delta)\sqrt{2(\overline{c}(2\delta) + \delta)} - \delta > 0.$$

Let $V^1_\varepsilon = \{\varphi_{T-\varepsilon^2 s}\varepsilon^{-2} + Y^\varepsilon_s < \overline{\sigma}(\delta)\delta M^{-1}\varepsilon^{-2}$, for all $s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}]\}$ and $V^2_\varepsilon = \{\overline{\sigma}(\delta)\sqrt{2(\overline{c}(2\delta) + \delta)}s < \varphi_{T-\varepsilon^2 s}\varepsilon^{-2} + Y^\varepsilon_s$, for all $s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}]\}$.

(8.7) yields that, for ε small enough,

$$P(\Omega\backslash V^1_\varepsilon) \leq P(\exists s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}] \quad Y^\varepsilon_s > cst\, \varepsilon^{-2}) \leq \exp - cst\, \varepsilon^{\gamma_2 - 4}$$
$$P(\Omega\backslash V^2_\varepsilon) \leq P(\exists s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}] \quad Y^\varepsilon_s < -l\, s) \leq \exp - cst\, \varepsilon^{\gamma_2 - 2\gamma_1}.$$

Let $C_\varepsilon = \int_{\varepsilon^{-\gamma_1}}^{\varepsilon^{-\gamma_2}} r(u^\varepsilon(\varepsilon^2 s, \varphi_{T-\varepsilon^2 s} + \varepsilon^2 Y^\varepsilon_s))ds$. Therefore, lemma 4.1 implies

$$EC_\varepsilon = E1_{V^1_\varepsilon \cap V^2_\varepsilon} C_\varepsilon + o(1) \leq \int_{\varepsilon^{-\gamma_1}}^{\varepsilon^{-\gamma_2}} r(4\exp - \delta s)\, ds + o(1) = o(1).$$

The same computation for Y'^{ε} ends the proof of lemma 6.5. □

Now, let $\gamma \in]0, \mu/2[$. By iterating lemma 6.5, we get $l^{\varepsilon}(\varepsilon^{-\gamma}, \varepsilon^{a-2}) = o(1)$.

It remains to prove $l^{\varepsilon}(0, \varepsilon^{-\gamma}) = o(1)$. Let $\tau \in]\gamma/2, \mu - 3\gamma/2[$.

(8.7) yields

$$P(\|Y^{\varepsilon}\|_0^{\varepsilon^{-\gamma}} \geq \varepsilon^{-\tau}) \leq \exp - cst\ \varepsilon^{-2\tau+\gamma}. \tag{6.8}$$

Define $k^{\varepsilon}(a_1, a_2)$ by replacing D_{ε} by

$$\{\|Y^{\varepsilon}\|_0^{\varepsilon^{-\gamma}} \leq \varepsilon^{-\tau})\} \cap \{\|Y'^{\varepsilon}\|_0^{\varepsilon^{-\gamma}} \leq \varepsilon^{-\tau})\} \cap \{\|Y^{\varepsilon} - Y'^{\varepsilon}\|_0^{\varepsilon^{-\gamma}} \leq \varepsilon^{\mu}\} \cap B_{\varepsilon} \cap B'_{\varepsilon}$$

in the definition of $l^{\varepsilon}(a_1, a_2)$. (6.8) yields $l^{\varepsilon}(0, \varepsilon^{-\gamma}) = k^{\varepsilon}(0, \varepsilon^{-\gamma}) + o(1)$.

Appendix 8.1 and $r \in C^1[0,1]$ imply

$$\sup_{\varepsilon^{\gamma} \leq s \leq \varepsilon^{-\gamma}} |\partial_x r(u^{\varepsilon}(\varepsilon^2 s, \varepsilon^2 x))| = O(\varepsilon^{-\gamma}) + O(\varepsilon^{-\gamma/2}|x|).$$

Therefore,

$$k^{\varepsilon}(0, \varepsilon^{-\gamma}) = k^{\varepsilon}(0, \varepsilon^{\gamma}) + k^{\varepsilon}(\varepsilon^{\gamma}, \varepsilon^{-\gamma}) = O(\varepsilon^{\gamma}) + O\big(\varepsilon^{\mu-\gamma}(\varepsilon^{-\gamma} + \varepsilon^{-\gamma/2}\varepsilon^{-\tau})\big) = o(1).$$

6.4. Proof of proposition 6.1.ii

Together with (6.2), we will use an explicit formula for q^{ε}, the transition density of Y^{ε}. Let

$$\gamma(x) = \sigma(\varepsilon^2 x), \quad \alpha_t = \varepsilon^{-2}\varphi_{T-\varepsilon^2 t}$$

$$G(t,x) = \int_0^x \frac{du}{\gamma(\alpha_t + u)}, \quad L(t,.) \text{ be the inverse function of } G(t,.)$$

$$C(t,x) = \partial_t G(t, L(t,x)) - \frac{1}{2}\gamma'(\alpha_t + L(t,x))$$

$$= \alpha'_t[\gamma(\alpha_t + L(t,x))^{-1} - \gamma(\alpha_t)^{-1}] - \frac{1}{2}\gamma'(\alpha_t + L(t,x))$$

$$D(t,x) = -\frac{1}{2}\partial_x C(t,x) - \frac{1}{2}C^2(t,x) - \int_0^x \partial_t C(t,u)du$$

$$H(t,x) = \int_0^{G(t,x)} C(t,u)du$$

$$= \int_0^x \big(\alpha'_t[\gamma(\alpha_t+v)^{-1} - \gamma(\alpha_t)^{-1}] - \frac{1}{2}\gamma'(\alpha_t+v)\big)\frac{dv}{\gamma(\alpha_t+v)}$$

$$V_u(a,b) = (1-u)a + ub$$

$$J(s,t,x,y) =$$

$$\mathbb{E}[\exp(t-s)\int_0^1 D\big(V_u(s,t), V_u(G(s,x), G(t,y)) + \sqrt{t-s}\,\mathbb{B}_u\big)du]$$

where $\mathbb{B}$ is a standard brownian bridge defined on another probability space Ω' and $\mathbb{E}[\ldots]$ refers to Ω' (as well as $\mathbb{P}(\ldots)$ later on).

LEMMA 6.8.

$$q^{\varepsilon}_{s,t}(x,y) = \frac{1}{\sqrt{2\pi(t-s)}} \frac{J(s,t,x,y)}{\gamma(\alpha_t + y)}$$
$$\exp[-\frac{(G(t,y) - G(s,x))^2}{2(t-s)} + H(t,y) - H(s,x)].$$

Proof. Straightforward extension of lemma 1 p. 268, Dacunha-Castelle and Florens-Zmirou (1986). □

We will need the following bounds.

LEMMA 6.9.
i) $\exists C_1 > 0 \ \ \forall\, t, x \qquad |H(t,x)| \leq C_1\, \varepsilon^2\ (1 + x^2)$
ii) $\exists C_2 > 0 \ \ \forall\, t, x \qquad D(t,x) \leq C_2\, \varepsilon^2\ (1 + |x|)$
iii) $\forall \lambda > 0\ \exists C_3(\lambda) > 0 \ \ |\varepsilon\, x_\varepsilon| \leq \lambda,\, \varepsilon^2 t_\varepsilon \leq \varepsilon^d \Rightarrow |D(t_\varepsilon, x_\varepsilon)| \leq C_3(\lambda)\, \varepsilon^2.$

Proof. i) We have $\|\gamma'\|_\infty = O(\varepsilon^2)$, $\|\gamma''\|_\infty = O(\varepsilon^4)$, and $\|\alpha'\|_\infty < \infty$. Hence

$$|H(t,x)| \leq cst\, \varepsilon^2 |x| + cst| \int_0^x [\gamma(\alpha_t + v) - \gamma(\alpha_t)]dv| \leq cst\, \varepsilon^2\ (1 + x^2).$$

ii) Straightforward.

iii) Let $\alpha_0 = \frac{\dot\varphi_T \sigma'(0)}{\sigma(0)}$. Easy computations yield the successive results:

$$\varepsilon^2 L(t_\varepsilon, x_\varepsilon) = O(\varepsilon),\ \varepsilon^2 \alpha_{t_\varepsilon} = p\varepsilon^2 t_\varepsilon + O(\varepsilon^{2d}),\ L(t_\varepsilon, x_\varepsilon) = \sigma(0) x_\varepsilon [1 + O(\varepsilon^d)]$$
$$C(t_\varepsilon, x_\varepsilon) = \alpha_0 \varepsilon^2 x_\varepsilon + O(\varepsilon^{2d}),\ \ \partial_x C(t_\varepsilon, x_\varepsilon) = \alpha_0 \varepsilon^2 + O(\varepsilon^{2+d})$$
$$\partial_t C(t_\varepsilon, x_\varepsilon) = cst\, \varepsilon^4 x_\varepsilon + O(\varepsilon^{2+2d}),\ \ D(t_\varepsilon, x_\varepsilon) = O(\varepsilon^2).$$

□

According to (6.2), (6.3) and lemma 6.6, we have

$$g^\varepsilon(y,z) - g^\varepsilon(y) = E 1_{B_\varepsilon} \exp(-I_\varepsilon)[J_\varepsilon \exp(G_\varepsilon + H_\varepsilon) - 1] \tag{6.9}$$
$$G_\varepsilon = \frac{\varepsilon^{2-d}}{2}[\big(G(\varepsilon^{d-2}, z\varepsilon^{-1}) - G(0,y)\big)^2$$
$$- (1 - \varepsilon^{a-d})^{-1}\big(G(\varepsilon^{2-d}, z\varepsilon^{-1}) - G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}})\big)^2]$$
$$H_\varepsilon = H(0,y) - H(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}})$$
$$J_\varepsilon = \frac{J^\varepsilon_1}{J^\varepsilon_2}$$
$$J^\varepsilon_1 = J(\varepsilon^{a-2}, \varepsilon^{d-2}, Y^\varepsilon_{\varepsilon^{a-2}}, z\varepsilon^{-1})$$
$$J^\varepsilon_2 = J(0, \varepsilon^{d-2}, y, z\varepsilon^{-1})$$

As we expected, $J_\varepsilon \exp(G_\varepsilon + H_\varepsilon) \to 1$ a.s., and more precisely,

LEMMA 6.10. *For $\eta > 0$ small enough, define* $X_\varepsilon = \varepsilon^{\frac{1}{2}(2-a+\eta)}\, |Y^\varepsilon_{\varepsilon^{a-2}}|$.

i) $X_\varepsilon \to 0$ *a.s. and* $E \exp X^2_\varepsilon \to 1$

ii) $|G_\varepsilon| + |H_\varepsilon| \leq \alpha_\varepsilon (1 + X^2_\varepsilon)$ *a.s. where* $\alpha_\varepsilon \to 0$.

iii) $J_1^\varepsilon \to 1$ *a.s. and* $J_2^\varepsilon \to 1$.

The proof of this lemma is postponed to the end of the section.

In order to get a dominated convergence, we introduce

$$\overline{B}_\varepsilon = B_\varepsilon \cap \{|Y^\varepsilon_{\varepsilon^{a-2}}| \leq \varepsilon^{-1}\}$$

as well as $\overline{g}^\varepsilon(y)$ and $\overline{g}^\varepsilon(y,z)$ defined from $g^\varepsilon(y)$ and $g^\varepsilon(y,z)$ by replacing B_ε by $\overline{B}_\varepsilon$.

It can be proved easily that $\overline{g}^\varepsilon(y) - g^\varepsilon(y) \to 0$.

Let us prove that $\overline{g}^\varepsilon(y,z) - g^\varepsilon(y,z) \to 0$. According to lemma 6.7.ii,

$$\begin{aligned} J_1^\varepsilon &\leq \mathbb{E}[\exp cst\ \varepsilon^d \int_0^1 (1 + |z|\varepsilon^{-1} + |Y^\varepsilon_{\varepsilon^{a-2}}| + \varepsilon^{d/2-1}|\mathbb{B}_u|)du] \text{ a.s.} \\ &\leq \exp[cst\ \varepsilon^{d-1} + \varepsilon^{d+a/2-1-\eta}X_\varepsilon]\mathbb{E}[\exp \|\mathbb{B}\|_0^1] \text{ a.s.} \end{aligned}$$

(notice that a.s. refers to the arguments of $J_1^\varepsilon, Y^\varepsilon_{\varepsilon^{a-2}}$ and X_ε).

Together with lemma 6.8.ii and iii, it yields, for ε small enough,

$$\begin{aligned} 0 \leq g^\varepsilon(y,z) - \overline{g}^\varepsilon(y,z) &\leq (J_2^\varepsilon)^{-1} E 1_{\{|Y^\varepsilon_{\varepsilon^{a-2}}| \geq \varepsilon^{-1}\}} J_1^\varepsilon \exp[\alpha_\varepsilon(1+X_\varepsilon^2)] \\ &\leq \exp cst\ \varepsilon^{d-1} E 1_{\{|Y^\varepsilon_{\varepsilon^{a-2}}| \geq \varepsilon^{-1}\}} \exp \beta_\varepsilon(1+X_\varepsilon^2) \end{aligned}$$

where $\beta_\varepsilon \to 0$. Moreover, $P(|Y^\varepsilon_{\varepsilon^{a-2}}| \geq \varepsilon^{-1}) \leq \exp -cst\ \varepsilon^{-a}$. Hence,

$$0 \leq g^\varepsilon(y,z) - \overline{g}^\varepsilon(y,z) \leq \exp(cst\,\varepsilon^{d-1} - cst\,\varepsilon^{-a})[E(\exp 2\beta_\varepsilon X_\varepsilon^2)]^{\frac{1}{2}}$$

and we can conclude by using lemma 6.8.i and $a > 1-d$.

It remains to prove that $\overline{g}^\varepsilon(y,z) - \overline{g}^\varepsilon(y) \to 0$. We introduced $\overline{B}_\varepsilon$ because

$$1_{\{|Y^\varepsilon_{\varepsilon^{a-2}}| \leq \varepsilon^{-1}\}} J_1^\varepsilon \leq cst \tag{6.10}$$

(actually, lemma 6.9.iii below is still valid when $z = O(\varepsilon^{1-\nu})$). Therefore

$$|\overline{g}^\varepsilon(y,z) - \overline{g}^\varepsilon(y)| \leq |\overline{g}^\varepsilon(y,z) - E 1_{\overline{B}_\varepsilon} J_\varepsilon \exp(-I_\varepsilon)| + |\overline{g}^\varepsilon(y) - E 1_{\overline{B}_\varepsilon} J_\varepsilon \exp(-I_\varepsilon)|.$$

The second term is smaller than $E 1_{\{|Y^\varepsilon_{\varepsilon^{a-2}}| \leq \varepsilon^{-1}\}} |J_\varepsilon - 1|$ which goes to 0 according to (6.10).

By (6.10),the first term is smaller than $cst\ E|\exp(G_\varepsilon + H_\varepsilon) - 1|$ which goes to 0 by lemma 6.8.i and 6.8.ii.

Proof of lemma 6.8. i) The first part is a consequence of (8.8).

The second part holds since $(\exp X_\varepsilon^2)_{\varepsilon>0}$ is uniformly integrable. Actually

$$\begin{aligned} E \exp 2X_\varepsilon^2 &= \int_0^{+\infty} P(\exp 2X_\varepsilon^2 \geq y)dy = 1 + \int_0^{+\infty} P(X_\varepsilon \geq \sqrt{\frac{r}{2}}) \exp r dr \\ &\leq 1 + \int_0^1 \exp r dr + \int_1^{+\infty} \exp(r - cst \frac{r}{\varepsilon^\eta})dr \leq cst \text{ for } \varepsilon \text{ small enough.} \end{aligned}$$

ii) Lemma 6.7.i and $|G(t,x)| \leq m^{-1}|x|$ yield

$$|H(\varepsilon^{a-2}, Y^\varepsilon - \varepsilon^{a-2})| \leq C_1\, \varepsilon^2 (1 + \varepsilon^{a-2-\eta} X_\varepsilon^2)$$
$$2\varepsilon^{2-d}|G_\varepsilon| \leq \frac{\varepsilon^{a-d} z^2 \varepsilon^{-2}}{1-\varepsilon^{a-d}} + cst\,|z\varepsilon^{-1}| + cst + cst\,(Y^\varepsilon_{\varepsilon^{a-2}})^2 + cst\,|z\varepsilon^{-1} Y^\varepsilon_{\varepsilon^{a-2}}|$$
$$|G_\varepsilon| \leq cst\,[\varepsilon^{a-2d+2-2\nu} + o(1) + \varepsilon^{\frac{1}{2}(a-2d-\eta)+1-\nu} X_\varepsilon + \varepsilon^{a-2d-\eta} X_\varepsilon^2].$$

iii) In this proof, we restrict ourselves to the event $\{\varepsilon Y^\varepsilon_{\varepsilon^{a-2}} \to 0\}$ (it has probability 1, see lemma 6.8.i). There exists c such that $d > 2c > 1-d$. Define

$$\widetilde{J}_1^\varepsilon = \mathbb{E}[1_{\{\|\mathbb{B}\|_0^1 \leq \varepsilon^{-c}\}} \exp(\varepsilon^{d-2} - \varepsilon^{a-2})$$
$$\int_0^1 D[V_u(\varepsilon^{a-2}, \varepsilon^{d-2}), V_u(G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}}), G(\varepsilon^{d-2}, \frac{z}{\varepsilon})) + \sqrt{\varepsilon^{d-2} - \varepsilon^{a-2}}\mathbb{B}_u]du].$$

Lemma 6.7.ii yields that $|J_1 - \widetilde{J}_1|$ is smaller than

$$\begin{aligned}\mathbb{E}[1_{\{\|\mathbb{B}\|_0^1 \geq \varepsilon^{-c}\}} &\exp C_2\, \varepsilon^d\, [1 + m^{-1}(Y^\varepsilon_{\varepsilon^{a-2}} + z\varepsilon^{-1}) + \varepsilon^{\frac{1}{2}d-1}\|\mathbb{B}\|_0^1]] \\ &\leq \exp cst\, \varepsilon^{d-1}\, \mathbb{E}[1_{\{\|\mathbb{B}\|_0^1 \geq \varepsilon^{-c}\}} \exp \|\mathbb{B}\|_0^1] \\ &\leq \exp(cst\, \varepsilon^{d-1} - cst\, \varepsilon^{-2c}) = o(1).\end{aligned}$$

We now prove that $\widetilde{J}_1^\varepsilon \to 1$ a.s..

There exists $\lambda > 0$ such that, a.s. on $\{\|\mathbb{B}\|_0^1 \leq \varepsilon^{-c}\}$,

$$\varepsilon\, |V_u(G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}}), G(\varepsilon^{d-2}, z\varepsilon^{-1})) + \sqrt{\varepsilon^{d-2} - \varepsilon^{a-2}}\mathbb{B}_u| \leq cst + \varepsilon^{\frac{1}{2}d-c} \leq \lambda.$$

Since $\varepsilon^2 V_u(\varepsilon^{a-2}, \varepsilon^{d-2}) \leq \varepsilon^d$, lemma 6.7.iii yields

$$|\widetilde{J}_1^\varepsilon - \mathbb{P}(\|\mathbb{B}\|_0^1 \leq \varepsilon^{-c})| \leq \exp(C_3(\lambda)\varepsilon^d) - 1,$$

which yields $\widetilde{J}_1^\varepsilon \to 1$ a.s.. We prove $J_2^\varepsilon \to 1$ in the same way.

6.5. Proof of proposition 6.2.ii.

We will need the following results

Lemma 6.11. *Define* $X_\varepsilon = \varepsilon^{\frac{1}{2}(2-a+\eta)}\, |Y^\varepsilon_{\varepsilon^{a-2}}|$ *as in lemma 6.8. Then*

i) $(\exp X_\varepsilon^2)_{\varepsilon>0}$ *is bounded in* L^q*, for all* $q \geq 1$

ii) $G_\varepsilon + H_\varepsilon \leq \alpha_\varepsilon(1 + X_\varepsilon^2)$ *where* $\alpha_\varepsilon \to 0$

iii) $J_1^\varepsilon \leq cst$ on $\{|Y^\varepsilon_{\varepsilon^{a-2}}| \leq \varepsilon^{-1}\}$

iv) $J_2^\varepsilon \geq \frac{1}{2}$ *for* ε *small enough*

v) $J_1^\varepsilon \leq cst \exp X_\varepsilon$ *for* ε *small enough.*

All these results remain valid if we replace $X_\varepsilon, G_\varepsilon, \ldots$ *by* $X'_\varepsilon, G'_\varepsilon, \ldots$.

Proof . i) and ii) The proofs are similar to that of lemmas 6.8.i and 6.8.ii. The fact that y is not fixed but satisfies only $|y| \le \varepsilon^{-\eta}$ does not alter the result. iii) There exists $k_0 > 0$ such that, on $\{|Y^\varepsilon_{\varepsilon^{a-2}}| \le \varepsilon^{-1}\}$, we have

$$|V_u(G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}}, G(\varepsilon^{d-2}, z\varepsilon^{-1}))| \le k_0\ \varepsilon^{-1}.$$

According to lemma 6.7.ii and iii, on $\{|Y^\varepsilon_{\varepsilon^{a-2}}| \le \varepsilon^{-1}\}$, we have

$$\begin{aligned} J_1^\varepsilon \le \mathbb{P}(\|\mathbb{B}\|_0^1 \le \varepsilon^{-d/2}) \exp C_3(k_0+1)\,\varepsilon^d \\ + \mathbb{E}[1_{\{\|\mathbb{B}\|_0^1 \ge \varepsilon^{-d/2}\}} \exp C_2\varepsilon^d(1 + cst\ \varepsilon^{-1} + \varepsilon^{d/2-1}\|\mathbb{B}\|_0^1)] \\ = O(1 + \exp(cst\ \varepsilon^{d-1} - cst\ \varepsilon^{-d})) = O(1). \end{aligned}$$

iv) We use lemma 6.7.iii to find a lower bound for D.

Since $|y|, |z\varepsilon^{-1}| \le \varepsilon^{-1}$, on $\{\|\mathbb{B}\|_0^1 \le \varepsilon^{-d/2}\}$,

$$|V_u(G(0,y), G(\varepsilon^{d-2}, z\varepsilon^{-1})) + \varepsilon^{d/2-1}\mathbb{B}_u| \le (m^{-1}+1)\varepsilon^{-1}.$$

Therefore: $J_2^\varepsilon \ge \mathbb{P}(\|\mathbb{B}\|_0^1 \le \varepsilon^{-\frac{1}{2}d}) \exp[-C_3(m^{-1}+1)\,\varepsilon^d] \ge \dfrac{1}{2}$.

v) By lemma 6.7.ii,

$$J_1^\varepsilon \le \exp C_2\varepsilon^d(1 + cst\, Y^\varepsilon_{\varepsilon^{a-2}} + cst\,|z|\varepsilon^{-1}) \le cst\ \exp X_\varepsilon.$$

□

We now prove prop. 6.2.ii., ie $g^\varepsilon(y,z) - g^\varepsilon(y',z') = o(1)$.

As $B_\varepsilon, I_\varepsilon, G_\varepsilon, H_\varepsilon, J_\varepsilon, J_1^\varepsilon$ and J_2^ε where defined from Y^ε and z, we define $B'_\varepsilon, I'_\varepsilon \ldots$ from Y'^ε and z' (see proof of prop. 6.2.i). Then,

$$g^\varepsilon(y,z) - g^\varepsilon(y',z') = E[1_{B_\varepsilon} J_\varepsilon \exp(-I_\varepsilon + G_\varepsilon + H_\varepsilon) - 1_{B'_\varepsilon} J'_\varepsilon \exp(-I'_\varepsilon + G'_\varepsilon + H'_\varepsilon)]. \tag{6.11}$$

We define $k^\varepsilon(y,y',z,z')$ by replacing B_ε and B'_ε by $B_\varepsilon \cap B'_\varepsilon$ in the right-hand side member of (6.11). Then,

$$\begin{aligned} |g^\varepsilon(y,z) - g^\varepsilon(y',z') - k^\varepsilon(y,y',z,z')| \le [P(\Omega\backslash B'_\varepsilon) E 1_{B_\varepsilon} J_\varepsilon^2 \exp 2(G_\varepsilon + H_\varepsilon)]^{\frac{1}{2}} \\ + [P(\Omega\backslash B_\varepsilon) E 1_{B'_\varepsilon} J'^2_\varepsilon \exp 2(G'_\varepsilon + H'_\varepsilon)]^{\frac{1}{2}}. \end{aligned}$$

(8.7) yields $P(\Omega \setminus B'_\varepsilon) = o(1)$ and $P(\Omega \setminus B_\varepsilon) = o(1)$. Lemma 6.9.i, ii, iv and v, yield that $E 1_{B_\varepsilon} K_\varepsilon^2 \exp 2(G_\varepsilon + H_\varepsilon)$ and $E 1_{B'_\varepsilon} K'^2_\varepsilon \exp 2(G'_\varepsilon + H'_\varepsilon)$ are bounded. Therefore, $g^\varepsilon(y,z) - g^\varepsilon(y',z') = k^\varepsilon(y,y',z,z') + o(1)$.

We now fix μ in $]0, \inf(b, 3a/2-1)[$ and define $g^\varepsilon(y,y',z,z')$ by replacing B_ε and B'_ε by B''_ε in (6.11), where

$$B''_\varepsilon = \{\|Y^\varepsilon - Y'^\varepsilon\|_0^{\varepsilon^{a-2}} \le \varepsilon^\mu\} \cap \{|Y^\varepsilon_{\varepsilon^{a-2}}| \le \varepsilon^{-1}\} \cap \{|Y'^\varepsilon_{\varepsilon^{a-2}}| \le \varepsilon^{-1}\} \cap B_\varepsilon \cap B'_\varepsilon.$$

Lemma 6.4 yields $P(B_\varepsilon \cap B'_\varepsilon \cap (\Omega \setminus B''_\varepsilon)) = o(1)$, and the same computation as before thus implies $k^\varepsilon(y, y', z, z') = g^\varepsilon(y, y', z, z') + o(1)$.

It is therefore enough to prove that $g^\varepsilon(y, y', z, z') = o(1)$. Since $a_1 \ldots a_n - a'_1 \ldots a'_n = \sum_{j=1}^{n} (\prod_{i=1}^{j-1} a'_i)(a_j - a'_j)(\prod_{i=j+1}^{n} a_i)$, lemma 6.9.iii and iv yield $|g^\varepsilon(y, y', z, z')| \le cst\,(g_1^\varepsilon + g_2^\varepsilon + g_3^\varepsilon + g_4^\varepsilon)$ where

$$g_1^\varepsilon = E1_{B''_\varepsilon}|I_\varepsilon - I'_\varepsilon| \exp(G_\varepsilon + H_\varepsilon), \quad g_2^\varepsilon = E1_{B''_\varepsilon}|\exp G_\varepsilon - \exp G'_\varepsilon| \exp H_\varepsilon$$
$$g_3^\varepsilon = E1_{B''_\varepsilon}|\exp H_\varepsilon - \exp H'_\varepsilon| \exp G'_\varepsilon, \quad g_4^\varepsilon = E1_{B''_\varepsilon}|J_\varepsilon - J'_\varepsilon| \exp(G'_\varepsilon + H'_\varepsilon).$$

We conclude by proving that, for $i = 1$ to 4, $g_i^\varepsilon = o(1)$.

i=1: We change slightly the proof of $E1_{D_\varepsilon}|I_\varepsilon - I'_\varepsilon| = o(1)$ (proof of prop. 6.2.i) by using lemma 6.9.i and ii.

i=2: $|(a-b)^2 - (a'-b')^2| \le 4 \sup(|a|, |a'|, |b|, |b'|)(|a-a'| + |b-b'|)$ yields

$$[G(\varepsilon^{d-2}, z\varepsilon^{-1}) - G(0, y)]^2 - [G(\varepsilon^{d-2}, z'\varepsilon^{-1}) - G(0, y')]^2 = O(\varepsilon^{\frac{3}{2}d-2}).$$

In the same way, on B''_ε,

$$[G(\varepsilon^{d-2}, z\varepsilon^{-1}) - G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}})]^2 - [G(\varepsilon^{d-2}, z'\varepsilon^{-1}) - G(\varepsilon^{a-2}, Y'^\varepsilon_{\varepsilon^{a-2}})]^2 = O(\varepsilon^{d+\eta-2}).$$

Therefore $1_{B''_\varepsilon}|G_\varepsilon - G'_\varepsilon| \le \varepsilon^{\mu'}$ where $\mu' > 0$. Hence

$$g_2^\varepsilon = E[1_{B''_\varepsilon}|\exp(G'_\varepsilon - G_\varepsilon) - 1| \exp(G_\varepsilon + H_\varepsilon)] \le cst\ (\exp \varepsilon^{\mu'} - 1).$$

i=3: Since

$$|H(t,x) - H(t,x')| \le \|C\|_\infty |G(t,x) - G(t,x')| \le m^{-1}\|C\|_\infty |x - x'|,$$

we have $1_{B''_\varepsilon}|H_\varepsilon - H'_\varepsilon| = O(\varepsilon^\mu)$ and thus $g_3^\varepsilon = O\big(\exp(cst\,\varepsilon^\mu) - 1\big)$.

i=4: Inequality $|D(t,x) - D(t,x')| \le cst\ \varepsilon^2 |x - x'|$ yields

$$|J_1^\varepsilon - J_1'^\varepsilon| \le J_1^\varepsilon[\exp(cst\,\varepsilon^d \int_0^1 A_u^\varepsilon du)\ - 1] \text{ where } A_u^\varepsilon \text{ equals}$$
$$|V_u(G(\varepsilon^{a-2}, Y^\varepsilon_{\varepsilon^{a-2}}), G(\varepsilon^{d-2}, z\varepsilon^{-1})) - V_u(G(\varepsilon^{a-2}, Y'^\varepsilon_{\varepsilon^{a-2}}), G(\varepsilon^{d-2}, z'\varepsilon^{-1}))|.$$

Let $k(\varepsilon) = \exp(cst\ \varepsilon^{2d+\eta-1}) - 1$. Then

$$1_{B''_\varepsilon} A_u^\varepsilon \le m^{-1}(\varepsilon^\mu + \varepsilon^{d+\eta-1}) \Rightarrow 1_{B''_\varepsilon}|J_1^\varepsilon - J_1'^\varepsilon| \le k(\varepsilon) J_1^\varepsilon.$$

Similarly, $|J_2^\varepsilon - J_2'^\varepsilon| \le k(\varepsilon) J_2^\varepsilon$ and $g_4^\varepsilon \le cst\ k(\varepsilon) E \exp(G'_\varepsilon + H'_\varepsilon) = o(1)$.

7. End of the proof

7.1. The linear problem

By lemma 3.1, we have to study

$$v_1^\varepsilon = E1_{\{Z_T^\varepsilon \le 0, \|Z^\varepsilon\| \le \rho\}} \exp[p\,\overline{Z}_T^\varepsilon\,\varepsilon^{-2} + K(g_1)].$$

Let $\gamma > 0$ small enough. Lemma 5.1 yields the existence of $Q = (Q_0, Q_1, K_0, K_1)$ independent of $Y = -g_1(T)$ and $Q_2, K_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\overline{Z}_T^\varepsilon = \varepsilon J_\varepsilon^Q(Y) \quad \text{where} \quad J_\varepsilon^Q(y) = -y + \varepsilon(Q_0 + Q_1 y + Q_2 y^2)\\ &K(g_1) = K_0 + K_1 Y + K_2 Y^2 \quad \text{and} \quad K_0 = K(g_1^0)\\ &P(\Omega \backslash D_\varepsilon Q) = O(\exp -\varepsilon^{-cst}) \end{aligned}$$

where $D_\varepsilon(A_1, \ldots, A_n) = \{|A_i| \le \varepsilon^{-\gamma},\ i = 1, \ldots, n\}$.

Lemma 7.1. *We have* $v_1^\varepsilon = v_2^\varepsilon + o(\varepsilon)$ *where*

$$v_2^\varepsilon = E1_{\{D_\varepsilon(Y,Q),\ -\varepsilon^{2-\gamma} \le \overline{Z}_T^\varepsilon \le 0\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)].$$

Proof. Let $\overline{D}_\varepsilon = D_\varepsilon(Y, Q, \|\Gamma^\varepsilon\|)$ and

$$\begin{aligned} v_3^\varepsilon &= E1_{\{\overline{D}_\varepsilon,\ Z_T^\varepsilon \le 0\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)]\\ \overline{v}_3^\varepsilon &= E1_{\{\overline{D}_\varepsilon,\ \overline{Z}_T^\varepsilon \le 0\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)]. \end{aligned}$$

Azencott ((1980-81), p. 270) proved that there exists $C > 0$ such that

$$\rho r \ge C \Longrightarrow P(\|Z^\varepsilon\| \le \rho, \|\Gamma^\varepsilon\| \ge r) \le \exp -C^{-1} r^{2/3}. \tag{7.1}$$

(7.1) and $P(\|Z^\varepsilon\| \ge \rho) = O(\exp -\varepsilon^{-cst})$ yield $P(\Omega \backslash \overline{D}_\varepsilon) = O(\exp -\varepsilon^{-cst})$.

Therefore, (3.10) implies $v_1^\varepsilon = v_3^\varepsilon + O(\exp -\varepsilon^{-cst})$. Moreover

$$\begin{aligned} |v_3^\varepsilon - \overline{v}_3^\varepsilon| &\le E1_{\overline{D}_\varepsilon} |1_{\{Z_T^\varepsilon \le 0\}} - 1_{\{\overline{Z}_T^\varepsilon \le 0\}}| \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)]\\ &\le E1_{\{\overline{D}_\varepsilon,\ |\overline{Z}_T^\varepsilon| \le \varepsilon^{3-\gamma}\}} \exp[p\varepsilon^{1-\gamma} + K(g_1)]\\ &\le cst\ E1_{\{D_\varepsilon(Y,Q),\ |J_\varepsilon^Q(Y)| \le \varepsilon^{2-\gamma}\}} \exp(K_0 + K_1 Y + K_2 Y^2). \end{aligned}$$

Let $\xi = \displaystyle\int_0^T \sigma_0^2(s)ds$ and $K_3 = K_2 - \dfrac{1}{2\xi}$. This last expression equals

$$E1_{D_\varepsilon Q} H_2^\varepsilon(Q) \exp K_0,$$

where

$$H_2^\varepsilon(Q) = \frac{1}{\sqrt{2\pi\xi}} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} 1_{\{|J_\varepsilon^Q(y)| \le \varepsilon^{2-\gamma}\}} \exp[K_1 y + K_3 y^2] dy.$$

We need the following lemma whose proof is straightforward.

LEMMA 7.2. *On $D_\varepsilon Q$, the function J_ε^Q is one-to-one from $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$ to an interval I_ε^Q which contains $[-1,1]$. Its inverse function Φ_ε^Q satisfies*

$$\forall u \in I_\varepsilon^Q \quad |\Phi_\varepsilon^Q(u) + u| \leq 3\,\varepsilon^{1-3\gamma} \ \ and \ \ |\frac{d\Phi_\varepsilon^Q}{du}(u) + 1| \leq 4\,\varepsilon^{1-2\gamma}.$$

Therefore, on $D_\varepsilon Q$, we have

$$H_2^\varepsilon(Q) = -\frac{1}{\sqrt{2\pi\xi}} \int_{I_\varepsilon^Q} 1_{\{|u| \leq \varepsilon^{2-\gamma}\}} \exp[K_1 \Phi_\varepsilon^Q(u) + K_3 \Phi_\varepsilon^Q(u)^2] d\Phi_\varepsilon^Q(u).$$

Therefore, $1_{D_\varepsilon Q} H_2^\varepsilon(Q) = O(\varepsilon^{2-\gamma})$ and $v_3^\varepsilon = \overline{v}_3^\varepsilon + o(\varepsilon)$. Finally,the presence of $\exp p\varepsilon^{-2}\overline{Z}_T^\varepsilon$ and (3.10) yield $\overline{v}_3^\varepsilon = v_2^\varepsilon + O(\exp -\varepsilon^{-cst})$. □

Since Q and Y are independent, we have $v_2^\varepsilon = E1_{D_\varepsilon Q} H_1^\varepsilon(Q) \exp K(g_1^0)$ where

$$\begin{aligned}
\sqrt{2\pi\xi} H_1^\varepsilon(Q) &= \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} 1_{\{-\varepsilon^{1-\gamma} \leq J_\varepsilon^Q(y) \leq 0\}} \exp[\frac{pJ_\varepsilon^Q(y)}{\varepsilon} + K_1 y + K_3 y^2] dy \\
&= -\int_{I_\varepsilon^Q \cap [-\varepsilon^{1-\gamma}, 0]} \exp[\frac{pu}{\varepsilon} + K_1 \Phi_\varepsilon^Q(u) + K_3 \Phi_\varepsilon^Q(u)^2] d\Phi_\varepsilon^Q(u) \\
&= -\varepsilon \int_{-\varepsilon^{-\gamma}}^{0} \exp\,[pv + K_1 \Phi_\varepsilon^Q(\varepsilon v) + K_3 \Phi_\varepsilon^Q(\varepsilon v)^2] \frac{d\Phi_\varepsilon^Q}{du}(\varepsilon v) dv.
\end{aligned}$$

Finally, joint dominated convergence in ω and v ends the proof of th. 2.1:

$$\frac{v_2^\varepsilon}{\varepsilon} \longrightarrow \frac{1}{\sqrt{2\pi\xi}} \int_{-\infty}^{0} \exp pv \ dv \ E \exp K(g_1^0).$$

7.2. THE NONLINEAR PROBLEM

In this section, the proofs of the lemmas are postponed to the end.

By lemmas 3.2 and 4.2, we have to study

$$u_2^\varepsilon = E1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \rho\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1) - \varepsilon^{-2} F_\varepsilon(T(\varepsilon), T, \varphi + Z^\varepsilon)].$$

In order to apply prop. 6.1 and 6.2, we first condition with respect to $\sigma(Z_s^\varepsilon, \, s \leq T'(\varepsilon); \, Z_T^\varepsilon)$ where $T'(\varepsilon) = T - \varepsilon^d$ and $d \in]2/3, a[$. Let $\alpha \in]0, 1-a[$ and

$$u_3^\varepsilon = E1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)] g^\varepsilon(\varepsilon^{-2} Z_T^\varepsilon, \varepsilon^{-1} Z_{T'(\varepsilon)}^\varepsilon).$$

The conditionning yields

LEMMA 7.3. $u_2^\varepsilon = u_3^\varepsilon + o(\varepsilon)$.

According to the strategy described in section 2.4, we introduce the following functional of g_1 and g_2

$$\begin{aligned}
u_4^\varepsilon = E1_{\{D_\varepsilon(V_\varepsilon, Q_\varepsilon), \, -\varepsilon^{2-\gamma} \leq \overline{Z}_T^\varepsilon \leq 0, \, |\overline{Z}_{T'(\varepsilon)}^\varepsilon| \leq \varepsilon^{1+d/2-\gamma}\}} \\
\exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)] g^\varepsilon(\varepsilon^{-2}\overline{Z}_T^\varepsilon, \varepsilon^{-1}\overline{Z}_{T'(\varepsilon)}^\varepsilon).
\end{aligned}$$

where the r.v. Q^ε is defined as follow.

By lemma 5.3, there exists $Q_\varepsilon = \{Q^\varepsilon_{jk}, R^\varepsilon_{jk}, K^\varepsilon_{jk},\ 0 \le j+k \le 1\}$ independent of $V_\varepsilon = (Y, Y_\varepsilon) = (-g_1(T), -g_1(T'(\varepsilon)))$, and there exists $\{Q^\varepsilon_{jk}, R^\varepsilon_{jk}, K^\varepsilon_{jk},\ j+k=2\} \in \mathbb{R}^9$ uniformly bounded in ε such that:

$$\overline{Z}^\varepsilon_T = \varepsilon M^Q_\varepsilon(Y, Y_\varepsilon) \quad \text{where} \quad M^Q_\varepsilon(y,z) = -y + \varepsilon \sum_{0 \le j+k \le 2} Q^\varepsilon_{jk} y^k z^j$$

$$\overline{Z}^\varepsilon_{T'(\varepsilon)} = \varepsilon N^Q_\varepsilon(Y, Y_\varepsilon) \quad \text{where} \quad N^Q_\varepsilon(y,z) = -z + \varepsilon \sum_{0 \le j+k \le 2} R^\varepsilon_{jk} y^k z^j$$

$$K(g_1) = \sum_{0 \le j+k \le 2} K^\varepsilon_{jk} Y^j_\varepsilon Y^k \quad \text{where} \quad K^\varepsilon_{00} = K(G^\varepsilon_{100})$$

$$P(\Omega \backslash D_\varepsilon Q_\varepsilon) = O(\exp - \varepsilon^{-cst}).$$

We now have to study u^ε_4 since

LEMMA 7.4. $u^\varepsilon_3 = u^\varepsilon_4 + o(\varepsilon)$.

Since Q_ε and V_ε are independent, $u^\varepsilon_4 = E 1_{D_\varepsilon Q_\varepsilon} H^\varepsilon_3(Q_\varepsilon) \exp K^\varepsilon_{00}$ where

$$\begin{aligned} H^\varepsilon_3(Q_\varepsilon) = \mu_\varepsilon \iint_{\mathcal{C}(\varepsilon^{-\gamma})} & 1_{\{-\varepsilon^{1-\gamma} \le M^Q_\varepsilon(y,z) \le 0,\ |N^Q_\varepsilon(y,z)| \le \varepsilon^{d/2-\gamma}\}} \\ & \exp(p\varepsilon^{-1} M^Q_\varepsilon(y,z) + \sum_{1 \le j+k \le 2} K^\varepsilon_{jk} y^k z^j)\ g^\varepsilon(\varepsilon^{-1} M^Q_\varepsilon(y,z), N^Q_\varepsilon(y,z)) \\ & \exp(-\frac{z^2}{2 I_0^{T'(\varepsilon)}} - \frac{(y-z)^2}{2 I^T_{T'(\varepsilon)}})\ dy\ dz \end{aligned}$$

$$\mathcal{C}(r) = \{(y,z) \mid |y|, |z| \le r\}, \quad \xi^t_s = \int_s^t \sigma^2_0(u) du, \quad \mu_\varepsilon = (2\pi)^{-1} [\xi_0^{T'(\varepsilon)} \xi^T_{T'(\varepsilon)}]^{-\frac{1}{2}}.$$

In order to compute this gaussian double integral, we use

LEMMA 7.5. *On $D_\varepsilon Q_\varepsilon$, $(M^Q_\varepsilon, N^Q_\varepsilon)$ is a C^∞-diffeomorphism from a neighborhood of $\mathcal{C}(\varepsilon^{-\gamma})$ to a neighborhood of $\mathcal{C}(1)$. Let $\Lambda^Q_\varepsilon = (\Phi^Q_\varepsilon, \Psi^Q_\varepsilon)$ be its inverse function. Then, for all $(u,v) \in \mathcal{C}(1)$,*

$$\mathrm{Jac}\Lambda^Q_\varepsilon(u,v) = 1 + O(\varepsilon^\gamma) \ \ and \ \ |\Phi^Q_\varepsilon(u,v) + u| + |\Psi^Q_\varepsilon(u,v) + v| = O(\varepsilon^{2\gamma}).$$

Let $\lambda_\varepsilon = \sqrt{\xi^T_{T'(\varepsilon)}}$. A change of variables yields (see lemma 7.5)

$$\begin{aligned} H^\varepsilon_3(Q_\varepsilon) = \mu_\varepsilon \iint & 1_{\{-\varepsilon^{1-\gamma} \le u \le 0,\ |v| \le \varepsilon^{d/2-\gamma}\}} \\ & \exp(\frac{pu}{\varepsilon} + \sum_{1 \le j+k \le 2} K^\varepsilon_{jk} \Phi^Q_\varepsilon(u,v)^k \Psi^Q_\varepsilon(u,v)^j)\ g^\varepsilon(\frac{u}{\varepsilon}, v) \\ & \exp(-\frac{\Psi^Q_\varepsilon(u,v)^2}{2\xi_0^{T'(\varepsilon)}} - \frac{(\Phi^Q_\varepsilon(u,v) - \Psi^Q_\varepsilon(u,v))^2}{2\xi^T_{T'(\varepsilon)}}) |\mathrm{Jac}\ \Lambda^Q_\varepsilon(u,v)|\, du\, dv \\ = \varepsilon \mu_\varepsilon \lambda_\varepsilon (1 + O(\varepsilon^\gamma)) \iint & 1_{\{-\varepsilon^{-\gamma} \le u \le 0,\ |v| \le \lambda_\varepsilon^{-1} \varepsilon^{d/2-\gamma}\}} \\ & \exp(pu + \sum_{1 \le j+k \le 2} K^\varepsilon_{jk} \Phi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^k \Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^j)\ g^\varepsilon(u, \lambda_\varepsilon v) \\ & \exp(-\frac{\Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^2}{2\xi_0^{T'(\varepsilon)}} - \frac{(\Phi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v) - \Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v))^2}{2\xi^T_{T'(\varepsilon)}})\ du\ dv. \end{aligned}$$

The following lemma will allow us to end the proof of theorem 2.3.

LEMMA 7.6. *Let* $D'_\varepsilon = D_\varepsilon \|g_1\| \cap \{|Y - Y_\varepsilon| \le \varepsilon^{d/2-\gamma}\}$. *Then,*

i) $P(\Omega \backslash D'_\varepsilon) = O(\exp -\varepsilon^{-cst})$ *and* $1_{D'_\varepsilon} |K^\varepsilon_{00} - K(g^0_1)| = O(\varepsilon^\gamma)$ a.s.

ii) There exists $\varepsilon_0 > 0$ *such that* $(\exp K^\varepsilon_{00})_{\varepsilon \le \varepsilon_0}$ *is uniformly integrable.*

iii) $\varepsilon^{-1} 1_{D_\varepsilon Q_\varepsilon} H^\varepsilon_3(Q_\varepsilon) \longrightarrow (2\pi\xi)^{-\frac{1}{2}} \int_{-\infty}^0 g(u) \exp pu \, du$ a.s.

iv) There exists $M_0 > 0$ *such that for all* ε, $\varepsilon^{-1} 1_{D_\varepsilon Q_\varepsilon} H^\varepsilon_3(Q_\varepsilon) \le M_0$ *a.s.*

Therefore,

$$\begin{aligned}
\varepsilon^{-1} u^\varepsilon_4 &= \varepsilon^{-1} E 1_{D_\varepsilon Q_\varepsilon} H^\varepsilon_3(Q_\varepsilon) \exp K^\varepsilon_{00} \\
&= \varepsilon^{-1} E 1_{D'_\varepsilon \cap D_\varepsilon Q_\varepsilon} H^\varepsilon_3(Q_\varepsilon) \exp K^\varepsilon_{00} + O(\exp -\varepsilon^{-cst}) \\
&= (2\pi\xi)^{-\frac{1}{2}} \int_{-\infty}^0 g(u) \exp pu \, du \; E \exp K(g^0_1) + o(1).
\end{aligned}$$

Proof of lemma 7.3. (3.10) and (8.7) yield $u^\varepsilon_2 = r^\varepsilon_1 + o(\varepsilon)$ where

$$\begin{aligned}
r^\varepsilon_1 &= E 1_{A_\varepsilon} \exp[p\varepsilon^{-2} \overline{Z}^\varepsilon_T + K(g_1) - \varepsilon^{-2} F_\varepsilon(T(\varepsilon), T, \varphi + Z^\varepsilon)] \\
A_\varepsilon &= \{Z^\varepsilon_T \le 0, \|Z^\varepsilon\| \le \varepsilon^{1-\alpha}\}.
\end{aligned}$$

Before conditioning with respect to $\mathcal{G}_\varepsilon = \sigma(Z^\varepsilon_s,\ s \le T'(\varepsilon);\ Z^\varepsilon_T)$, we introduce a $\mathcal{G}_\varepsilon$-measurable r.v. close to $p\varepsilon^{-2}\overline{Z}^\varepsilon_T + K(g_1)$. Integration by part in (2.6) yields

$$K(g_1) = \int_0^T \psi_1(s) g^2_1(s) ds + \int_0^T \psi_2(s) g_1(s) dg_1(s)$$

with continuous ψ_1 and ψ_2. Therefore, $K(g_1) = \kappa_\varepsilon + \kappa'_\varepsilon$ where κ_ε (resp. κ'_ε) corresponds to the integrals on $[0, T'(\varepsilon)]$ (resp. $[T'(\varepsilon), T]$). Hence

$$p\varepsilon^{-2} \overline{Z}^\varepsilon_T + K(g_1) = p\varepsilon^{-2} Z^\varepsilon_T + \kappa_\varepsilon + U_\varepsilon \quad \text{where} \quad U_\varepsilon = -p\varepsilon \Gamma^\varepsilon_T + \kappa'_\varepsilon.$$

(7.1) yields that, for $\lambda \in \,]-\infty, \alpha[$,

$$P(\|Z^\varepsilon\| \le \varepsilon^{1-\alpha}, p\varepsilon|\Gamma^\varepsilon_T| \ge \varepsilon^\lambda) \le \exp -cst\, \varepsilon^{\frac{2}{3}(\lambda-1)}. \tag{7.2}$$

Besides, for $\eta > 0$ small enough, $P(|\kappa'_\varepsilon| \ge 2\varepsilon^\lambda)$ is smaller than

$$\begin{aligned}
&P(|\int_{T'(\varepsilon)}^T \psi_1(s) g^2_1(s) ds| \ge \varepsilon^\lambda) + P(|\int_{T'(\varepsilon)}^T \psi_2(s) g_1(s) \sigma_0(s) dW_s| \ge \varepsilon^\lambda) \\
&\quad \le P(\|g_1\| \ge cst\; \varepsilon^{\frac{1}{2}}(\lambda - d)) + P(\|g_1\| \ge \varepsilon^{-\eta}) \\
&\qquad + P(|\int_{T'(\varepsilon)}^T \psi_2(s) g_1(s) \sigma_0(s) dW_s| \ge \varepsilon^\lambda, \|g_1\| \le \varepsilon^{-\eta}) \\
&\quad \le \exp -cst\, \varepsilon^{\lambda-d} + \exp -cst\, \varepsilon^{-2\eta} + \exp -cst\, \varepsilon^{2\lambda-d+2\eta}
\end{aligned} \tag{7.3}$$

($\lambda < \alpha < 1/3$ and $d > 2/3$). (7.2) and (7.3) with $\lambda \in]0, \alpha[$ yield

$$P(\|Z^\varepsilon\| \le \varepsilon^{1-\alpha}, |U_\varepsilon| \ge 3\varepsilon^\lambda) = O(\exp -\varepsilon^{-cst}). \tag{7.4}$$

(7.2) and (7.3) with $\lambda = -\eta$ yield the existence of $\eta' > \eta$ such that

$$P(\|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}, |U_\varepsilon| \geq 3\varepsilon^{-\eta}) = O(\exp -\varepsilon^{-\eta'}). \quad (7.5)$$

(7.5) imply that, for all $q \in \mathbb{R}$,

$$\sup_{\varepsilon \leq \varepsilon_0} E1_{\{\|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}\}} \exp q|U_\varepsilon| \ < \infty. \quad (7.6)$$

Let $r_2^\varepsilon = E1_{A_\varepsilon} \exp[p\varepsilon^{-2} Z_T^\varepsilon + \kappa_\varepsilon - \varepsilon^{-2} F_\varepsilon(T(\varepsilon), T, \varphi + Z^\varepsilon)]$. Then,

$$|r_2^\varepsilon - r_1^\varepsilon| \leq E1_{A_\varepsilon} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)]\big(\exp|U^\varepsilon| - 1\big).$$

Therefore, (3.10), (7.4) and (7.6) imply $r_2^\varepsilon - r_1^\varepsilon = o(\varepsilon)$.

With the same techniques, if we replace in r_2^ε the event A_ε by $\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\|_0^{T'(\varepsilon)} \leq \varepsilon^{1-\alpha}, \|Z^\varepsilon\|_{T(\varepsilon)}^T \leq \varepsilon^{1-\alpha}\}$, then the difference is $o(\varepsilon)$.

Finally, we condition on $\mathcal{G}_\varepsilon$: the Markov property yields that r_2^ε equals

$$E1_{\{Z_T^\varepsilon \leq 0, \|Z^\varepsilon\|_0^{T'(\varepsilon)} \leq \varepsilon^{1-\alpha}\}} \exp[p\varepsilon^{-2} Z_T^\varepsilon + \kappa_\varepsilon] g^\varepsilon(\varepsilon^{-2} Z_T^\varepsilon, \varepsilon^{-1} Z_{T'(\varepsilon)}^\varepsilon) + o(\varepsilon).$$

We conclude by proving $r_2^\varepsilon = u_3^\varepsilon + o(\varepsilon)$ (same techniques). □

Proof of lemma 7.4. Let

$$S_\varepsilon = 1_{D_\varepsilon(V_\varepsilon, Q_\varepsilon, \|\Gamma^\varepsilon\|)} \exp K(g_1)$$
$$s_1^\varepsilon = E1_{\{\|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}, Z_T^\varepsilon \leq 0\}} S_\varepsilon \exp(p\varepsilon^{-2}\overline{Z}_T^\varepsilon) g^\varepsilon(\varepsilon^{-2} Z_T^\varepsilon, \varepsilon^{-1} Z_{T'(\varepsilon)}^\varepsilon)$$
$$s_2^\varepsilon = E1_{\{\|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}, \bar{Z}_T^\varepsilon \leq 0\}} S_\varepsilon \exp(p\varepsilon^{-2}\overline{Z}_T^\varepsilon) g^\varepsilon(\varepsilon^{-2} Z_T^\varepsilon, \varepsilon^{-1} Z_{T'(\varepsilon)}^\varepsilon).$$

(3.10) and (7.1) yield $u_3^\varepsilon = s_1^\varepsilon + O(\exp -\varepsilon^{-cst})$. We have

$$1_{D_\varepsilon\|\Gamma^\varepsilon\|} |1_{\{Z_T^\varepsilon \leq 0\}} - 1_{\{\overline{Z}_T^\varepsilon \leq 0\}}| \exp p\varepsilon^{-2}\overline{Z}_T^\varepsilon \leq \exp p\varepsilon^{1-\gamma}$$
$$|s_2^\varepsilon - s_1^\varepsilon| \leq cst\, E1_{\{D_\varepsilon Y,\ |\overline{Z}_T^\varepsilon| \leq \varepsilon^{3-\gamma}\}} \exp K(g_1).$$

We divide this last expectation into two parts by introducing $1_{D_\varepsilon Q}$ and $1_{\Omega \setminus D_\varepsilon Q}$ (see section 7.1). The first part is $o(\varepsilon)$ (see proof of lemma 7.2), and the second one is $O(\exp -\varepsilon^{-cst})$. Hence $s_1^\varepsilon = s_2^\varepsilon + o(\varepsilon)$.

Write $s_2^\varepsilon = s_3^\varepsilon + s_4^\varepsilon$ where s_3^ε (resp. s_4^ε) corresponds to $1_{\{-\varepsilon^{2-\gamma} \leq \overline{Z}_T^\varepsilon\}}$ (resp. $1_{\{-\varepsilon^{2-\gamma} > \overline{Z}_T^\varepsilon\}}$). Then $s_4^\varepsilon = O(\exp -\varepsilon^{-cst})$. Moreover, (8.7) yields

$$P(|Z_T^\varepsilon - Z_{T'(\varepsilon)}^\varepsilon| \geq \varepsilon^{1+d/2-\gamma}) \leq \exp -cst\, \varepsilon^{-2\gamma}. \quad (7.7)$$

Since $\{D_\varepsilon\|\Gamma^\varepsilon\|, -\varepsilon^{2-\gamma} \leq \overline{Z}_T^\varepsilon \leq 0\} \subset \{|Z_T^\varepsilon| \leq 2\varepsilon^{2-\gamma}\}$, inequality (7.7) yields

$$s_3^\varepsilon = E1_{\{\|Z^\varepsilon\| \leq \varepsilon^{1-\alpha}, -\varepsilon^{2-\gamma} \leq \overline{Z}_T^\varepsilon \leq 0, |Z_T^\varepsilon| \leq 2\varepsilon^{2-\gamma}\}}$$
$$1_{\{|Z_{T'(\varepsilon)}^\varepsilon| \leq 2\varepsilon^{1+\frac{d}{2}-\gamma}, |\overline{Z}_{T'(\varepsilon)}^\varepsilon| \leq 3\varepsilon^{1+d/2-\gamma}\}}$$
$$S_\varepsilon \exp(p\varepsilon^{-2}\overline{Z}_T^\varepsilon) g^\varepsilon(\varepsilon^{-2} Z_T^\varepsilon, \varepsilon^{-1} Z_{T'(\varepsilon)}^\varepsilon) + o(\varepsilon).$$

Define s_5^ε by replacing $g^\varepsilon(\varepsilon^{-2}Z_T^\varepsilon, \varepsilon^{-1}Z_{T'(\varepsilon)}^\varepsilon)$ by $g^\varepsilon(\varepsilon^{-2}\overline{Z}_T^\varepsilon, \varepsilon^{-1}\overline{Z}_{T'(\varepsilon)}^\varepsilon)$ in the previous expectation. According to proposition 6.2.ii,

$$|s_3^\varepsilon - s_5^\varepsilon| \le o(1)\ E1_{\{\overline{Z}_T^\varepsilon \le 0,\ \|Z^\varepsilon\| \le \varepsilon^{1-\alpha}\}} \exp[p\varepsilon^{-2}\overline{Z}_T^\varepsilon + K(g_1)].$$

This last expectation is smaller than v_1^ε if $\varepsilon^{1-\alpha} \le \rho$. Therefore, theorem 2.1 yields $s_3^\varepsilon = s_5^\varepsilon + o(\varepsilon)$. Finally, easy computations imply $s_5^\varepsilon = u_4^\varepsilon + o(\varepsilon)$. □

Proof of lemma 7.5. On $D_\varepsilon Q_\varepsilon$, for $(y,z) \in \mathcal{C}(\varepsilon^{-\gamma})$,

$$\begin{aligned} &\partial_y M_\varepsilon^Q(y,z) = -1 + O(\varepsilon^\gamma) \quad && \partial_z M_\varepsilon^Q(y,z) = O(\varepsilon^\gamma) \\ &\partial_y N_\varepsilon^Q(y,z) = O(\varepsilon^\gamma) && \partial_z N_\varepsilon^Q(y,z) = -1 + O(\varepsilon^\gamma). \end{aligned}$$

Let $L_\varepsilon^Q = (M_\varepsilon^Q, N_\varepsilon^Q)$. Hence, Jac $L_\varepsilon^Q(y,z) = 1 + O(\varepsilon^\gamma)$. According to the local inversion theorem, there exists V_ε^Q, open neighborhood of $\mathcal{C}(\varepsilon^{-\gamma})$, such that L_ε^Q is a C^∞-diffeomorphism from V_ε^Q to $L_\varepsilon^Q(V_\varepsilon^Q)$. We can choose V_ε^Q simply connected. Thus $L_\varepsilon^Q(V_\varepsilon^Q)$ is also simply connected. Consider a closed path whose support is the boundary of $\mathcal{C}(\varepsilon^{-\gamma})$. Its image by L_ε^Q is a closed path whose winding number with respect to $(0,0)$ is not zero and whose intersection with $\mathcal{C}(1)$ is empty (easy check). Hence $L_\varepsilon^Q(V_\varepsilon)$ contains $\mathcal{C}(1)$. The rest of the proof is straightforward. □

Proof of lemma 7.6.i. The first part is a consequence of (8.7). According to (5.2), we have

$$\begin{aligned} G_{100}^\varepsilon(t) - g_1^0(t) &= \Big(\frac{\xi_0^t}{\xi_0^{T'(\varepsilon)}} - \frac{\xi_0^t}{\xi_0^T}\Big)Y_\varepsilon - \frac{\xi_0^t}{\xi_0^T}(Y - Y_\varepsilon) \quad \text{if } t \le T'(\varepsilon) \\ &= \Big(1 - \frac{\xi_0^t}{\xi_0^T}\Big)Y_\varepsilon - \Big(\frac{\xi_0^t}{\xi_0^T} - \frac{\xi_{T'(\varepsilon)}^t}{\xi_{T'(\varepsilon)}^T}\Big)(Y - Y_\varepsilon) \quad \text{if } t \ge T'(\varepsilon), \end{aligned}$$

and $\|G_{100}^\varepsilon - g_1^0\| = O(\varepsilon^d\|g_1\| + |Y - Y_\varepsilon|) = O(\varepsilon^{2\gamma})$ on D'_ε.

(2.11) yields $K_{00}^\varepsilon - K(g_1^0) = O\big(\|(G_{100}^\varepsilon)^2 - (g_1^0)^2\|\big)$. Therefore,

$$1_{D'_\varepsilon}|K_{00}^\varepsilon - K(g_1^0)| = O\big(\varepsilon^{2\gamma}(\|G_{100}^\varepsilon + g_1^0\|)\big)$$

and we conclude by using $\|G_{100}^\varepsilon\| + \|g_1^0\| = O(\|g_1\|)$. □

Proof of lemma 7.6.ii. Let β given by lemma 2.2.

A conditioning on $\{K_{00}^\varepsilon, K_{01}^\varepsilon, K_{10}^\varepsilon\}$ yields

$$\begin{aligned} &E\exp(1+\beta)K(g_1) = E\Lambda_\varepsilon \exp(1+\beta)K_{00}^\varepsilon < \infty \\ &\Lambda_\varepsilon = \mu_\varepsilon \iint \exp(1+\beta)\Big(\sum_{1\le j+k\le 2} K_{jk}^\varepsilon y^k z^j\Big)\exp\Big(-\frac{z^2}{2\xi_0^{T'(\varepsilon)}} - \frac{(y-z)^2}{2\xi_{T'(\varepsilon)}^T}\Big)\, dy\, dz. \end{aligned}$$

Since the deterministic $K_{02}^\varepsilon, K_{11}^\varepsilon, K_{20}^\varepsilon$ are bounded uniformly in ε,

$$\sum_{j+k=2} K_{jk}^\varepsilon y^k z^j \ge -cst\,(z^2 + |yz| + y^2) \ge -cst\,(z^2 + (y-z)^2).$$

It yields

$$\Lambda_\varepsilon \geq \mu_\varepsilon \iint \exp(1+\beta)(K_{01}y + K_{02}z) \exp(-cst\, z^2 - (y-z)^2 \, (\xi^T_{T'(\varepsilon)})^{-1}) dydz.$$

If X is a one-dimensional gaussian centered r.v., then $E \exp \lambda X \geq 1$ for all $\lambda \in \mathbb{R}$. Hence, there exists $m_0 > 0$ such that $\inf_\varepsilon \Lambda_\varepsilon > m_0$ a.s., and $\sup_\varepsilon E \exp(1+\beta) K^\varepsilon_{00} < \infty$. □

Proof of lemma 7.6.iii and 7.6.iv. Let $\nu \in]1 - d/2 + \gamma, 1 + a/2 - d[$. Since $|v| \leq \lambda_\varepsilon^{-1} \varepsilon^{d/2-\gamma}$, we have $\lambda_\varepsilon v = O(\varepsilon^{1-\nu})$ and prop. 6.1 yields $g^\varepsilon(u, \lambda_\varepsilon v)$ tends to $g(u)$. Assume $\omega \in \cap_\varepsilon D_\varepsilon Q_\varepsilon$ (countable intersection, see remark 6.1). Assume $-\varepsilon^{-\gamma} \leq u \leq 0$, $1 \leq |v| \leq \lambda_\varepsilon^{-1} \varepsilon^{d/2-\gamma}$. Lemma 7.5 yields

$$| \sum_{1 \leq j+k \leq 2} K^\varepsilon_{jk} \Phi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^k \Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^j | \leq \varepsilon^\gamma$$

$$- \frac{(\Phi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v) - \Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v))^2}{2\xi^T_{T'(\varepsilon)}} \leq -\frac{1}{3} v^2$$

$$\frac{(\Phi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v) - \Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v))^2}{2\xi^T_{T'(\varepsilon)}} \longrightarrow \frac{1}{2} v^2$$

$$\frac{\Psi^Q_\varepsilon(\varepsilon u, \lambda_\varepsilon v)^2}{2\xi_0^{T'(\varepsilon)}} \longrightarrow 0$$

and we can conclude easily. □

8. Appendix

Appendix 8.1. *There exists $C > 0$ such that, for all ε, t, x*

$$|\partial_x u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x)| \leq C \, (t^{-1/2} + t^{1/2}|x| + t) \tag{8.1}$$

$$|\partial^2_{xx} u^\varepsilon(\varepsilon^2 t, \varepsilon^2 x)| \leq C \, (t^{-1/2} + t^2 x^2 + t^{5/2}). \tag{8.2}$$

Proof . A similar result is known if σ and c are constant (see Uchiyama (1978)), and we use the same line of proof. Let $f(u) = 1_{\{0 \leq u \leq 1\}} u(1 - r(u))$.

Let $p_t(x, y)$ be the transition density of diffusion $dX_s = \sigma(X_s) dW_s$. Then, the following problems 1 and 2 both have a unique solution, and they coincide.

Problem 1

$\partial_t u = \dfrac{1}{2} \sigma^2(x) \, \partial^2_{xx} u + c(x) \, f(u)$ if $t > 0$

$\lim_{t \to 0} u(t, x) = 1_{\{x \, \leq \, 0\}}$ uniformly on every compact of $\mathbb{R}^*$.

$u \in C^{1,2}(]0, +\infty[\times \mathbb{R}, [0, 1])$

Problem 2

$$u(t, x) = \int_{-\infty}^0 p_t(x, y) dy + \int_0^t ds \int_{-\infty}^{+\infty} p_{t-s}(x, y) c(y) f(u(s, y)) dy \;\text{ if }\; t > 0$$

$u(0,x) = 1_{\{x \leq 0\}}$
$u \in C(]0,+\infty[\times\mathbb{R},[0,1])$

The proof of this equivalence is known if c is bounded (see Bramson (1982)) and it can be extended when c has, at most, a linear growth by using classical arguments (Feynman-Kac formula, maximum principle ...). We get

$$u^\varepsilon(\varepsilon^2 t,\varepsilon^2 x) = \int_{-\infty}^0 p_t^\varepsilon(x,y)dy + \int_0^t ds \int_{-\infty}^{+\infty} p_{t-s}^\varepsilon(x,y)c(\varepsilon^2 y)f(u^\varepsilon(\varepsilon^2 s,\varepsilon^2 y))dy \tag{8.3}$$

where p^ε is the transition density of the diffusion $dX_s^\varepsilon = \sigma(\varepsilon^2 X_s^\varepsilon)dW_s$.

Since $\|\sigma\|_\infty < \infty$ and $\sup_{\varepsilon\leq\varepsilon_0} \|\partial_x\sigma(\varepsilon^2.)\|_\infty < \infty$, Friedman's estimates yield (see Friedman(1964)): $\exists C_1, C_2 > 0\ \forall\varepsilon, t, x, y$

$$p_t^\varepsilon(x,y) \leq C_1\ (2\pi t)^{-1/2}\exp[-C_2\,(x-y)^2(2t)^{-1}] \tag{8.4}$$
$$|\partial_x p_t^\varepsilon(x,y)| \leq C_1\ (2\pi t^2)^{-1/2}\exp[-C_2\,(x-y)^2(2t)^{-1}] \tag{8.5}$$
$$|\partial_y p_t^\varepsilon(x,y)| \leq C_1\ (2\pi t^2)^{-1/2}\exp[-C_2\,(x-y)^2(2t)^{-1}]. \tag{8.6}$$

Then, we get (8.1) by differentiating (8.3) and applying (8.4).

(8.2) cannot be derived by differentiating twice (8.3) (no dominated convergence). Let A^ε and $(P_s^\varepsilon)_{s\geq 0}$ be the infinitesimal generator and the semi-group of X^ε.

Let $f_s^\varepsilon(x) = c(\varepsilon^2 x)f(u^\varepsilon(\varepsilon^2 s,\varepsilon^2 x))$ and $w^\varepsilon(t,x) = \int_0^t P_{t-s}^\varepsilon f_s^\varepsilon(x)ds$.

Then, since A^ε and P_{t-s}^ε commute,

$$\begin{aligned} A^\varepsilon w^\varepsilon(t,x) &= \int_0^t A^\varepsilon P_{t-s}^\varepsilon f_s^\varepsilon(x)ds = \int_0^t P_{t-s}^\varepsilon A^\varepsilon f_s^\varepsilon(x)ds \\ &= -\frac{1}{2}\int_0^t ds \int_{-\infty}^{+\infty} \partial_y[p_{t-s}^\varepsilon(x,y)\sigma^2(\varepsilon^2 y)]\partial_y[c(\varepsilon^2 y)f(u^\varepsilon(\varepsilon^2 s,\varepsilon^2 y))]dy, \end{aligned}$$

(last equality is a consequence of integration by part formula). (8.1), (8.4) and (8.6) imply $|A^\varepsilon w^\varepsilon(t,x)| = O(1+t^2x^2+t^{5/2})$. It allows us to conclude since $\sigma \geq m > 0$. □

Appendix 8.2. *Consider a brownian martingale* $Z_t = x + \int_0^t A_s dW_s$ *where* A *satisfies* $||A||_0^\infty \leq M$ *a.s.* $(M \in \mathbb{R})$. *Then*

$$\forall r > |x| \quad P(||Z||_0^t \geq r) \leq \exp -\frac{(r-x)^2}{2tM^2} + \exp -\frac{(r+x)^2}{2tM^2} \tag{8.7}$$

$$\limsup_{s\to+\infty} \frac{|Z_s|}{\sqrt{2s\log\log s}} \leq \frac{1+M^2}{2} \quad a.s. \tag{8.8}$$

Proof. (8.7) is a classical consequence of Doob inequality. Concerning (8.8), we copy the proof of the law of iterated logarithm for the brownian motion (see for instance Revuz and Yor (1991)). □

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