C. V. Stanojevic

**Modified Mielnik's Axioms and Reflexivity**

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# Modified Mielnik's Axioms and Reflexivity

C.V. Stanojevic.

1. Introduction. Mielnik's [1] geometric approach to the foundation of general quantum mechanics revived the interest in characterization of inner product spaces. A natural form of the generalized parallelogram law [2] came out of studying geometric properties of the concrete representation space of Mielnik's quantum states. This generalized parallelogram law was related to that of D.A. Senechalle [3], through the functional equation $f + f \circ g = 1$, where

$$f \in F = \{f | f \in C[0,2],\ f\uparrow,\ f(0) = 0,\ f(2) = 1\}$$
$$g \in G = \{g | g \in C[0,2],\ g\downarrow,\ g(0) = 2,\ g(2) = 0\}.$$

The generalized parallelogram law

$$f(||x + y||) + f(||x - y||) = 1$$

where $f \in F$, and $||x|| = ||y|| = 1$ turned out to be a concrete form of the well-known condition of E.R. Lorch, [4].
Before we show how by modifying Mielnik's axioms we can get other geometric properties of the concrete representation space, we shall give a brief account of the results mentioned above.

2. Mielnik's probability spaces and characterization of inner product-spaces.

Let S be a non-empty set and p a real-valued function defined on S × S such that

(A) $0 < p(a,b) \leq 1$ and $a = b \Leftrightarrow p(a,b) = 1$

(B) $p(a,b) = p(b,a)$,

for all $a,b \in S$.

Definition 2.1. Two elements a and b in S are orthogonal if $p(a,b) = 0$. A subset R of S is an orthogonal system if any two distinct elements of R are orthogonal.

It is easy to show that there exists a maximal orthogonal system.

Definition 2.2. A maximal orthogonal system is called a basis B in S. Let $\mathcal{F}_B$ be the class of all finite subsets F of B, then

$$p(a,F) = \sum_{b\in F} p(a,b)$$

is defined for all $a \in S$, and all $F \in \mathcal{F}_B$.

The following property of B is postulated.

(C) For each basis B and for each $a \in S$

$$\sup_{F\in\mathcal{F}_B} p(a,F) = 1$$

Definition 2.3. Any pair (S,p) satisfying axiom (A), (B) and (C) is called a probability space.

Theorem 2.1. Let $B_1$ and $B_2$ be two basis, then $B_1$ and $B_2$ have the same cardinal number.

Definition 2.4. The common cardinal number of all basis is called the dimension of (S,p)

Theorem 2.2. Let $f \in F$. Then there exists a $g \in G$ such that

$$f + f \circ g = 1 \tag{2.1}$$

Let $g \in G$. Then (2.1) has a solution $f \in F$ if and only if g is an involution, i.e. $g = g^{-1}$.

Example 2.1. Let $h \in G$. If

$$g(t) = h^{-1}(2 - \lg[e^2 - e^{2-h(x)} + 1])$$

then

$$f(t) = \frac{e^{2-h(t)-1}}{e^2 - 1}$$

is a solution of (2.1).

Example 2.2. For

$$h(t) = -\lg[\frac{t^2}{4}(1 - e^{-2}) + e^{-2}]$$

we have

$$f(t) = \frac{t^2}{4} .$$

Theorem 2.3. Let N be normed real linear space and $S = \{x | ||x|| = 1\}$. Then N is an inner product space if and only if

$$(2.2) \qquad f(||x + y||) + f(||x - y||) = 1$$

for some $f \varepsilon F$ and all $x,y \varepsilon S$.

Example 2.3. Refering to Example 2.1 we have that a necessary and sufficient condition for N to be an inner product space is that

$$e^{-h(||x+y||)} + f(||x-y||) = 1$$

for some $h \varepsilon G$ and all $x,y \varepsilon S$.

Example 2.4. Let h be as in Example 2.2. Then (2.2) becomes the well-known condition of M.M. Day [5].

Theorem 2.4. Let N be a normed real linear space, $S = \{x | ||x|| = 1\}$, and let $p(x,y) = f(||x+y||)$, where $f \varepsilon F$. Then N is an inner product space if and only if for some $f \varepsilon F$, (S,p) is a probability space of dimension 2.

Example 2.5. Let h be as in Example 2.1. Then N is an inner product space if and only if

$$(S, \frac{e^{2-h(||x+y||)} - 1}{e^2 - 1})$$

is a probability space of dimension 2, for some $h \varepsilon G$.

Example 2.6. For

$$h(t) = -\lg[\frac{t^2}{4}(1-e^{-2}) + e^{-2}]$$

we have result given in [6].

3. <u>Modified Mielnik's axioms and geometry of representation spaces</u>.

First we shall change axiom (A) as follows

(A*) $0 \leq p^*(a,b) \leq 1 \qquad p^*(a,b) = 1 \Longrightarrow a = b$

and keep (B) and (C) as in the Mielnik system of axioms. A pair $(S,p^*)$ satisfying axioms (A*), (B) and (C) we shall call *-probability space. As before S is the unit sphere of a normed real linear space N.

<u>Lemma 3.1</u>. Let $(S,p^*)$ be a *-probability space. If

(3.1) $$p^*(x,y) \geq f(||x+y||), \quad x,y \in S,$$

where $f \in F^* = \{f|f \in C[0,]; f(t) \Leftrightarrow t=0$, then $(S,p^*)$ is a probability space

<u>Proof</u>. We have to show that $x = y \Longrightarrow p^*(x,y) = 1$.

From (3.1) we have

$$p^*(x,x) \geq f(2) = 1$$

But $p^*(x,y) \leq 1$, thus $p^*(x,x) = 1$.

<u>Lemma 3.2</u>. If $(S,p^*)$ is a *-probability space of dimension 2 and (3.1) holds, then every basis is of the form $\{y,-y\}$, $\forall y \in S$.

<u>Proof</u>. By Lemma 3.1 $(S,p^*)$ is a probability space. Let x and y be any two orthogonal elements. Then

$$0 = p^*(x,y) \geq f(||x+y||)$$

and

$$f(||x+y||) = 0$$

However $f(t) = 0 \Leftrightarrow t = 0$. Therefore

$$||x+y|| = 0$$

or

$$x + y = 0$$

Finally y is orthogonal to -y, and since (S,p*) is of dimension 2, we have that every basis is of the form {y,-y}

Corollary 3.1. If

$$f(||x+y||) = \frac{||x+y||^2}{4}$$

and (S,p*) is a *-probability space of dimension 2 with (3.1), then N is an inner product space.

Proof. By Lemma 3.2 every basis of the form {y,-y}. From the axiom (C)

$$1 = p^*(x,y) + p^*(x,-y) \geq \frac{||x+y||^2}{4} + \frac{||x-y||^2}{4}$$

for all x,y ε S. Applying a result of Schoenberg [7] we conclude that N is an inner product space.

Now we shall modify axiom C to read: For every basis B and each a ε S

(C*) $\sup_{F \varepsilon F_B} p(a,F) \leq 1$

A pair (S,p*) that satisfies (A*), (B), (C*) we shall call modified probability space. Some of the above results may be reformulated for a modified probability space.

Lemma 3.3. If for some $f \in F^* = \{f | f \varepsilon C[0,2];\ f(t) = 0 \Leftrightarrow t=0;\ f(2) = 1\}$

$$f(||x+y||) + f(||x-y||) \leq 1$$

and all x,y ε S, then N is uniformly convex.

Proof. Let $\{x_n\}$, $\{y_n\} \subset S$. We have to show that

$$||x_n + y_n|| \to 2 \Rightarrow ||x_n - y_n|| \to 0.$$

From

$$f(||x_n + y_n||) + f(||x_n - y_n||) \leq 1$$

we have

$$f(\lim||x_n + y_n||) + f(\lim||x_n - y_n||) \leq 1$$

or

$$f(2) + f(\lim||x_n - y_n||) \leq 1$$

But $f(2) = 1$, so

$$f(\lim||x_n - y_n||) \leq 0$$

i.e.

$$f(\lim||x_n - y_n||) = 0$$

For any $f \in F^*$ we have that

$$f(t) = 0 \Leftrightarrow t = 0$$

Thus

$$\lim_n ||x_n - y_n|| = 0$$

Example 3.1. The well-known Clarkson's inequality [8] states

$$\left|\left|\frac{f+g}{2}\right|\right|^{p'} + \left|\left|\frac{f-g}{2}\right|\right|^{p'} \leq \left(\frac{1}{2}\,||f||^p + \frac{1}{2}\,||g||^p\right)^{\frac{1}{p-1}},$$

for $1 < p < 2$, and for $p \geq 2$ .

$$\left|\left|\frac{f+g}{2}\right|\right|^{p} + \left|\left|\frac{f-g}{2}\right|\right|^{p} \leq \frac{1}{2}\,||f||^p + \frac{1}{2}\,||g||^p .$$

The norm $||\cdot||$ is the standard Lp norm (or lp), and $p' = 1 - p$. Let S be the unit sphere of Lp. Then

$$\left|\left|\frac{f+g}{2}\right|\right|^{p'} + \left|\left|\frac{f-g}{2}\right|\right|^{p'} \leq 1, \; 1 < p < 2,$$

and

$$\left|\left|\frac{f+g}{2}\right|\right|^{p} + \left|\left|\frac{f-g}{2}\right|\right|^{p} \leq 1, \; p \geq 2 .$$

It is easy to recognize two last inequality as special case of the inequality (3.1), by taking

$$f(t) = (\frac{t}{2})^{p'} \quad \text{or} \quad f(t) = (\frac{t}{2})^{p} .$$

It follows that Lp (or lp) are uniformly convex.

Lemma 3.3. says that every normed real linear space on whose unit sphere (3.1) (generalized Clarkson's inequality) holds, is uniformly convex.

Corollary 3.2. If N is a Banach space and (3.1) holds then N is reflexive.

Proof. According to Milman's [9] (see also Dieudonne [10]) every uniformly convex Banach space is reflexive.

Theorem 3.1. Let N be a normed real linear space and S its unit sphere i.e. $S = \{x \mid ||x|| = 1\}$. If $(S,p^*)$ is a modified probability space of dimension 2, and

$$p^*(x,y) \geq f(||x+y||)$$

where $f \in F^* = \{f \mid f \in C[0,2];\ f(t)=0 \Leftrightarrow t=0;\ f(2)=1\}$, then N is uniformly convex

Proof. By Lemma 3.2 every basis of $(S,p^*)$ is of the form $\{y,-y\}$. From the axiom $(C^*)$ we have

$$p^*(x,y) + p^*(x,-y) \leq 1$$

That implies

$$f(||x+y||) + f(||x-y||) \leq 1.$$

Applying Lemma 3.3 we get that N is uniformly convex.

Corollary 3.3. In addition to the conditions of Theorem 3.1 assume that N is a Banach space. Then N is reflexive.

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University of Missouri-Rolla
Rolla, Missouri 65401