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RECONSTRUCTION OF SCATTERING AMPLITUDES
FROM DIFFERENTIAL CROSS-SECTION

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1. FORMULATION OF THE PROBLEM

One of the most familiar problem in particle physics is what is called "Phase shift analysis" and which is more exactly "amplitude analysis", namely the problem of finding scattering or reactions amplitudes from measured cross-sections polarisations, etc... What is incredible but true is that even if we see hundreds of phase shift analyses performed and published this problem is not really solved, even "in principle", i.e., even if you start from extremely accurate measurements.

Here we shall restrict ourselves to a particularly simple case which is that of spin zero elastic scattering $A + B \rightarrow A + B$ at an energy which is below the first inelastic threshold. We assume that we know with perfect accuracy the differential cross-section at one given energy and for all physical scattering angles

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2}\left|F(\cos\theta)\right|^2 \qquad (1)$$

What we want is to find

$$\operatorname{Arg} F(\cos\theta) = \phi(\cos\theta) \qquad (2)$$

F is normalized in such a way that, when the partial wave expansion converges:

$$F = \sum (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta) \qquad (3)$$

where δ_ℓ is real because there is no inelastic channel open; $P_\ell(\cos\theta)$ is a Legendre polynomial.

The problem is indeed the problem of phase shift analysis because once you know $\phi(\cos\theta)$ by Eq. (2) you can obtain the phase shifts by

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{1}{2}\int_{-1}^{+1}\left[\frac{k^2}{2\pi}\frac{d\sigma}{d\cos\theta}\right]^{1/2} \exp\left[i\,\phi(\cos\theta)\right] P_\ell(\cos\theta)\, d\cos\theta \qquad (4)$$

To conclude this section I would like to say that I am perfectly aware of the fact that this idealized problem is rather far from reality. It is, however, my belief that studying it should bring some light. The only thing which is more favourable in practice is that one uses continuity and possibly analyticity in energy. Here we decided to disregard this constraint because of the instability of continuations along the energy cut. In fact if one assumes perfect knowledge of $d\sigma/d\cos\Theta$ at all energies and all angles one finds that the amplitude is unique as shown by Bessis and myself [1] for the pion-pion case and by Alvarez-Estrada [2] for the general case, but the result seems to me rather academic.

2. SUMMARY OF THE RESULTS

The problem even in its simplest form is not completely solved.

i) There is an obvious ambiguity

If $F(\cos\Theta)$ is an acceptable amplitude $-F^*(\cos\Theta)$ is also an acceptable amplitude; going from one amplitude to the other amounts to reverse the sign of all the phase shifts in Eq. (4). The only way to remove this ambiguity is to use analyticity with respect to energy, which is outside our programme. In what follows we shall only be interested in non-trivial ambiguities. All the statements we shall make will be modulo the trivial ambiguity.

ii) There exists a sufficient condition for the existence and uniqueness of the solution

If we define $F(12)$ as the scattering amplitude in which 1 and 2 designate unit vectors in the initial and final directions of the particles we can construct the quantity

$$\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3 \qquad (5)$$

this is the spherical convolution of $|F|$ with itself. It resembles very much the unitarity integral

$$\operatorname{Im} F(12) = \frac{1}{4\pi}\int F(13)\,F^*(23)\,d\Omega_3$$

except for the fact that F and F^* are replaced by their moduli.

Then we define

$$\sin\mu = \sup_{\text{all directions 1 and 2}} \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\, d\Omega_3}{|F(12)|} \qquad (6)$$

What can be proved is this [3),4),5)].

- If $\sin\mu < 0.79$, there is one and only one amplitude corresponding to the differential cross-section.
- If $\sin\mu < 1$ there is at least one solution and, probably only one (but for the latter point, the proof is not complete).

It must be realized, however, that this condition $\sin\mu < 1$ is very restrictive. In particular it implies automatically $|\delta_\ell| < \pi/6$ for all $\ell > 0$, which means that an amplitude with a resonating $\ell \neq 0$ wave will never fulfill the requirement $\sin\mu < 1$.

iii) Existence of non-trivial ambiguities

Crichton [6)] has produced a very simple example in which two sets of phase shifts give exactly the same differential cross-section. In this example the maximum angular momentum is 2. Then, for instance, the two sets

$$\begin{cases} \delta_0 = -23°60' & \delta_1 = -43°27' & \delta_2 = 20° \\ \delta'_0 = 98°50' & \delta'_1 = -26°33' & \delta_2 = 20° \end{cases}$$

give exactly the same cross-section. This is not a numerical accident. More generally, if $12°32' < \delta_2 < 24°9'$, to a given δ_2 correspond two couples (δ_0, δ_1), (δ'_0, δ'_1) which give the same cross-section. One has therefore a one-parameter family of cross-sections which give rise to two distinct amplitudes.

iv) Ambiguities in the general polynomial case

A very simple but non-rigorous counting argument leads to believe that there are never more than two solutions differing in a non-trivial way. The case $L_{Max} = 2$ has already been explicitly solved by Crichton as we have seen. It has recently been checked that for $L_{Max} = 3$ [7)] and $L_{Max} = 4$ [8)] the maximum number of solutions is indeed 2. What has been shown in general [9)] for the polynomial case is that the solution is unique if the cross-section is small enough:

$$\frac{k^2}{4\pi}\sigma_{Total} < 1.4 \tag{7}$$

v) Ambiguities if the amplitude is an entire function of $\cos\theta$ (but not a polynomial)

In that case we have explicitly shown [9] that there are never more than two solutions if the entire function is of finite order, i.e., if

$$\lim_{|z|\to\infty} \frac{\log\log|F(z)|}{\log|z|}$$

is finite. If the order is non-integer there is no ambiguity.

vi) The question of amplitudes analytic in ellipses

This is of course the realistic case. A very useful tool for the study of this problem has been built by Atkinson, Mahoux and Yndurain [10]. They have been able to prove the existence and local uniqueness under certain sufficient conditions on the cross-sections which differ from what is described in ii), but so far they have no statement about the maximum number of solutions. Atkinson [11] has recently been able to prove that there exists at least two fold ambiguities in that case by making small perturbations to a cross-section giving rise to the Crichton ambiguity.

Because of lack of time and lack of competence I shall not give any more details on this case.

3. THE CASE $\sin\mu < 1$

We have introduced the quantity

$$\sin\mu = \sup_{\substack{\text{over all directions}\\ 1 \text{ and } 2}} \frac{\frac{1}{4\pi}\int d\Omega_3\, |F(13)|\,|F(23)|}{|F(12)|}$$

If this integral over solid angles seems unfamiliar we can produce an equivalent definition

$$\sin\mu = \sup \frac{\int |F(\cos\theta')|\,|F(\cos\theta'')|\,K(\cos\theta,\cos\theta',\cos\theta'')\,d\cos\theta''\,d\cos\theta'}{|F(\cos\theta)|}$$

with
$$K = \frac{\Theta\left[1-(\cos\theta)^2-(\cos\theta')^2-(\cos\theta'')^2+2\cos\theta\cos\theta'\cos\theta''\right]}{\sqrt{1-(\cos\theta)^2-(\cos\theta')^2-(\cos\theta'')^2+2\cos\theta\cos\theta'\cos\theta''}}$$

Now, let us try to explain in an intuitive way how this quantity appears naturally. The unitarity condition reads

$$\operatorname{Im} F(12) = \frac{1}{4\pi}\int F^*(13)\,F(23)\,d\Omega_3$$
$$= \frac{1}{4\pi}\int |F(13)|\,|F(23)|\cos\left[\phi(13)-\phi(23)\right]d\Omega_3 \qquad (8)$$

Now assume that either ϕ is small or ϕ is not varying appreciably over the physical region. Then, as a first approximation we could neglect $\cos\left[\phi(13)-\phi(23)\right]$ in the right-hand side of (8) and get

$$\operatorname{Im} F_0(12) = |F(12)|\sin\phi_0(12)$$
$$= \frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3 \qquad (9)$$

Then one would like to extract ϕ_0 from this equation and to substitute ϕ_0 in the right-hand side of (8). To be able to do this we must make sure that we never get from (9) a quantity $\sin\phi_0$ which is larger than unity! This is precisely the case if $\sin\mu<1$! Then, in addition, $\sin\phi_0$ is strictly less than 1 at all angles, and if we decide that the scattering amplitude should be <u>continuous</u>, and in addition that $0<\phi_0\ (\theta=0)<\pi/2$ (which removes the trivial ambiguity), we have $0<\phi_0<\pi/2$ at all angles and there is no ambiguity in going from $\sin\phi_0$ to ϕ_0. The condition $\sin\mu<1$ not only ensures that the first approximation is meaningful, it also guarantees that we shall be able to continue the iteration procedure. Suppose now that ϕ is not a solution but a trial phase. Then inserting it in the right-hand side of the equation we get ϕ' in the left-hand side

$$|F(12)|\sin\phi' = \frac{1}{4\pi}\int |F(13)|\,|F(23)|\cos\left[\phi(13)-\phi(23)\right]d\Omega_3 \qquad (10)$$

Clearly, if $\sin\mu < 1$, for any ϕ we get

$$|\sin\phi'| < \sin\mu < 1$$

Equation (10) defines a non-linear mapping of the space of continuous functions ϕ on itself. Solutions of Eq. (8) are "fixed points" of the mapping. What can be established, using standard mathematical techniques [12], is that this mapping has at least one fixed point if $\sin\mu < 1$, such that $0 \leq \phi \leq \mu$. I.e., if $\sin\mu < 1$ there exists at least one unitary amplitude with modulus $|F|$.

Let us now return, however, to the iteration procedure. We know that we shall never be stopped, but what about convergence? To study this, consider two trial amplitudes

$$\begin{cases} \phi \longrightarrow \phi' \\ \psi \longrightarrow \psi' \end{cases}$$

$$|F(12)| \sin\phi'(12) = \frac{1}{4\pi}\int |F|\,|F| \cos[\phi(13)-\phi(23)]\, d\Omega_3$$

$$F(12) \sin\psi'(12) = \frac{1}{4\pi}\int |F|\,|F| \cos[\psi(13)-\psi(23)]\, d\Omega_3$$

Take now the difference:

$$|F(12)| \sin\frac{\phi'-\psi'}{2} \cos\frac{\phi'+\psi'}{2}$$

$$= -\frac{1}{4\pi}\int |F(13)|\,|F(23)| \sin\left[\frac{\phi(13)-\psi(13)}{2} - \frac{\phi(23)-\psi(23)}{2}\right]$$

$$\times \sin\left[\frac{\phi(13)+\psi(13)}{2} - \frac{\phi(23)+\psi(23)}{2}\right]$$

then, since $\cos(\phi' + \psi'/2) > \cos\mu$, we easily get

$$\sup\left(\sin\left|\frac{\phi'-\psi'}{2}\right|\right) < 2\,\frac{(\sin\mu)^3}{\cos\mu}\, \sup\left(\sin\left|\frac{\phi-\psi}{2}\right|\right)$$

Therefore we see that if

$$\frac{2(\sin\mu)^3}{\cos\mu} \leqslant \alpha < 1 \tag{11}$$

$$\sup \sin\left|\frac{\phi'-\psi'}{2}\right| < \alpha \sup \sin\left|\frac{\phi-\psi}{2}\right|$$

Now, if we iterate n times we get

$$\sup \sin\left|\frac{\phi^{(n)}-\psi^{(n)}}{2}\right| < \alpha^n \sup \sin\left|\frac{\phi-\psi}{2}\right| \tag{12}$$

So we conclude

i) if we start from any two trial functions

$$\psi \quad \phi \quad , \quad 0 < \psi < \mu \quad 0 < \phi < \mu$$

the difference between the n^{th} iterates approaches zero;

ii) the successive iterates of ϕ form a Cauchy sequence because (if $m > n$)

$$\sup \sin\left|\frac{\phi^{(m)}-\phi^{(n)}}{2}\right| < \alpha^n \sup \sin\left|\frac{\phi^{(m-n)}-\phi}{2}\right|$$

which goes to zero with $n \to \infty$. So the $\phi^{(n)}$'s converge.

iii) they converge to a solution;

iv) the solution is unique among the continuous functions varying between zero and μ. On the other hand, we can easily prove that there are no solutions outside this domain.

<u>Conclusion</u>: If (11) is true there is one and only one solution, and it can be obtained by an iterative procedure. Notice, in passing, that this iterative procedure is perfectly tractable from a computer's point of view.

With extra work I succeeded [3] to prove that the solution is unique under the weaker condition

$$\frac{2(\sin\mu)^4}{1-\cos^2\mu\,\sin^2\mu} < 1 \tag{13}$$

which gives $\sin\mu < 0.79$. In addition Atkinson and Johnson succeeded to prove that if (13) holds the iterative method works.

So far we are happy since under a well defined condition on the differential cross-section we find one and only one solution modulo the trivial ambiguity. Now, however, I shall try to show you that the condition $\sin\mu < 1$ is, in fact, very restrictive. Remember first of all that $\sin\mu < 1$ implies

$$0 < \phi(\cos\theta) < \mu < \frac{\pi}{2}$$

This means that

$$\begin{aligned} \mathrm{Re}\,F(\cos\theta) &> 0 \\ \mathrm{Im}\,F(\cos\theta) &> 0 \end{aligned} \tag{14}$$

and hence for $\ell \geq 1$

$$\mathrm{Re}\,f_0 \pm \mathrm{Re}\,f_\ell = \int_{-1}^{+1} \frac{dx}{2}\,\mathrm{Re}\,F(x)\left[1 \pm P_\ell(x)\right] > 0$$

since $|P_\ell(x)| < 1$ and similarly $\mathrm{Im}\,f_0 \pm \mathrm{Im}\,f_\ell > 0$. Since f_0 and f_ℓ lie on the unitarity circle this automatically implies that $\mathrm{Im}\ f_\ell < \frac{1}{2}$ for $\ell \geq 1$, i.e., if we make the convention to take $-\pi/2 < \delta_\ell < \pi/2$ (the phase shifts are defined modulo π), we have

$$|\delta_\ell| < \frac{\pi}{4} \quad \text{for} \quad \ell \geq 1 \tag{15}$$

This has been improved recently by the following argument: we have

$$|f_\ell| = \left| \int \frac{dx}{2} F\, P_\ell^+(x) - \int \frac{dx}{2} F\, |P_\ell^-(x)| \right| \tag{16}$$

where P_ℓ^+ designates the positive part of $P_\ell(x)$ for $-1 < x < +1$, P_ℓ^- designates the negative part.

For the two separate pieces in (16) one can apply the Weierstrass mean value theorem (F is complex!) and find the maximum of (16) to be

$$|f_\ell|^2 \leqslant \left[\int |F|\, P_\ell^+(x)\frac{dx}{2}\right]^2 + \left[\int |F|\, P_\ell^-(x)\frac{dx}{2}\right]^2$$
$$= \frac{1}{2}\left[\left[\int_{-1}^{+1}\frac{dx}{2}|F|\, P_\ell(x)\right]^2 + \left[\int_{-1}^{+1}\frac{dx}{2}|F|\,|P_\ell(x)|\right]^2\right] \tag{17}$$

Now, it is easy to prove that for $\ell \geq 1$

$$|P_\ell(x)| \leqslant \frac{1+x^2}{2} = \frac{2}{3} + \frac{P_2(x)}{3} \tag{18}$$

Hence, if we expand $|F(\cos\theta)|$ in partial waves

$$|F| = \sum (2\ell+1)\, c_\ell\, P_\ell(\cos\theta)$$

we get

$$|f_\ell|^2 \leqslant \frac{1}{2}\left[c_\ell^2 + \left(\frac{2c_0}{3} + \frac{c_2}{3}\right)^2\right] \tag{19}$$

Now, from $|F| > 0$ we get

$$c_0 \pm c_\ell = \int \frac{dx}{2}|F|\,(1 \pm P_\ell) > 0$$

Hence,

$$c_0 > |c_\ell| \tag{20}$$

From $\sin\mu < 1$

$$|F(12)| - \int |F(13)|\,|F(23)| \frac{d\Omega_3}{4\pi} > 0$$

Hence, projecting over partial waves

$$c_0 - c_0^2 > |c_\ell - c_\ell^2| \tag{21}$$

It is easy to see that if $C_0 > \frac{1}{2}$ the combination of (20) and (21) implies

$$1 - c_0 > |c_\ell|$$

Therefore, if $C_0 > \frac{1}{2}$ we get

$$|f_\ell|^2 \leqslant \frac{1}{2}\left[(1-c_0)^2 + \left(\frac{1+c_0}{3}\right)^2\right]$$

This is maximum for $C_0 = \frac{1}{2}$ and gives $|f_\ell| < \frac{1}{2}$. If $C_0 < \frac{1}{2}$ then

$$|f_\ell| < \int |F|\,|P_\ell| \frac{dx}{2} < \int |F| \frac{dx}{2} < \frac{1}{2}$$

so that in both cases

$$|\delta_\ell| < \frac{\pi}{6} \quad \text{for} \quad \ell \geqslant 1 \tag{22}$$

The conclusion is that the condition $\sin\mu < 1$ implies automatically that all partial waves, except possibly the S wave are non-resonating and small. This makes the physical interest of the condition rather doubtful, since what is really exciting is resonating amplitudes.

Let us point out, however, that if scattering amplitudes have normal threshold behaviour, the condition $\sin\mu < 1$ will always be satisfied at energies sufficiently close to threshold, because since the phase shifts behave like

$$\delta_\ell \sim a_\ell k^{2\ell+1}$$

the S wave will be dominant for k small enough if the scattering length a_0 is not zero. Then the angular distribution will be approximately flat, and this implies $\sin\mu < 1$. This has some importance for questions of principle and we shall come back on that in the concluding remarks.

4. THE CASE OF POLYNOMIALS

At the end of Section 3 we had a feeling of dissatisfaction because resonances had to be excluded. Ideally one would like to get rid of the condition $\sin\mu < 1$ without making any additional assumption except, perhaps, that the amplitude is analytic inside an ellipse. However, nobody has so far been able to make any statement about the multiplicity of the solutions in that case. So we shall now examine another face of the problem in which it is no longer assumed that $d\sigma/d\cos\theta$ has some smoothness or that partial waves are small, but, on the other hand we restrict ourselves to amplitudes in which the angular momentum is bounded, i.e., amplitudes which are polynomials in $z = \cos\theta$. After all, in practical phase shift analysis, this is what is always assumed, even if it is not strictly correct because of the existence of an exponential tail of the partial wave amplitude distribution.

A. The zeros and the counting argument

A convenient way to write a polynomial amplitude is to express it as a product over its zeros:

$$F^{1} = \lambda (z - z_1)(z - z_2) \dots (z - z_L) \tag{23}$$

λ is proportional to the highest partial wave amplitude:

$$\lambda = e^{i\delta_L} \sin\delta_L \times (2L+1) \frac{(2L)!}{2^L (L!)^2} \tag{24}$$

Now, given that F^1 is an acceptable amplitude associated with the differential cross-section

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2} |F^1|^2 = \frac{2\pi}{k^2} |\lambda|^2 \prod |\cos\theta - z_i|^2$$

what are the other possible amplitudes which produce the same cross-section? If we disregard unitarity except for the L^{th} wave we see that we can replace a subset of the zeros $z_1 \ldots z_L$ by their complex conjugates. Unitarity of the L^{th} wave forces us to keep λ fixed because $|\sin \delta_L|$ is fixed and we choose $0 < \delta_L < \pi/2$ to remove the trivial ambiguity [this convention differs from the one of Section 3, where we had decided to take $\mathrm{Re}\, F(\theta = 0) > 0$].

So, a priori, there seems to be 2^L possible amplitudes. However, unitarity has not been imposed on the $\ell = 0, 1, 2, \ldots L-1$ partial wave amplitudes. Suppose there are N acceptable amplitudes:

$$F^1 = \lambda (z - z_1)(z - z_2) \cdots (z - z_L)$$
$$F^2 = \lambda (z - z_1^*) \cdots$$
$$\cdots$$
$$F^N =$$

These N amplitudes depend on $2L + 1$ parameters: δ_L, the L real parts of the zeros, the L moduli of the imaginary parts of the zeros. We must impose unitarity on the partial wave amplitudes for $\ell = 0 \ldots L-1$;

$$\mathrm{Im}\, f_\ell^k = \left| f_\ell^k \right|^2 \tag{25}$$

$k = 1, 2, \ldots, N$

$\ell = 0, 1, \ldots, L - 1$

Equations (25) constitute a system of NL non-linear algebraic equations. If these equations are really independent, the number of equations should be less or equal to the number of parameters

$$NL \leq 2L + 1 \tag{26}$$

i.e., $$N \leq 2$$

If $N = 2$ there is one parameter left. So, if there is a two-fold ambiguity for a given maximum angular momentum L this ambiguity arises on a one dimensional variety of differential cross-sections. This is exactly what happens in the Crichton case which corresponds to $L = 2$.

Admittedly this counting argument is not rigorous because we have no general proof so far that the system (25) is made of independent equations. However, in addition to the case $L=2$, the equations have been studied explicitly for $L=3$ [7] and $L=4$ [8]. In both cases it has been shown that no more than two simultaneous solutions can be present. The case $L=4$ is so complicated that Cornille and Drouffe had to use a computer. Their result, however, is rigorous, because they get strict inequalities which show the incompatibility of the system (25) for $N=3$. For $L \geq 5$ all we can say is, following Golberger, that nature would be unkind if the system (25) were not independent.

B. The descending construction

We would like to stress now another aspect of the polynomial case. In A we first imposed to the various amplitudes to have the same modulus and then tried to check that unitarity was satisfied. One can do the reverse and first try to find all possible unitary amplitudes compatible with a given $d\sigma/d\cos\theta$ and see later which one has the correct modulus. Here we shall do something very unphysical but very straightforward from a mathematical point of view when you have perfect knowledge of $d\sigma/d\cos\theta$

$$\frac{k^2}{2\pi}\frac{d\sigma}{d\cos\theta} = \left|\sum_0^L (2\ell+1) f_\ell P_\ell(\cos\theta)\right|^2 = \sum_0^{2L} (2\ell+1)\sigma_\ell P_\ell(\cos\theta) \tag{26}$$

with

$$\begin{aligned} \sigma_{2L} &= C_{2L}^{LL} |f_L|^2 \\ \sigma_{2L-1} &= C_{2L-1}^{L\,L-1} \operatorname{Re}\left(f_{L-1}^* f_L\right) \\ &\cdots \\ \sigma_{2L-k} &= \sum C_{2L-k}^{\ell m} \operatorname{Re} f_\ell^* f_m \\ &\cdots \end{aligned} \tag{27}$$

the $C^{\ell m}_{2L-K}$ are squares of Clebsch Gordan coefficients. 2L-K, ℓ and m satisfy triangular inequalities. It is easy to see that in σ_{2L-K}, for $0 \leq K < L$, f_{L-K} appears only in $\mathrm{Re}\, f^*_{L-K} f_L$ and f_{L-K-n} never appears. Suppose we already know f_L, f_{L-1}, f_{L-K+1} then

$$\sigma_{2L-k} = C^{L\; L-k}_{2L-k}\, \mathrm{Re}\left(f^*_L f_{L-k}\right) + \text{known terms}$$

$\mathrm{Re}\,(f^*_L f_{L-K}) = \text{const.}$ defines a straight line in the Argand diagram, which is perpendicular to f_L. It intersects the unitarity circle in two points and gives therefore two possible values for f_{L-K}. Then, choosing f_{L-K} one can find f_{L-K-1} and so on. In the end, from the knowledge of σ_{2L} $\sigma_{2L-1}, \ldots, \sigma_{L+1}$ one gets 2^{L-1} possible unitary amplitudes. Then one has to compute σ_L $\sigma_{L-1}, \ldots, \sigma_0$ with these various sets and see if they agree with the values given in advance. Here the maximum multiplicity is less transparent, but there are other interesting aspects.

First of all we shall show that if the condition $\sin\mu < 1$ of Section 3 holds and if the number of partial waves, however large, is finite the amplitude is unique. Indeed from (22)

$$|\delta_L| < \frac{\pi}{6} \quad \text{and} \quad |\delta_\ell| < \frac{\pi}{6}$$

Now, suppose there are two solutions. For $\ell = L$, $L-1$, $L-k+1$ the partial waves are common. For $\ell = L-k$ they differ, and from the geometrical construction (see the Figure)

$$\frac{1}{2}\left(\delta^1_{L-k} + \delta^2_{L-k}\right) = \frac{1}{2}\left(\delta_L + \frac{\pi}{2}\right) \tag{28}$$

It is easy to see that if $|\delta^1_{L-k}|$, $|\delta^2_{L-k}|$, $|\delta_L|$ are all less than $\pi/6$ condition (28) cannot be satisfied, and the solution is therefore unique. In fact, in Ref. 3) it is shown that if $\mathrm{Re}\, F > 0$, $\mathrm{Im}\, F > 0$ (which is weaker than $\sin\mu < 1$) and if the number of partial waves is <u>finite</u> the solution is unique.

Second we shall show that if the total cross-section is small enough the solution is again unique. Looking again at the geometrical construction we see that $\frac{1}{2}(\mathrm{Im}\, f^1_{L-k} + \mathrm{Im}\, f^2_{L-k})$ is certainly larger than $\frac{1}{2}\left[1 - |\sin\delta_L|\right]$.

Therefore, the common cross-section is certainly larger than the mean contribution of the $L-k^{th}$ wave plus that of the L^{th} wave

$$\frac{k^2}{4\pi}\,\sigma > \left[2(L-k)+1\right]\frac{1-|\sin\delta_L|}{2}+(2L+1)\sin^2\delta_L \tag{29}$$

This reaches its minimum for $L-k=1$, $L=2$, $(L-k=0$ is excluded) and one finds

$$\frac{k^2}{4\pi}\,\sigma_{total} > 1.38 \tag{30}$$

Hence if

$$\frac{k^2}{4\pi}\,\sigma_{total} < 1.38 \tag{31}$$

the amplitude is <u>unique</u>. It must be realized that this condition is fundamentally different from the condition $\sin\mu < 1$. If (31) holds the differential cross-section may have very violent oscillations, very deep minima which would produce a violation of $\sin\mu < 1$. On the contrary $\sin\mu < 1$ does not exclude very large total cross-sections, provided the partial wave distribution is smooth enough.

5. THE CASE OF ENTIRE FUNCTIONS

As we have seen, the case of polynomial amplitudes is not completely settled. It represents anyway a rather extreme case in which the partial wave distribution stops abruptly. A case which seems closer to physical reality is that in which there are infinitely many partial waves. As we already said we have not been able to treat this case in general and in particular we have no answer for the case of exponentially decreasing partial wave amplitudes, which corresponds to $F(z)$ analytic inside an ellipse. The case we want to treat here is that of partial wave amplitude which decreases with ℓ faster than any exponential, i.e., the case where $F(z)$ is an entire function of z. It may be worth pointing out that polynomial amplitudes cannot be reproduced by short range potentials while entire functions naturally appear with potentials decreasing faster than any exponential. In particular finite range potentials correspond to $F(z)$ of order $\frac{1}{2}$, i.e.,

$$\sup_{|z|\leq r} |F(z)| \sim \exp r^{1/2} .$$

What miraculously happens is that the case of entire functions of finite order, which, a priori, would look more complicated than that of polynomials, is in fact much easier. Why is it so? First of all, if the partial wave expansion does not stop:

$$F = \sum_0^{\infty} (2\ell+1)\, f_\ell\, P_\ell(z)$$

the only way for this expansion to converge in the entire complex plane is to have $f_\ell \to 0$ for $\ell \to \infty$. That means that for large ℓ unitarity simplifies: $\operatorname{Im} f_\ell = |f_\ell| \simeq |\operatorname{Re} f_\ell|^2$. On the other hand, it is the large ℓ behaviour which determines the dominant behaviour of $F(z)$ for $z \to \infty$, because the small ℓ behaviour will only affect $F(z)$ by polynomials. In the limit of z large, therefore, $F(z)$ is dominated by the dispersive part, i.e.,

$$\sum (2\ell+1)\, \operatorname{Re} f_\ell\, P_\ell(z)$$

So, the quantity

$$\frac{2\pi}{k^2} \frac{d\sigma}{dz} = F(z)\, F^*(z^*)$$

is dominated almost everywhere by

$$\left[\sum (2\ell+1)\, \operatorname{Re} f_\ell\, P_\ell(z)\right]^2$$

This obviously should simplify things since, "approximately" we are only left with a sign ambiguity.

Before giving details let us recall what is the "order" of an entire function. In mathematical terms the order ρ is given by

$$\rho = \limsup_{r\to\infty} \frac{\log\log\left[\operatorname{Max} |F(z)|, |z|=r\right]}{\log r} \tag{32}$$

This means that a function of order ρ behaves approximately like $\exp(r)^{\rho}$ in some direction. For instance $\exp z$ is of order 1. What is also needed is some property of the unitarity integral.

The absorptive part of the scattering amplitude A is given by

$$A(12) = \frac{1}{4\pi} \int F(13)\, F^*(23)\, d\Omega_3 \qquad (33)$$

[This equation coincides with (8)], or

$$A(\cos\theta_{12}) = \sum (2\ell+1)\, \mathrm{Im}\, f_\ell \; P_\ell(\cos\theta_{12}) \qquad (34).$$

The problem is to continue the unitarity condition (33) outside the physical region. Let us choose as z axis the bissector of the angle 12. Then

$$A(\cos\theta_{12}) = A(\cos 2\theta_0) =$$

$$\int \frac{d\phi\, d\cos\theta}{4\pi} F(\cos\theta_0\cos\theta + \sin\theta_0 \sin\theta\cos\phi) \times \\ \times F^*(\cos\theta_0^*\cos\theta - \sin\theta_0^*\sin\theta\cos\phi) \qquad (35)$$

Make now θ_0 complex and assume that F is analytic inside an ellipse E_{θ_0} with foci $-1, +1$ going through $\cos\theta_0$. Then without any contour deformation we see that the argument of F lies on the segment connecting the points $\cos(\theta_0+\theta)$ and $\cos(\theta_0-\theta)$ which both lie on the ellipse E_{θ_0}. The same holds for the argument of F^*. So both arguments are inside the ellipse of analyticity and $A(\cos 2\theta_0)$ has a meaningful expression. In particular, if we take $\theta_0 = i\psi$

$$A(\cosh 2\psi) = \frac{1}{4\pi}\int d\phi\, d\cos\theta \left| F(\cos\theta\cosh\psi + i\sin\theta\sinh\psi\cos\phi)\right|^2 \qquad (36)$$

With the notation $M_F(x)$ = maximum of $|F|$ inside the ellipse of semi-major axis x, and foci -1, $+1$, we get from (36)

$$M_A(2x^2-1) \leq |M_F(x)|^2 \quad x \text{ real} > 1 \tag{37}$$

(remember that A, a sum of Legendre polynomials with positive coefficients is maximum at the right extremity of an ellipse).

Inequality (37) is the key for the study of amplitudes which are entire functions. In the limit of x very large the ellipses become very close to circles (with an error of the order of $1/x$). Equation (37) shows that if F, is an entire function of order ρ A is entire function of order $\rho/2$. Indeed

$$M_A(Y) \lesssim M_F\left(\sqrt{\frac{1+Y}{2}}\right) \lesssim \exp\left[\left(\frac{1+Y}{2}\right)^{\rho/2+\epsilon}\right],$$

ϵ arbitrarily small.

A. <u>The case of entire functions of order $0 < \rho < 1$</u>

Suppose we have two amplitudes F and F' giving the same cross-section:

$$F = D + iA$$
$$F' = D' + iA'$$

$$D^2(z) + A^2(z) \equiv D'^2(z) + A'^2(z) \tag{38}$$

if F is of order $\rho > 0$ A is of order $\rho/2$, and D is of order ρ. This implies that D' is of order ρ and A' of order $\rho/2$. F and F' have to have the same order. Now (38) can be written as

$$(D+D')(D-D') = -(A+A')(A-A') \tag{39}$$

The right-hand side is of order $\rho/2$. The left-hand side must also be of order $\rho/2$.

Here comes an important decomposition theorem of entire functions [13]: let $f(z)$ be an entire function of order $\sigma<1$ such that $f(0)\neq 0$. Then $f(z) = c\prod(1-(z/z_i))$, where the product, extending over all zeros of f is absolutely convergent. Furthermore, any product over a subset of the zeros is also an entire function of order $\leq\sigma$.

Now $D-D'$ and $D+D'$ are separately entire functions of order $\rho < 1$ and can be written as absolutely convergent products over zeros. The zeros of $D-D'$ are a subset of the zeros of $(A-A')(A+A')$. Hence $D-D'$ is necessarily of order $\rho/2$. So is $D+D'$. Therefore D and D' are separately of order $\rho/2$, which contradicts the assumption unless $\rho=0$. Hence there is at most <u>one</u> amplitude of order $0<\rho<1$ reproducing a given differential cross-section. We have excluded of course the case $A^2-A'^2\equiv 0$ which admits as a solution $D'=-D$ which is the trivial ambiguity.

B. <u>The case of entire functions of order $1\leq\rho<2$ and its generalization to arbitrary finite order >0.</u>

Again, if there are two solutions F and F' they are both of the same order ρ. A and A' are of order $\rho/2<1$ and hence $A-A'$ and $A+A'$ can be written as <u>convergent</u> products over zeros:

$$A^2 - A'^2 = C\prod\left(1-\frac{z}{z_i}\right)$$

From (39) we see that the zeros of $D+D'$ from a subset of the z_i's:

$$D+D' = E(z)\prod\left(1-\frac{z}{z_j}\right)$$

where the z_j's form a subset of the z_i's and $E(z)$ is an entire function without zeros of order <2. The only such function is $c\exp\lambda z$. Hence $D+D'=\exp(\lambda z)\prod(1-(z/z_j))$, where $\prod(1-(z/z_j))$ is of order $\rho/2<1$, $D+D'=\exp(\lambda z)\phi(z)$ where $\phi(z)$ is of order <1.

Similarly, $D-D'=\exp-\lambda z\,\psi(z)$ when $\psi(z)$ is of order <1. Therefore,

$$2D = \exp(\lambda z)\,\phi(z) + \exp(-\lambda z)\,\psi(z)$$
$$2D' = \exp(\lambda z)\,\phi(z) - \exp(-\lambda z)\,\psi(z) \tag{41}$$

We see that D and D' are of order 1. So, if $2 > \rho > 1$ we meet again a contradiction, and there can be only one amplitude. On the other hand, for $\rho = 1$ we cannot exclude the possibility of two solutions. Then is it possible to have more than two solutions? If there are three solutions there are two decompositions of the type (41):

$$2\ D = \exp(\lambda z)\,\phi(z) + \exp(-\lambda z)\,\psi(z) \tag{42}$$
$$2D = \exp(\lambda' z)\,\phi'(z) + \exp(-\lambda' z)\,\psi'(z)$$

where ϕ, ψ, ϕ', ψ' are of order <1. Then, it is easy to see that (assume $\lambda > 0$ $\lambda' > 0$) for z real $\to +\infty$ the first terms in (42) dominate and one is forced to have $\lambda = \lambda'$. After this is established it is not difficult to show that $\phi \equiv \phi'$ $\psi \equiv \psi'$. Therefore, the decomposition of D is <u>unique</u> and there are only <u>two</u> solutions at most.

It is not difficult, as was done in Ref. 9), to generalize this argument to arbitrary finite order >0. Indeed, if $\rho \geq 2$ it is still possible to write $(A - A')(A + A')$ as a product over zeros provided extra convergence factors are added [13]. For instance, if $2 \leq \rho < 4$, i.e., $1 \leq \rho/2 < 2$ $A - A'$ will be written as a product of factors $(1 - (z/z_i))\exp(z/z_i)$, instead of $(1-(z/z_i))$ previously, times a pure exponential. We do not want to give details. The conclusions are the same: if ρ is not integer there is only one amplitude of order ρ. If ρ is integer D can be decomposed in a unique way into

$$2D = \exp(z^\rho)\,\phi(z) + \exp\left[-(z)^\rho\right]\psi(z) \tag{43}$$

where ϕ and ψ are of order strictly less than ρ . There are never more than two solutions.

The fact that this holds for arbitrary finite order makes us believe that this might also be true in more general cases but we have no proof of this.

C. The case of functions of order zero

This is perhaps the most difficult case, probably because it is close to the polynomial case. Here inequality (36) connecting the growth of $|F|$ and the growth of $|A|$ for $z \to \infty$ must be used in a more careful way. We have no simpler proof to offer than the one proposed in Ref. 9). So we just state the result: if F and F' are two unitary amplitudes of order zero producing the same cross-section, $D - D'$ or $D + D'$ is a polynomial. Assume that $D - D'$ is a polynomial. Then

$$d^0(D-D') \leqslant d^0(A-A') - 1 \tag{44}$$

Once this result is accepted it is not very difficult to see that there cannot exist more than two solutions.

Let $D + iA$ be one solution, $D + \Delta D + i(A + \Delta A)$ the second, $D + \Delta D' + i(A + \Delta A')$ the third. Then

$$(2A + \Delta A)\Delta A + (2D + \Delta D)\Delta D \equiv 0$$
$$(2A + \Delta A')\Delta A' + (2D + \Delta D')\Delta D' \equiv 0$$

hence

$$2\left(\frac{\Delta A}{\Delta D} - \frac{\Delta A'}{\Delta D'}\right) + \frac{\Delta A^2}{A\Delta D} + \frac{\Delta D}{A} - \frac{\Delta A'^2}{A\Delta D'} - \frac{\Delta D'}{A} = 0 \tag{45}$$

A, for $z \to +\infty$, increases, by assumption, faster than any power. Therefore, the last four terms in (45) decrease faster than any inverse power of z. On the other hand $\Delta A/\Delta D - \Delta A'/\Delta D'$ is a rational function. It has therefore to be identically zero:

$$\frac{\Delta A}{\Delta D} \equiv \frac{\Delta A'}{\Delta D'}$$

Then it is not difficult to get $\Delta A = \Delta A'$ and $\Delta D = \Delta D'$. The second and the third solutions have to coincide.

The case of order 0 makes therefore no exception. There cannot be more than two solutions. The major difference with $\rho \neq 0$ is that if $\rho > 0$, integer, the difference between two solutions is <u>not</u> a polynomial (a fact which is also true for amplitudes analytic in a cut plane, as shown by Burkhardt [14]), while for $\rho = 0$ the difference <u>must</u> be a polynomial.

6. <u>CONCLUDING REMARKS</u>

We have presented the situation as it is today. Our findings suggest that there are never more than two unitary amplitudes producing the same differential cross-sections. This will be true if entire functions are not exceptional animals. The reason of our success is, on the one hand, the fact that in that case only the large angular momentum partial wave amplitudes matter, on the other hand, that entire functions have been very well studied by mathematicians (especially old mathematicians) with these beautiful decomposition theorems of Weierstrass, Hadamard, etc...

There are many problems left among which I want to mention three:

1) the extension to particles with spin. This is in part treated in the papers of Atkinson-Mahoux-Yndurain [15] and Carreras and Alvarez-Estrada [16] and Berends and Ruijsenaars [17];

2) the stability with respect to experimental errors. At least in one case we can give a positive answer: if $\sin\mu < 0.79$, one can prove that small perturbations of $d\sigma/d\cos\theta$ produce small perturbations of $\phi(\cos\theta)$, the unique solution;

3) if we know the amplitude not only at one given energy but on a range of energies, does this fix the amplitude in a unique way? Does this eliminate the trivial ambiguity $F(s,\cos\theta) \to -F^*(s,\cos\theta)$?

It has been shown long ago that if $|F|$ is known in the whole physical region F is determined uniquely [1),2)]. It must be realized, however, that this determination is rather unstable: $F(s,\cos\theta)$ is <u>inside</u> the analyticity domain in $\cos\theta$ but on the energy cut in s. What is easy to establish is that there are never more than two amplitudes because, as we said, the condition $\sin\mu < 1$ holds if we are close enough to threshold. Then modulo the trivial ambiguity there is only one amplitude inside a two-dimensional region in s and t and this fixes the amplitude everywhere else by analytic continuation. It is more tedious to remove the trivial ambiguity. Here one must use, in particular, fixed t dispersion relations for $t \geq 0$ together with the positivity of the absorptive part. For completeness let us sketch a proof.

We work with a s-u crossing symmetric amplitude $A + B \rightarrow A + B$, with masses M and μ (like $K\pi^0 \rightarrow K\pi^0$!). Then

$$z = \left[s - (M+\mu)^2\right]\left[(M+\mu)^2 - u\right] \tag{46}$$

is a convenient variable.

Assume that we know

i) $|F|$ in the elastic region
$$(M+\mu)^2 < s < (M+2\mu)^2$$

ii) $|F|$ and σ_{total} in the forward direction for arbitrarily large energies.

From the previous remarks we know that we have only the trivial ambiguity when s is close enough to threshold, and since F is analytic and can be continued to the second sheet for $(M+\mu)^2 < s < (M+2\mu)^2$, this will persist till the first inelastic threshold.

So if F and G are solutions

$$\left.\begin{aligned} \mathrm{Re}\,F &= -\mathrm{Re}\,G \\ \mathrm{Im}\,F &= \mathrm{Im}\,G \end{aligned}\right\} \quad \text{for} \quad (M+\mu)^2 < s < (M+2\mu)^2 \tag{47}$$

Let us look now at the forward amplitude. F and G are both analytic in the z plane with a cut starting at $z = 0$. Since σ_{total} is given at all energies, $\mathrm{Im}\,F = \mathrm{Im}\,G$ and $F-G$ is real analytic for $z < 0$ and also real for $z > 0$. Therefore, it is an entire function. But from high energy bounds we have $|F-G| < \sqrt{z}(\log z)^2$. Therefore, $F-G = C$, C real. On the other hand,

$F+G$ is purely imaginary for $0<z<z_1$ where z_1 is the first inelastic threshold. Hence, $F+G=\sqrt{-z}\,\phi(z)$ where $\phi(z)$ has a possible cut starting at z_1. Now, we impose $|\mathrm{Re}\,F| = |\mathrm{Re}\,G|$ at all energies:

$$\left|\sqrt{z}\,\mathrm{Im}\,\phi(z)+C\right| = \left|\sqrt{z}\,\mathrm{Im}\,\phi(z)-C\right|$$

which for a given z admits the solutions

$$\mathrm{Im}\,\phi(z)=0 \qquad C=0$$

$C=0$ is excluded, because then there is no problem: F and G coincide. So we are left with $\mathrm{Im}\,\phi(z)=0$ which easily gives $\phi=\text{const}=D$

$$\begin{cases} F = \dfrac{C}{2}+\dfrac{D}{2}\sqrt{-z} \\ G = -\dfrac{C}{2}+\dfrac{D}{2}\sqrt{-z} \end{cases} \tag{48}$$

at all energies. Outside this exceptional structure there is no room for the trivial ambiguity. Now: this structure itself seems to be unacceptable because there is no inelastic cut in F. It seems extremely difficult to make the inelastic cut everywhere zero in the forward direction. What we know is that above the inelastic threshold the total cross-section must be strictly larger than the elastic cross-section. For an amplitude $F=\sum(2\ell+1)f_\ell P_\ell$ the discontinuity $(F^{I}-F^{II}/2i)(\cos\theta=1)$ of the amplitude across the inelastic cut is given by

$$\Delta = \sum(2\ell+1)\frac{\mathrm{Im}\,f_\ell-|f_\ell|^2}{1-2i\,f_\ell^*} \tag{49}$$

In the case where for all ℓ, $\mathrm{Im}\,f_\ell<\frac{1}{2}-\epsilon$ it is easy to see that

$$|\Delta| > \text{Const}\cdot\frac{k^2}{4\pi}\left(\sigma_{\text{Total}}-\sigma_{\text{elastic}}\right) \tag{50}$$

while in (48) Δ is zero. We see therefore that if D is small enough we get an obvious contradiction since

$$\mathrm{Im}\,F = \sum(2\ell+1)\,\mathrm{Im}\,f_\ell = D\sqrt{z}$$

Hence

$$\mathrm{Im}\,f_\ell < \frac{D\sqrt{z}}{2\ell+1}\,.$$

Even for large D, it seems anyway extremely difficult to make Δ identically zero at all energies. Indeed, let us place ourselves close enough to the first inelastic threshold. Then the inelastically produced particles will appear in the lowest angular momentum state allowed by selection rules, and the sum (49) will reduce to a single term which will differ from zero.

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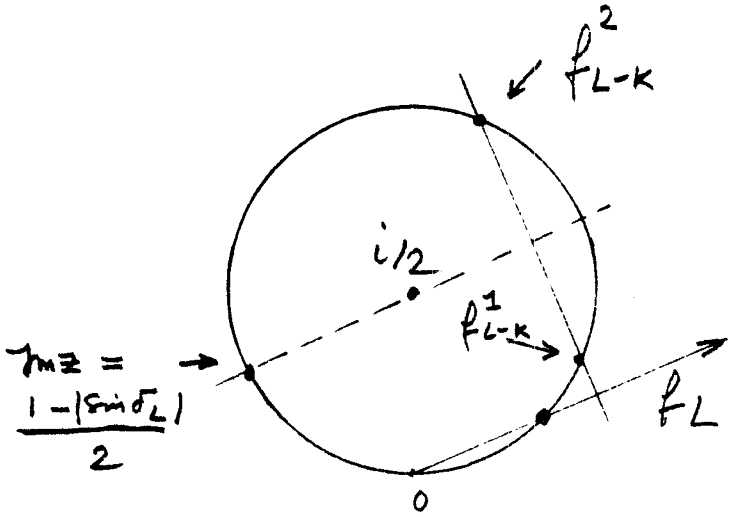

Figure 1

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