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Relativistic Wave Equations

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Introduction

In general quantum field theory, formulation of dynamics is one of the most important yet most difficult problems. Experience with classical field theories - e.g. Maxwell theory - indicates that relativistic wave equations, in one form or another, are going to play an important role. This optimism is furthermore supported by the analysis of models, e.g. Thirring model [1] or φ^4 theory both in 2 space time dimensions, and of renormalized perturbation theory.

As a step towards full quantum theory one might consider the so called external field problems. There the quantum field is coupled to a classical field given at the outset. No selfinteraction is included. The dynamical equations are linear in the quantum field. Nothing is known rigorously in what sense such theories are approximations of a full quantum field theory. This is believed to be the case.

It is well known that there is an uncountable set of dynamical equations which lead in the limit of vanishing external field to a free field theory of given mass and spin. Not all of them lead to a satisfactory field theory if an external field is included. Those passing this test are called stable [2].

Problems and results in quantum field theory with external fields will be reviewed. Since the talk addresses in particular the mathematicians in the audience, I will start with a fairly long historical introduction. For a very clear review of mainly the mathematical aspects of the external field approximation we refer to recent work by A.S. Wightman [3].

I. On the History of Relativistic Wave Equations

In the very beginning of this century Einstein [4] reproduced Planck's result on the black body radiation in quantizing the electromagnetic field in a cavity Starting from Maxwell's equations

$$\mathrm{rot}\, E + \dot{B} = 0 \qquad \mathrm{div}\, E = 0 \qquad \dot{} = \frac{d}{dt}$$

$$\mathrm{div}\, B = 0 \qquad \mathrm{rot}\, B - \dot{E} = 0$$

for the radiation field inside a cubic cavity with the boundary conditions:

Component of E parallel to the wall vanishes,

Component of B perpendicular to the wall vanishes.

In terms of the vector potential A (in the so called Coulomb gauge i. e. $A_0 = 0$, div $A = 0$) the equation of motions for the Fourier components of A reads

$$\ddot{q}(k,t,\alpha) + \omega^2 q(k,t,\alpha) = 0 \quad , \quad \omega^2 = k^2 .$$

k is a vector of the reciprocal lattice $\Gamma = \{ k = \frac{\pi}{L}(n_1, n_2, n_3)$, (n_i natural number, L length of the cavity)$\}$. $\alpha = 1, 2$ characterizes the internal degree of freedom (the spin). For each point in the reciprocal lattice one gets a classical harmonic oscillator. At this point Einstein introduced the quantum assumption, setting the energy of each oscillator as a multiple of a minimal quantum. In an updated version the discrete energy would be produced by imposing on the Fourier components the commutation rules

$$[p(k,t,\alpha), q(k',t,\alpha')] = i\delta_{kk'}\delta_{\alpha\alpha'} \quad , \quad p(k,t,\alpha) = \ddot{q}(k,t,\alpha) .$$

From a classical field theory Einstein produced through quantization a many body theory for the photon gas.

In 1926 Schrödinger [5] wrote the equation for a nonrelativistic particle

$$i\partial_t \varphi = -\frac{\hbar}{2m}\Delta\varphi + V\varphi \qquad (1)$$

Δ Laplace operator

V potential function

m mass of particle .

$\hbar$ Planck's constant

$\varphi \in \mathbb{C}$.

under the substitution

$$x \to \tilde{x} = Gx \quad , x = (t, \underline{x}) \in \mathbb{R}^4$$
$$\varphi \to \tilde{\varphi} \, , \; \tilde{\varphi}(x) = \varphi(G^{-1}x) \quad , V \to \tilde{V},$$

where G denotes a Galilean transformation. Therefore the set of (I. 1) with V = o is invariant under the transformation $\varphi \to \tilde{\varphi}$. This transformation is unitary with respect to the scalar product $(\varphi, \varphi) = \int d^3x \, |\varphi(x)|^2$.
Due to the continuity equation $\dot{\varrho} + \operatorname{div} j = 0$, $\varrho = |\varphi|^2$, $j = \frac{1}{2m}(\bar{\varphi} \operatorname{grad} \varphi - \varphi \operatorname{grad} \bar{\varphi})$ the density $\varrho(x) = |\varphi(x)|^2 \geq 0$ naturally lends itself for a probability interpretation for localization in a volume element d^3x at time t. In addition the limit $\hbar \to 0$ reproduces Newton's equation of motion for a point particle of mass m [6], (Correspondance principle).

Again in 1926 the Klein Gordon equation was introduced [7]:

$$(\partial_t^2 - \Delta)\varphi(x) = 0 \quad , \quad x \in \mathbb{R}^4 , \quad \varphi(x) \in \mathbb{C} . \tag{2}$$

To the dismay of physicists it was a partial differential equation of second order in the time derivative. Two functions are required for the Cauchy problem. It was not noticed that this was just a consequence of the fact that the set of solutions of (2) carries a reducible representation of the inhomogeneous Lorentz group or it's covering group $iSL(2,\mathbb{C})$, one irreducible part being characterized by a positive mass m and spin s=o, the other by -m and s = o . The solutions of (2) with positive particle energies are solutions of the first order equation

$$i\partial_t \varphi = \sqrt{m^2 - \Delta}\, \varphi .$$

(We call particle energies the zero components of the p's in the plane wave solution $e^{-ipx} u(p)$ of the wave equation). Of course $\sqrt{m^2-\Delta}$ is no longer a local operator in the sense that the values of $\sqrt{m^2-\Delta}\,\varphi$ at x are no longer given by those of φ and it s derivatives at x . The fact that the solutions of (2) carry a representation of $iSL(2,\mathbb{C})$ which is not irreducible manifests itself once more. The density $\varrho(x) = \bar{\varphi}(x)\partial_t\varphi(x) - \varphi(x)\partial_t\bar{\varphi}(x)$ is not positivedefinite. Together with $j = \bar{\varphi}\,\mathrm{grad}\,\varphi - \varphi\,\mathrm{grad}\,\bar{\varphi}$ it suffices a continuity equation

$\dot{\rho} + \mathrm{div}\, j = 0$.

However the Klein Gordon field has a well defined positive field energy

$$E(\varphi) = \frac{1}{2}\left(m^2|\varphi|^2 + |\partial_t\varphi|^2 + |\mathrm{grad}\,\varphi|^2\right) \tag{3}$$

leading to a usefull norm.

The situation is therefore quite similar to the one for the electromagnetic field where one has again no density which allows for a probability interpretation of localization but a positive energy density. However at that time it was thought that the Klein-Gordon equation would have to give a one particle theory of a relativistic mass point, an interpretation inconsistant with the principles of relativistic equivalence of mass and energy as it became clear only much later.

One might argue that just the positive energy solutions of (2) could be used for a one particle theory the same way as we considered only real solutions of the Maxwell equations. This does not work for the following reason. It is believed that the coupling of a particle or field to the electromagnetic field A_μ is given by the substitution $\partial_\mu \to D_\mu = \partial_\mu + iA_\mu$ in the equation of motion for the particle or field [6). In our case this leads to

$$\left[(\partial_t + iA_0)^2 - \sum_{\ell=1}^{3}(\partial_\ell + iA_\ell)^2 + m^2\right]\varphi = 0, \tag{4}$$

Consider a situation where A_μ has compact supp ort in space time and the solution of (4) is of the positive energy type for $t \to -\infty$, i.e. $\mathrm{supp}\,\tilde{\varphi} \subset V_+$ ($\tilde{\varphi}$ is the Fourier transformation of φ). Then $\varphi(x)$ is given by integration of (4) and this will in general lead to a wave function φ which for $t \to +\infty$ is not of the positive energy type anymore. This is the so called Klein phenomenon which makes such a theory unstable in the sense of the introduction. We will discuss problems of this type in chapter III.

Dirac [9], in his belief that the origin of all troubles with the Klein-Gordon equation would come from the second order structure, invented his famous first order equation for a vector valued function $\psi(x) \in \mathbb{C}^4$

$$(\gamma^\mu \partial_\mu + m)\,\psi(x) = 0. \tag{5}$$

It was considered a great success that this theory allows for a positive density $\rho(x) = \psi^*(x)\,\psi(x)$ (star denoting complex conjugation and transposition) and a current $j^\rho = \psi^* \gamma^0 \gamma^\rho \psi$ such that

$$\partial_t \rho + \operatorname{div} j = 0.$$

$\rho(x)$ was considered to be a probability of localization, an interpretation which turnes out to be wrong [8]. The field energy density

$$E(\psi) = \frac{i}{2}\left(\psi^* \partial_t \psi - \psi\, \partial_t \psi^*\right)$$

as well as the particle energy - energies which appear in plane wave solutions - are indefinite. The Klein phenomenon holds again.

The Dirac equation still had it's striking success: It's nonrelativistic limit turned out to be the previously known Pauli equation for an electron with spin $1/2$ and the correct magnetic moment $-\frac{e\hbar}{2m}$. The energy spectrum of the hydrogen atom was given correctly.

Summarizing, the situation at the end of the twenties looked rather hopeless for relativistic theories. Neither the Klein-Gordon nor the Dirac wave functions produced satisfactory probability distributions. Dirac's equation was even so pathological as it did not even allow for a classical field interpretation because of it's indefinite field energy.

In 1930 came a real breakthrough when Dirac proposed the quantization of his theory analogously to Einstein's quantization of the black body radiation, however with the corresponding operators not commuting but rather anticommuting [10]. This he proposed in order to account for Pauli's exclusion principle for electrons. It was an other great success as it turned out that the anticommutation rules are necessary for a theory with support of the energy momentum vector in the forward light cone V_+ and Einstein causality [11) 12)].

In 1936 Fierz generalized the quantization procedure to free fields with arbitrary spin. He used wave equations given earlier by Dirac and demonstrated the connection between spin and statistics. He already noted the instability of these wave equations. In a latter paper by Fierz and Pauli [13)] a general method is given for the construction of relativistic wave equations which were hoped to be free from pathologies considered at that time. The technics of constructing relativistic wave equations blossomed for more then 30 years.

Yet in the beginning of the 40's, relativistic wave equations had still another surprise for physicists. This time so shocking that they were not ready to accept it when it came up for the first time [14)]. But finally, due to the convincing arguments of Velo and Zwanziger [15)] people got used to this sort of crazy behaviour. What I am thinking of is the non-causal propagation for relativistic waves (Chapter III).

Let me finish this introduction with some remarks:
Most useful to the understanding of relativistic wave equations was Wigner's analysis of unitary representation of iSL (2, C), because it allowed for a mathematical concise notion of a fundamental particle: an irreducible unitary representation of i SL (2, C) with $m \geq 0$ and finite spin.

Only recently some mathematicians returned to the problem of classical relativistic field theories, in particular to the selfcoupled massive spin zero field [16)],

$$(\partial_t^2 - \Delta)\varphi + m^2\varphi + g\varphi^3 = 0 \qquad (6)$$

In 1961 Jürgens showed global existence for the Cauchy problem using energy inequalities and in 1972 Morawetz and Strauss considered the asymptotes to a solution φ of (6), i.e.

$$
\begin{aligned}
&(\partial_t^2 - \Delta)\varphi_{\pm} + m^2 \varphi_{\pm} = 0 \\
&\varphi_{\pm} = \lim_{t \to \pm\infty} \varphi_t \\
&\lim_{t \to \pm\infty} \| \varphi_{\pm} - \varphi_t \| = 0,
\end{aligned}
\tag{7}
$$

where $\| \cdot \|$ denotes the energy norm defined in (3). The mapping is one to one and isometric in the E-norm. Energy inequalities have been most usefull in the analysis of Maxwell's equations and the interaction of the radiation field with a spin 1/2 field [17].

II. Quantized Fields, Reduction to c-Number Problem

Following Fierz [4] we look first at the free fields leading through quantization to a many body theory of particles and antiparticles of mass m and spin s . One could as well argue the other way around and start with a many body theory for particles and antiparticles, - the Fock-space construction over the one particle states for particles and antiparticles - and define a local field in terms of particle and antiparticle operators satisfying a field equation.

We will use wave equations which differ slightly from those used by Fierz. They are however equivalent as long as we do not introduce any interaction [18) 19)]. Let $[2s,0]$ ($[0,2s]$) denote the linear space of tensors with 2s undotted (2 s dotted indices), totally symmetric in the undotted (dotted) indices. Let furthermore $\Phi = \binom{\varphi}{\chi}$ be a $[2s,0] \oplus [0,2s]$ valued function of $x \in \mathbb{R}^4$, $\varphi \in [2s,0]$, $\chi \in [0,2s]$. The following equation is a straight forward generalization of the Dirac equation

$$\begin{pmatrix} 0 & D(\frac{p}{m}) \\ D(\frac{\tilde{p}}{m}) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{8}$$

where we used the standard notation:

$$p = \sum p^\mu \sigma_\mu \qquad p^\mu = \sum g^{\mu\nu} p_\nu , \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$$

$$\tilde{p} = \sum p_\mu \sigma_\mu \qquad \sigma^\mu \text{ Pauli matrices,} \quad p^\mu = i\partial^\mu$$

$$D(\tfrac{p}{m}) = \left(\tfrac{p}{m}\right)^{\otimes 2s} , \quad D(\tfrac{\tilde{p}}{m}) = \left(\tfrac{\tilde{p}}{m}\right)^{\otimes 2s} .$$

In addition we impose on the solution Φ of (8) the condition

$$(\Box + m^2)\Phi(x) = 0 . \tag{9}$$

Remarks:

1. (8) and (9) are consistent.
2. With $\phi(x)$ also $\tilde{\phi}(x)$,

$$\tilde{\phi}(x) = \begin{pmatrix} S(A)^{-1} & 0 \\ 0 & S(\varepsilon\bar{A}\varepsilon)^{-1} \end{pmatrix} \phi(\Lambda(A)x + a), \quad \varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \underline{S}(A)^{-1} \phi(\Lambda(A)x + a),$$

is a solution for every $A \in SL(2,\mathbb{C})$. $\Lambda(A)$ denotes the standard representation of SL $(2,\mathbb{C})$ by proper orthochronous Lorentztransformations.

It is readily seen that every solution of (8) and (9) can be written in the form:

$$\phi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d^3p}{\omega(p)} \sum_{\sigma=-s}^{s} \left(u(p,\sigma) a(p,\sigma) e^{-ipx} + v(p,\sigma) b^+(p,\sigma) e^{ipx} \right)$$

where $\omega(p) = \sqrt{m^2 + \vec{p}^2}$ and a, b^+ are Fourier components. u and v are matrices related to the above defined D's. They are most simply given in terms of the so called boosts [p] ,

$$u(p,\sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} D_{\cdot\sigma}([p]) \\ D_{\cdot\sigma}([p]^{+-1}) \end{pmatrix}$$

$$v(p,\sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} D_{\cdot\sigma}([p]\varepsilon^{-1}) \\ D_{\cdot\sigma}([p]^{+-1}\varepsilon^{-1}) \end{pmatrix}$$

$$[p] = \left(2m(m+p^0)\right)^{-1/2} \left((m+p^0) + \vec{p}\vec{\sigma} \right)$$

$$D([p]) = [p]^{\otimes 2s}.$$

In analogy to the radiation field, quantization changes the Fourier components a and b^+ into particle destruction and antiparticle creation operators acting on the Hilbert space of states $\mathfrak{F}$, whose vacuum state $|0\rangle$ satisfies

$$a(p,\sigma)|0\rangle = b(p,\sigma)|0\rangle = 0$$

In addition, the commutation resp. anticommutation relations hold

$$[a(p,\sigma), a^+(p',\sigma')]_\pm = [b(p,\sigma), b^+(p',\sigma')]_\pm = 2\omega(\vec{p})\,\delta(\vec{p}-\vec{p}')\,\delta_{\sigma\sigma'}$$

$$[a(p,\sigma), b(p',\sigma')]_\pm = [a(p,\sigma), b^+(p',\sigma')]_\pm = 0$$

The symmetry relation mentioned in remark 2 above reflects itself in the quantized version as follows: There exists a unitary representation $U(a,\Lambda)$ of $iSL(2,\mathbb{C})$ in $\mathfrak{F}$ implementing the transformation $\psi(x) \to \tilde{\psi}(x)$,

$$U(a,\Lambda)\,\psi(x)\,U^{-1}(a,\Lambda) = S(\Lambda)^{-1}\,\psi(\Lambda x + a).$$

U is highly reducible, however, on the two one-particle subspaces generated by vectors $\psi(f)|0\rangle$ respectively $\psi(f)^+|0\rangle$, the representation is irreducible, characterized by mass m and spin s.

Remarks:

1. $\psi(x)$ is an operator valued distribution.
2. $\psi(x)$ is local, i.e. the only nontrivial commutator resp. anticommutator $[\psi(f), \psi^+(g)]_\pm = 0$, if supp f spacelike to supp g.
3. The spectrum of the energy momentum vector P_μ -generator of translation -is in the forward light cone.
4. The identification of a and b^+ with particle destruction and antiparticle creation operators is the only one consistent with the above remarks. (Theorem of spin and statistics) [11) 12)]
5. $\mathfrak{F}$ is equal to the Fock space constructed over the one-particle state spaces for particles and antiparticles.

6. Starting from a free field satisfying (8) and (9) we can of course construct plenty of other local fields which act on $\mathcal{F}$ such that $|0\rangle$ is cyclic.

With $\phi(x)$ also e. g.

$$\psi(x) = \begin{pmatrix} \frac{1}{m}(\varepsilon p)\varphi \\ -\frac{1}{m}(p\varepsilon)\chi \end{pmatrix}$$

((εp acts on the first index of φ .)

is a local field and it satisfies a similar wave equation as ϕ .

In particular let us construct a free field as follows: Consider the vector valued function

$$u^{(n)}(p,s) = \begin{pmatrix} 0 & \left(\frac{p}{m}\right)^{\otimes(n-1)} \\ \left(\frac{\tilde{p}}{m}\right)^{\otimes(n-1)} & 0 \end{pmatrix} u(p,s) ,$$

where $\left(\frac{p}{m}\right)^{\otimes(n-1)}$ resp. $\left(\frac{\tilde{p}}{m}\right)^{\otimes(n-1)}$ act on the first $n-1$ indices of $u(p,s)$.
Analogously one defines $v^{(n)}(p,s)$. By construction,the field

$$\phi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d^3p}{2\omega(p)} \sum_{s=-s}^{+s} \left(\tilde{u}(p,s)a(p,s)e^{-ipx} + \tilde{v}(p,s)b^+(p,s)e^{ipx}\right) \tag{10}$$

$$\tilde{u}(p,s) = \begin{pmatrix} u^{(1)}(p,s) \\ \vdots \\ u^{(2s)}(p,s) \end{pmatrix} , \quad \tilde{v}(p,s) = \begin{pmatrix} v^{(1)}(p,s) \\ \vdots \\ v^{(2s)}(p,s) \end{pmatrix}$$

is local and satisfies a first order wave equation.

So much for free fields. They are extraordinary nice because they are local quantum fields and at the same time obey relativistic wave equations.

Let us now consider a wave equation for a quantized field $\phi(x)$ coupled to an external electromagnetic field. Specifically we look for a field $\phi(x)$ acting on the Fock space $\mathcal{F}_{in}$ of a free field ϕ_{in} and a generalized Dirac equation of the form

$$\left(\beta^\mu(\partial_\mu + i A_\mu) + m\right)\phi(x) = 0 \; . \tag{11}$$

We shall also require the existence of an antihermitionizing matrix η such that

$$\beta^{\mu *} = -\eta\beta^\mu\eta^{-1} .$$

For the Dirac equation $\eta = \gamma^0$. Then (11) implies

$$\phi^+(x)\left[\beta^\mu(\overleftarrow{\partial_\mu} - i A_\mu) - m\right] = 0$$

for the operator $\phi^+(x) = \phi^*(x)\eta$. The importance of η is based on the fact that the current,

$$j^\mu(x) = \psi^+(x)\beta^\mu\psi(x),$$

which is only formally defined, has vanishing divergence.

For the discussion of wave equations with external fields, it is particularly convenient to consider linear partial differential equations of first order The wave equations of free fields (8) (9) used previously can easily be rewritten in this form, introducing of course more components than in the higher order formulation (remark 6).

In order to make (11) a sensible equation for an operator-valued distribution $\psi(x)$, the external field $A_\mu(x)$ has to be sufficiently regular. We will assume more than necessary and suppose A_μ is a real valued test function of compact support in space-time.

We will assume $\psi(x)$ to be essentially a free field for x^0 negative and sufficiently large. By this we mean that in this region, $\psi(x)$ has a set of trivial components identically vanishing and a complementary set idential to a free field. The redundant components have been introduced by Fierz and Pauli [13] in order to make the theory stable. We will come back to this point. If $\psi(x)$ is a solution of (11) and satisfies the conditions mentioned above it is a solution of the Cauchy problem with data at $x^0 = -\infty$.

More general couplings could of course be considered but they do not lead to any new phenomenon. For an exhaustive discussion they have all to be analyzed. This has only been carried through for a few simple wave equations [20].

Let $S_R(x-y)$ be the retarded fundamental solution of the free wave equation

$$\beta^\mu(\partial_\mu + m) S_R(x-y) = \delta(x-y)$$

$$\operatorname{supp} S_R(x) \subset V_+ .$$

Then the Cauchy problem for $\psi(x)$ can be rewritten:

$$\psi(x) = \psi_{in}(x) - i \int d^4y \, S_R(x-y) \not{A}(y) \psi(y) , \quad \not{A} = \beta^\mu A_\mu .$$

Of course this equation, as well as every other wave equation for a quantum field, has to be considered as a relation between distributions, i.e.

$$\psi(f) = \psi_{in}(f) - i\int d^4x\, d^4y\; f(x)\, S_R(x-y)\, A(y)\, \psi(y) \qquad (12)$$

Notice that we have identified with the field having possibly more components than the free field that we started with. They coincide however as far as the nontrivial components are concerned.

Introducing with Capri [21] the operator on the test function space

$$(T_{R,A} f)(x) = f(x) + i\int d^4y\; f(y)\, S_{R,A}(y-x)\, A(x), \qquad (13)$$

one can rewrite (12) as follows

$$\psi_{in}(f) = \psi(T_R f).$$

Analogously one gets

$$\psi_{out}(f) = \psi(T_A f),$$

where ψ_{out} is supposed to be a free field defined in $\mathfrak{F}$ which might have a vacuum or not. Obviously the discussion of the problem mentioned above is now reduced to the discussion of the operators T_R and T_A. In particular the existence of T_R^{-1} as an operator, mapping the whole test-functions space into itself, is necessary for the existence of ψ. Formally the relation between ψ_{in} and ψ_{out} is given by (see also (27))

$$\psi_{out}(f) = \psi_{in}(T_R^{-1} T_A f) \qquad (14)$$

The operator T_R^{-1} and T_A^{-1} are intimately linked to the fundamental solutions of the interacting wave equation for functions. Let $S_R(x,y;A)$ be the fundamental solution of (11)

$$\left(\gamma^\mu(\partial_\mu + iA_\mu) + m\right) S_{\substack{R\\A}}(x,y,A) = \delta(x-y)$$

$$\operatorname{supp} S_{\substack{R\\A}}(x,y,A) \subset \{ (x,y) \mid x^0 - y^0 \gtrless 0 \} ,$$

We will say something on the existence of fundamental solution later (Chapter III).

Then one gets

$$(T_R^{-1} f)(x) = f(x) - i \int d^4y \; f(y) \, S_R(y,x,A) \, \not{A}(x) , \tag{15}$$

and analogously

$$(T_A^{-1} f)(x) = f(x) - i \int d^4y \; f(y) \, S_A(y,x,A) \, \not{A}(x) .$$

Using the above relations one shows that existence and uniqueness of the Cauchy problem with data at $x^0 = -\infty$ for the operator-valued distribution $\psi(x)$ is equivalent to the analogue c-number problem.

By a straight forward computation, one gets for the commutator resp. anticommutator of ψ the expression

$$[\psi(x), \bar{\psi}(y)] = S_R(x,y,A) - S_A(x,y,A) = S(x,y;A) . \tag{16}$$

III. Fundamental Solutions

Fundamental solutions S_R resp. S_A of relativistic wave equations are important for quantum field theory with external fields as it was explained in the last chapter. Here we wish to comment on the existence and uniqueness problem of S_R resp. S_A in some particular examples. A systematic approach does not exist presently because - among other reasons - there is no systematic approach to the problem of constructing free relativistic wave equations [22].

For some relativistic wave equations the question of existence of a fundamental solution can easily be answered negatively. For these the system of partial differential equations turns out to be inconsistent in the presence of an external field. A typical example of such an algebraically unstable equation is the Dirac equation for spin $3/2$ [13]. Consider a function of $x \in \mathbb{R}^4$ with values in the 12 dimensional linear space of tensors $T = \{\varphi_\mu \mid \mu = 0,1,2,3;\ \varphi_\mu$ Dirac spinor, $\gamma^\mu \varphi_\mu = 0$, γ^μ Dirac matrices$\}$. $\varphi_\mu \in T$ shall be a solution of

$$\begin{aligned} &(\gamma^\mu \partial_\mu + m)\,\varphi_\nu(x) = 0 \\ &\partial^\mu \varphi_\mu(x) = 0. \end{aligned} \tag{17}$$

Minimal coupling $\partial_\mu \to D_\mu = \partial_\mu + i A_\mu$ leads to the equation

$$\begin{aligned} &(\gamma^\mu D_\mu + m)\,\varphi^\nu(x) = 0 \\ &D_\mu \varphi^\mu(x) = 0 \end{aligned} \tag{18}$$

Combination of the two equations yields:

$$D_\mu(\gamma^\nu \partial_\nu + m)\,\varphi^\mu(x) = \gamma^\nu [D_\mu, D_\nu]\,\varphi^\mu(x) = i F_{\mu\nu}\gamma^\mu \varphi^\nu(x) = 0; \tag{19}$$

where we have used the abreviation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the field strength. However (19) is, in general, incompatible with (17) and (18).

Another example of algebraic instability arises from the wave equation

$$(\not\partial + m)\varphi_\mu - \gamma_\mu \partial_\nu \varphi^\nu = 0, \qquad \not\partial = \gamma^\mu \partial_\mu, \tag{20}$$

for functions of $x \in \mathbb{R}^4$ with values in T. (20) is constructed such as to imply (16) and (17). If we introduce in (20) minimal coupling $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, where A_μ is as usual a test function with compact support in space-time, we can state the Cauchy problem at $x_0 = -\infty$. However it will in general not be true that $\gamma^\mu \varphi_\mu(x) = 0$ for x^0 sufficiently large, hence $\varphi_\mu \notin T$.

There are many more examples of spin equations exhibiting algebraic instability [23]. The mechanism which produces this algebraic instability is the large number of equations (20) for a function with values in a vector space of only a few dimensions (12). Introducing functions with values in higher dimensional spaces, and using systems of equations of the type (11) - at least for have integer spin- and quadratic coefficient matrices, Fierz and Pauli were able to avoid algebraic inconsistency.

Wave equations of a second class are algebraically stable but they develop some subtle problems if one wants to construct fundamental solutions. The simplest example which can be carried through and shows the problem of noncausality is the spin one field with external symmetric tensor coupling $T_{\mu\nu}(x)$ in 2 space-time dimensions [24],

$$\begin{aligned} &\partial^\mu \varphi_{\mu\nu} + m\varphi_\nu + T_{\nu\rho}\varphi^\rho = 0 \\ &\partial_\nu \varphi_\mu - \partial_\mu \varphi_\nu + m\varphi_{\mu\nu} = 0, \qquad \partial_\nu = g_{\nu\mu}\partial^\mu, \ g \text{ Minkowski metric.} \end{aligned} \tag{21}$$

Fundamental solutions can be constructed and their support analyzed. It turns out that the support is in general not the light cone but rather a bigger or smaller cone depending on $T_{\mu\nu}$. This is the noncausality phenomenon of Velo and Zwanziger [15] Their result was such a surprise because through (16) the corresponding quantum theory is now nonlocal. The external field approximation was not believed to be in conflict with the principle of locality.

Notice that the noncausality phenomenon comes in through a piece of the partial differential operator not belonging to the principal part.

A second example of the class with subtle instabilities is the Fierz-Pauli equation for spin 3/2 [25]. For the construction of fundamental solutions we use an algebraic relation which is most usefull in other contexts too. [26) 27)]. It avoids the difficulties coming from the singularity of β° which is part of the origin of subtle instabilities. In the formulation of Rarita and Schwinger the wave equation reads

$$(L\varphi)^{\mu} = (\not{D} + m)\varphi^{\mu} - (\gamma^{\mu} D_{\nu} + D^{\mu}\gamma_{\nu})\varphi^{\nu} - \gamma^{\mu}\not{D}\gamma_{\nu}\varphi^{\nu} + m\gamma^{\mu}\gamma_{\nu}\varphi^{\nu} = 0 \qquad (22)$$

Multiplying the partial differential operator L from the right with d' yields

$$d'^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \tfrac{1}{3}\gamma^{\mu}\gamma_{\nu} + \tfrac{1}{3m}(\gamma^{\mu}D_{\nu} - D^{\mu}\gamma_{\nu}) + \tfrac{2}{3m^2} D^{\mu}D_{\nu}$$
$$(M_1^{+})^{\mu}{}_{\nu} = (Ld')^{\mu}{}_{\nu} = (\not{D} + m)\, g^{\mu}{}_{\nu} - g\,\tilde{F}^{\mu\rho}\gamma_{\rho}\gamma_5(D_{\nu} + \tfrac{1}{2}m\gamma_{\nu}), \quad g = {}^{2i}/_{3m^2}. \qquad (23)$$

The operator M_1^+ is still not of a very simple type, but it falls into a class of non stricly hyperbolic differential operator considered by Leray and Ohya [22] as long as the external field strength is not too big [29]. The crucial point to be checked is the determinant operator

$$\det M_1^{+} = \left((p^2 - m^2)^2 + \frac{4p^2}{9m^4}(p_{\mu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\nu}p^{\rho})\right)(p^2 - m^2)^6, \quad p_{\mu} = i\partial_{\mu}.$$

It has to be a product of strictly hyperbolic operators. This is obviously correct if the field strength is sufficiently small. Having now constructed a fundamental solution S^{c}_{R} for M_1^+ which is by the way not a distribution but rather a hyperdistribution (functional on a Gevrey test function space) one can use the algebraic relation (23) to write down a fundamental solution for L,

$$S_R = d' S_R^{o} \quad , \quad (L S_R)^{\mu}{}_{\nu}(x, y; A) = \delta(x - y) \, \mathbb{1}$$

The support of S_R is given by that of $S_R{}^{o}$ and will in general not be in the forward light cone V_+ .

IV. Quantum Mechanics

We now return to the Yang-Feldman equation (12)

$$\phi(x) = \phi_{in}(x) - i \int d^4y \, S_R(x-y) \, A\!\!\!/(y) \, \phi(y) , \tag{25}$$

for the quantized fields ϕ_{in}, ϕ_{out} and $\phi(x)$. In the previous chapter we looked at the fundamental solutions which allows us to solve the above equation

$$\phi(x) = \phi_{in}(x) - i \int d^4y \, S_R(x,y;A) \, A\!\!\!/(y) \, \phi_{in}(y) . \tag{26}$$

It leads also to an expression for ϕ_{out} in terms of ϕ_{in} (14)

$$\begin{aligned} \phi_{out}(x) = \phi_{in}(x) & - i \int d^4y \, S_R(x,y;A) \, A\!\!\!/(y) \, \phi_{in}(y) + \\ & + i \int d^4y \, S_A(x-y) \, A\!\!\!/(y) \, \phi_{in}(y) + \\ & + \int d^4y \, d^4z \, S_A(x-y) \, A\!\!\!/(y) \, S_R(y,z;A) \, A\!\!\!/(z) \, \phi_{in}(z) . \end{aligned} \tag{27}$$

Consider now the algebra $\mathfrak{A}_{in}$ of the free field, i. e. the bounded functions of the free field ϕ_{in} acting in $\mathfrak{F}_{in}$, the Fockspace of $\phi_{in}(x)$ with vacuum $|0\rangle_{in}$. Suppose that with every testfunction f of ϕ_{in}

$$\begin{aligned} & f(x) - i \int d^4y \, f(y) \, S_R(y,x) \, A\!\!\!/(x) + i \int d^4y \, f(y) \, S_A(y-x) \, A\!\!\!/(x) + \\ & + \int dy \, dz \, f(y) \, S_A(y-z) \, A\!\!\!/(z) \, S_R(y,x;A) \, A\!\!\!/(x) \end{aligned}$$

is a testfunction, there (27) induces a mapping of the field algebra $\mathfrak{A}_{in}$ into itself. The physically most interesting question is the existence and uniqueness of the S-matrix S [30]

$$S \phi_{in} S^{-1} = \phi_{out}$$

$$S |0\rangle_{in} = |0\rangle_{out}$$

$$\phi_{out}(x) |0\rangle_{out} = 0$$

For several cases the problem can be solved. A closed expression for $|0\rangle_{out}$ can be given in terms of the fundamental solutions $S_{12}(x,y;A)$.[31) 32) 33)]. For others only existence is known [27)]. We will summarize the result in a table.

It is perhaps interesting to note that the expression for $|0\rangle_{out}$ - for the cases which have been carried through - is identical to the expression which one gets from renormalized perturbation theory [34)] if ${}_{in}\langle 0|S|0\rangle_{in}$ is properly defined.

In this framework the S-matrix is constructed up to an arbitrary phase. The phase has to be fixed if one wants the S-matrix to be causal in the sense of Bogoliubov[35)]. This problem is very much related to the one mentioned previously: the definition of a conserved current. This is traditionally done (in quantum electrodynamics) by introducing a charge renormalization.

Wave equation with minimal coupling	s=0, Klein-Gordon or Petian Duffin Kemmer	s=1/2 Dirac	s=1	s=3/2 Fierz-Pauli
Question				
Existence and uniqueness of fundamental solution	yes in $\mathcal{S}'$	yes in $\mathcal{S}'$	yes in $\mathcal{S}'$ [27/35)]	yes, as hyperdistribution [29)]
Causality	yes	yes	yes [15)27) 34)]	no [15)29)]
Existence and uniqueness of S up to a phase				
a) for external fields		yes [27)32)]	?	?
b) for arbitrary fields	yes [3)27)30)]	yes, uniqueness for electric fields only [3)27)32)]	?	?

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33) Existence already follows from classical theorems by Scale resp. Scale and Stinespring.

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$$S = {}_{in}\langle 0|S|0\rangle_{in} :\exp\int dx\,dy\ \phi^{T}_{in}(x)\, I(x,y)\, \phi_{in}(y): ,$$

and the fundamental solution $S_R(x,y;A)$ resp. $S_A(x,y;A)$ of the wave equation (11). $G = G^{0} + G^{0} I G^{0}$ where G^{0} denotes the fundamental solution of the free equation with Feynman boundary conditions. Then G is a solution of the equation

$$G = G^{0} - i\, G^{0} \not{A}\, G$$

On the other hand S_R resp. S_A are solutions of the Yang-Feldman equation (12)

$$S_{\substack{R\\A}}(x,y,A) = S^{0}_{\substack{R\\A}}(x-y) - i \int dz \; S^{0}_{\substack{R\\A}}(x-z) \not{A}(z) \, S_{\substack{R\\A}}(z,y;A).$$

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