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RENORMALIZATION OF GAUGE THEORIES

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II

Abstract : Gauge theories are characterized by the Slavnov identities which express their invariance under a family of transformations of the supergauge type which involve the Faddeev Popov ghosts. These identities are proved to all orders of renormalized perturbation theory, within the BPHZ framework, when the underlying Lie algebra is semi-simple and the gauge function is chosen to be linear in the fields in such a way that all fields are massive. An example, the SU2 Higgs Kibble model is analyzed in detail : the asymptotic theory is formulated in the perturbative sense, and shown to be reasonable, namely, the physical S operator is unitary and independent from the parameters which define the gauge function.

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1 - Introduction

In a previous article [1] devoted to the renormalization of the Abelian Higgs Kibble model, we have developed a number of technical tools which will be applied here to the renormalization of non abelian gauge models.

Our analysis relies on the Bogoliubov Paraziuk Hepp Zimmermann [2] version of renormalization theory. The extensive use of the renormalized quantum action principle of Lowenstein [3] and Lam [4] allows to push the analysis of the algebraic structure of gauge field models.

Very few properties of the perturbation series are actually used here, namely, nothing more than the general consequences of locality [5], sharpened by the theory of power counting [5], which, through the fundamental theorem of renormalization theory [5] insure the existence of a basis [2][6] of local operators of given dimension and of linear relationships [2] between local operators of different dimensions. More precisely, we shall never use the information contained in the detailed structure of the coefficients involved in such relations, thus forbidding ourselves to envisage no renormalization type theorems [7].

The algebraic structure of classical gauge theories [8] is then deformed by quantum corrections in a way which can be completely analyzed with the above mentioned economical means, through a systematic use of the implicit function theorem for formal power series [1], provided that it is rigid enough : in technical terms, if some of the cohomology groups [9] of the finite Lie algebra which characterizes the gauge theory, vanish. The analysis can thus be successfully carried out when this Lie algebra is semi-simple - if the obstruction provided by the Adler Bardeen [10] anomaly is absent. We shall thus carry out most of our analysis in the semi-simple case, and only make a few remarks concerning some of the phenomena which occur when abelian components are involved. For instance, we have noted that the fulfillment of discrete symmetries allows favourable simplifications which in particular lead to complete analyses of massive electrodynamics [11] in the Stueckelberg gauge and of the abelian Higgs Kibble model in charge conjugation odd gauges [1] . A complete analysis of abelian cases would require proving non renormalization theorems [7] , which would take us far beyond the technical level of the present work.

Another conservative limitation, due to the present status of renormalization theory, is to discard the study of models in which massless fields are involved, which does not pose new algebraic problems [12] but would force us to rely on work in progress [13] and to go into analytic details which would obscure the main line of reasoning. In the same spirit, we have not included here the analysis needed to deal with non linear renormalizable gauge functions [1] , which would have made this article considerably longer without adding much to our understanding. Similarly, although the main subject of this paper has to do with algebraic properties, we have decided to include one important application which we have illustrated on a specific example.

The heart of the matter is to prove that one can fulfill to all orders of renormalized perturbation theory a set of identities, the so-called Slavnov identities,[1] [14], which express the invariance of the Faddeev Popov $(\Phi\Pi)$ [15] Lagrangian under a set of non linear field transformations [1] which explicitly involve the Faddeev Popov Fermi scalar ghost fields. Furthermore the theory should be interpreted whenever possible as an operator theory within a Fock space with indefinite metric involving the $\Phi\Pi$ ghost fields. When this interpretation is possible, the Slavnov identities allow to define a "physical" subspace of Fock space within which the norm is positive definite. The restriction of the S operator to this subspace is then both independent from the parameters which label the gauge function [11], and unitary in the perturbative sense.

This article is divided into two main sections, and a number of appendices devoted to some technical details.

Section 2 covers the algebraic discussion of the Slavnov identities.

Section 3 deals with a specific model (the SU2 Higgs Kibble model [16]) for which the operator theory is discussed.

Appendix A summarizes a number of well known definitions and facts about the cohomology of Lie algebras [17].

Appendix B is devoted to the resolution of some trivial cohomologies encountered in Section 2 .

2 - Slavnov Identities

A. The tree approximation

Let G be a compact Lie group, $\mathfrak{h}$ its Lie algebra. Let $\{\varphi_a\}$ be a matter field multiplet corresponding to a fully reduced unitary representation D of G, with the infinitesimal version :

$$\{\omega_\alpha\} \to t^\alpha \omega_\alpha \quad , \quad \{\omega_\alpha\} \in \mathfrak{h} \tag{1}$$

Let $\{a_{\alpha\mu}\}$ be a gauge field associated with $\mathfrak{h}$, $\{c_\alpha\}$, $\{\bar{c}_\alpha\}$, the corresponding Faddeev Popov ghost fields.

We start with a classical Lagrangian of the form :

$$\begin{aligned} \mathcal{L} &\equiv \mathcal{L}\left(\{\varphi_a\}, \{a_{\alpha\mu}\}, \{c_\alpha\}, \{\bar{c}_\alpha\}\right) \\ &= \mathcal{L}_{inv.}\left(\{\varphi_a\}, \{a_{\alpha\mu}\}\right) - \\ &\quad - \frac{1}{\kappa}\left[\frac{1}{2} \mathcal{G}_\alpha \mathcal{G}^\alpha - c_\alpha (m\bar{c})^\alpha\right] . \end{aligned} \tag{2}$$

$\mathcal{L}_{inv.}$ is the most general dimension four local polynomial invariant under the local gauge transformation :

$$\delta_\omega \varphi_a(x) = \int \left[\delta_{\omega_\alpha(y)} \varphi_a(x)\right] \omega_\alpha(y)\, dy \tag{3}$$

$$\delta_\omega a_{\alpha\mu}(x) = \int \left[\delta_{\omega_\beta(y)} a_{\mu\alpha}(x)\right] \omega_\beta(y)\, dy$$

with

$$\delta_{\omega_\alpha(y)} \varphi_a(x) = \delta(x-y)\, t^\alpha_{ab} \left[\varphi_b(x) + F_b\right] \tag{4}$$

$$\delta_{\omega_\beta(y)} a_{\alpha\mu}(x) = \partial^x_\mu \delta(x-y)\, \delta^\beta_\alpha + g f_\alpha{}^{\beta\gamma} a_{\gamma\mu}(x)\, \delta(x-y)$$

where $\{ f_\alpha{}^{\beta\gamma} \}$ are the structure constants of $\mathfrak{h}$, $\{ F_b \}$ some field translation parameters. The gauge function $\{ \mathcal{G}_\alpha \}$ will be chosen to be linear in the fields and their derivatives :

$$\mathcal{G}_\alpha = g^\beta_\alpha \partial^\mu a_{\beta\mu} + g^a_\alpha \varphi_a \tag{5}$$

Indices of $\mathfrak{h}$ are raised and lowered by means of an invariant non degenerate symmetric tensor. The role of the gauge term is to remove the degeneracy of the quadratic part of $\mathcal{L}_{inv.}$, which is connected with gauge invariance. The field dependent differential operator $\mathcal{M}$ involved in the Faddeev Popov part of $\mathcal{L}$ is defined through the kernel

$$\mathcal{M}^{\alpha\beta}_{xy} = \delta_{\omega_\beta(y)} \mathcal{G}^\alpha(x) \tag{6}$$

The essential property of this Lagrangian is its invariance under the following infinitesimal transformations which we shall call the Slavnov transformations :

$$\delta\varphi_a(x) = \delta\lambda\, \delta_{\omega_\alpha(y)} \varphi_a(x)\, \bar{c}_\alpha(y) \equiv \delta\lambda\, s\, \varphi_a(x)$$

$$\delta a_{\alpha\mu}(x) = \delta\lambda\, \delta_{\omega_\beta(y)} a_{\alpha\mu}(x)\, \bar{c}_\beta(y) \equiv \delta\lambda\, s\, a_{\alpha\mu}(x) \tag{7}$$

$$\delta c_\alpha(x) = \delta\lambda\, \mathcal{G}_\alpha(x) \equiv \delta\lambda\, s\, c_\alpha(x)$$

$$\delta \bar{c}_\alpha(x) = \frac{\delta\lambda}{2} f_\alpha{}^{\beta\gamma} \left(\bar{c}_\beta \bar{c}_\gamma\right)(x) \equiv \delta\lambda\, s\, \bar{c}_\alpha(x)$$

where the summation over repeated indices and the integration over repeated space time variables are understood.

$\delta\lambda$ is a space time independent infinitesimal parameter which commutes with $\{\varphi_a\}$, $\{a_{\alpha\mu}\}$, but anticommutes with $\{c_\alpha\}$, $\{\bar{c}_\alpha\}$ and, for two transformations labelled by $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_1$ and $\delta\lambda_2$ anticommute.

This invariance can be checked immediately, by using the composition law for gauge transformations

$$(8)\qquad \left[\,\delta_{\omega_\alpha(x)}\,,\ \delta_{\omega_\beta(y)}\right] = f^{\alpha\beta}{}_{\gamma}\ \delta(x-y)\ \delta_{\omega_\gamma(y)}$$

Conversely, it is interesting to know whether $\mathcal{L}$ is up to a divergence the most general Lagrangian leading to an action invariant under Slavnov transformations, and carrying no Faddeev Popov charge $Q^{\Phi\Pi}$:

$$(9)\qquad \begin{aligned} &Q^{\Phi\Pi}\, c = c \\ &Q^{\Phi\Pi}\, \bar{c} = -\bar{c} \\ &Q^{\Phi\Pi}\varphi_a = Q^{\Phi\Pi} a_{\alpha\mu} = 0 \end{aligned}$$

Let

$$(10)\qquad \underline{\Psi} = \left(\{\varphi_a\}\,,\{a_{\alpha\mu}\},\{c_\alpha\},\{\bar{c}_\alpha\}\right)$$

and given a functional $\mathcal{F}(\underline{\Psi})$, let us denote

$$(11)\qquad \delta\lambda\, s\, \mathcal{F}(\underline{\Psi}) = \delta\underline{\Psi}(x)\ \delta_{\underline{\Psi}(x)}\, \mathcal{F}(\underline{\Psi})$$

where $\delta\underline{\Psi} = \delta\lambda\, s\,\underline{\Psi}$ is the variation of $\underline{\Psi}$ under the Slavnov transformation of parameter $\delta\lambda$ (cf. Eq.7) . A remarkable property of s is :

$$(12) \qquad s^2 \mathcal{F}(\underline{\Psi}) = (m\,\bar{c})_\alpha(x)\, \delta_{c_\alpha(x)} \mathcal{F}(\underline{\Psi})$$

This property actually summarizes the group law as follows :
Let

$$(13) \qquad \begin{aligned} \delta\varphi_a &= \delta\lambda \left(\theta^\alpha_{ab}\varphi_b + q^\alpha_a \right)\bar{c}_\alpha \equiv \delta\lambda\, s\,\varphi_a \\ \delta a_{\alpha\mu} &= \delta\lambda \left(\theta_\alpha^{\beta\gamma} a_{\gamma\mu}\,\bar{c}_\beta + q^\beta_\alpha\, \partial_\mu \bar{c}_\beta \right) \equiv \delta\lambda\, s\, a_{\alpha\mu} \\ \delta\bar{c}_\alpha &= \delta\lambda\, \tfrac{1}{2}\, \varepsilon_\alpha^{\ \beta\gamma}\, \bar{c}_\beta\,\bar{c}_\gamma \equiv \delta\lambda\, s\,\bar{c}_\alpha \\ \delta c_\alpha &= \delta\lambda\; \mathcal{G}_\alpha(\underline{\varphi}, \underline{a}) \equiv \delta\lambda\, s\, c_\alpha \end{aligned}$$

Eq.(12) implies :

$$(14) \qquad s^2\varphi_a = s^2 a_{\alpha\mu} = s^2\bar{c}_\alpha = 0$$

which in terms of Eq.(13) reads :

$$(15) \qquad \begin{aligned} &(a) \quad \varepsilon_\alpha^{\ \beta\lambda}\varepsilon_\lambda^{\ \gamma\delta} + \varepsilon_\alpha^{\ \gamma\lambda}\varepsilon_\lambda^{\ \delta\beta} + \varepsilon_\alpha^{\ \delta\lambda}\varepsilon_\lambda^{\ \beta\gamma} = 0 \\ &(b) \quad \left[\theta^\alpha, \theta^\beta\right]_{ab} - \varepsilon_\gamma^{\ \alpha\beta}\,\theta^\gamma_{ab} = 0 \\ &(c) \quad \left[\theta^\alpha, \theta^\beta\right]_{\delta\eta} - \varepsilon_\gamma^{\ \alpha\beta}\,\theta^\gamma_{\delta\eta} = 0 \end{aligned}$$

$$(15)\quad \begin{aligned} &(d)\quad \left(\theta^{\alpha} q^{\beta} - \theta^{\beta} q^{\alpha} - \varepsilon_{\gamma}{}^{\alpha\beta} q^{\gamma}\right)_a = 0 \\ &(e)\quad \left(\theta^{\alpha} q^{\beta} - \varepsilon_{\gamma}{}^{\alpha\beta} q^{\gamma}\right)_{\delta} = 0 \end{aligned}$$

Eq.(15a) is the Jacobi identity. If we choose a solution corresponding to $\mathfrak{h}$, Eq.(15b) and Eq.(15c) assert that θ^{α} is a representation of $\mathfrak{h}$.

If $\mathfrak{h}$ is semi-simple, then all solutions (cf. Appendix A) of Eq.(15d) are of the form

$$(16)\qquad q^{\alpha}_{a} = \left(\theta^{\alpha} q\right)_a$$

for some fixed $\{q_a\}$.

Finally Eq.(15e) states that q^{α}_{β} intertwines $\theta^{\alpha}_{\beta\gamma}$ and $\varepsilon^{\alpha}_{\beta\gamma}$ (the adjoint representation of $\mathfrak{h}$). In the semi-simple case $\theta^{\alpha}_{\beta\gamma}$ is thus equivalent to the adjoint representation if $\{q^{\alpha}_{\beta}\}$ does not vanish identically, a requirement which belongs to the definition of $\{a\}$ as a gauge field. We may then choose in this case without any loss of generality Eq.(13) to be identical with Eq.(3,4,7).

Let us now come back to the Lagrangian $\mathcal{L}$.

If $\mathcal{L}$ is Slavnov invariant namely such that

$$(17)\qquad s \int \mathcal{L}(x)\, dx = 0$$

a fortiori

$$(18)\qquad s^2 \int \mathcal{L}(x)\, dx = 0$$

Now, $\mathcal{L}$ is of the form

$$(19)\qquad \begin{aligned} \mathcal{L} = {} & \mathcal{L}_{\text{inv.}}\left(\{\varphi_a\},\{a_{\alpha\mu}\}\right) + c_{\alpha}(x)\, K^{\alpha\beta}_{xy}\, \bar{c}_{\beta}(y) \\ & + \Delta\mathcal{L}\left(\{\varphi_a\},\{a_{\alpha\mu}\}\right) + L^{\alpha\beta\gamma\delta}\left(c_{\alpha} c_{\beta} \bar{c}_{\gamma} \bar{c}_{\delta}\right)(x) \end{aligned}$$

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where $\mathcal{L}_{inv.}$, invariant under gauge transformations, is by itself a solution of Eq.(17). By Eq.(12) it is obvious from Eq.(18) that

(20) $$L^{\alpha\beta\gamma\delta} = 0$$

and that

(21) $$\int (m\bar{C})_{\alpha,x} (K\bar{C})^{\alpha}_{x} \, dx = 0$$

Writing out the general form of $K^{\alpha\beta}_{xy}$ yields if $\mathcal{h}$ is semi-simple

(22) $$C_\alpha(x) K^{\alpha\beta}_{xy} \bar{C}_\beta(y) = C_\alpha(x) \Gamma^{\alpha}{}_{\alpha'} m^{\alpha'\beta}_{xy} \bar{C}_\beta(y)$$

where $\Gamma^{\alpha}{}_{\alpha'}$ is a numerical symmetrical matrix. If $\mathcal{h}$ has an abelian invariant part $\mathcal{A}$ this is not the general solution, the $\Phi\Pi$ mass term being left undetermined.

Going back to Eq.(17,19) yields in the semi-simple case :

(23) $$\mathcal{L} = \mathcal{L}_{inv.}\left(\{\varphi_a\},\{a_{\alpha\mu}\}\right) + \\ + C_\alpha \Gamma^{\alpha}{}_{\alpha'} (m^{\alpha'\beta} \bar{C}_\beta) - \frac{1}{2} \mathcal{G}_\alpha \Gamma^{\alpha\alpha'} \mathcal{G}_{\alpha'}$$

which is identical with Eq.(2) modulo a redefinition of $\mathcal{G}_\alpha$ and C_α. In the abelian case, there may arise an ambiguity unless the gauge function $\mathcal{G}_\alpha$, ($\alpha \in \mathcal{A}$) contains besides the $\partial^\mu a_\mu$ terms a part which is invariant under $\mathcal{A}$.

For instance in quantum electrodynamics in the Stueckelberg gauge [11], there may arise the Slavnov invariant photon- $\Phi\Pi$ mass term :

$$\frac{a_\mu a^\mu}{2} + c\,\bar{c} \tag{24}$$

This is however not the case in the t'Hooft-Veltman [18] gauge which contains an $a_\mu a^\mu$ term.
A similar phenomenon, which occurs in the abelian Higgs Kibble model produces quite spectacular complications [1] if one makes a comparison with its SU2 analog (cf. Section 3) !

This is but one pathology associated with the abelian parts of $\mathfrak{h}$ which make it unstable under deformations.
For the reasons explained in the Introduction we shall from now on assume that $\mathfrak{h}$ is semi-simple (and compact !)

B. Perturbation theory : the Slavnov Identities.

The problem is to find a renormalizable effective Lagrangian

$$\mathcal{L}_{eff}\left(\{\varphi_a\},\{a_{\alpha\mu}\},\{c_\alpha\},\{\bar{c}_\alpha\}\right) \tag{25}$$

whose lowest order term in $\hbar$ is given by Eq.(2), assuming that all the parameters in Eq.(2) have been chosen in such a way that all mass parameters are strictly positive, and which is furthermore invariant in the renormalized sense [1] under the Slavnov transformation

$$\begin{aligned}
\delta\phi_i &= \delta\lambda\, N_2\left[(T^\alpha_{ij}\phi_j + Q^\alpha_i)\bar{c}_\alpha\right] \equiv \delta\lambda\, P_i \\
\delta\bar{c}_\alpha &= \delta\lambda\,\frac{1}{2}\, N_2\left[F_\alpha^{\ \beta\gamma}\bar{c}_\beta\bar{c}_\gamma\right] \equiv \delta\lambda\, P_\alpha \\
\delta c_\alpha &= \delta\lambda\, G_\alpha \equiv \delta\lambda\, G^i_\alpha \phi_i
\end{aligned} \tag{26}$$

where

$$\{\Phi_i\} = \{\{\varphi_a\}, \{a_{\alpha\mu}\}\}$$

(27)

$$Q^{\alpha}_{\beta\mu} = Q^{\alpha}_{\beta}\,\partial_\mu\,, \quad \mathcal{G}^{\beta\mu}_{\alpha} = \mathcal{G}^{\beta}_{\alpha}\,\partial^\mu$$

and $\mathcal{G}^i_\alpha$, $F_\alpha^{\beta\gamma}$, T^α_{ij} , Q^α_i are to be found as formal power series in $\hbar$ [1] whose lowest order terms are the corresponding tree parameters, namely

(28)
$$\mathcal{P}_i = \overset{\circ}{\mathcal{P}}_i + O(\hbar)$$
$$\mathcal{P}_\alpha = \overset{\circ}{\mathcal{P}}_\alpha + O(\hbar)$$
$$\mathcal{G}^i_\alpha = \overset{\circ}{\mathcal{G}}{}^i_\alpha + O(\hbar)$$

with

(29)
$$\overset{\circ}{\mathcal{P}}_i = s\,\Phi_i$$
$$\overset{\circ}{\mathcal{P}}_\alpha = s\,\bar{c}_\alpha$$
$$\overset{\circ}{\mathcal{G}}{}^i_\alpha = g^i_\alpha$$

cf. Eqs.(4,5,7).

In order to deal with radiative corrections, let us add to $\mathcal{L}_{eff}$ the external field and source terms [19].

$$\gamma^i\mathcal{P}_i + \zeta^\alpha\mathcal{P}_\alpha + J^i\Phi_i + \bar{\xi}^\alpha c_\alpha + \xi^\alpha\,\bar{c}_\alpha \tag{30}$$

where γ^i is assigned dimension two, $\Phi\Pi$ charge + 1 , Fermi statistics, ζ^α is assigned dimension two, $\Phi\Pi$ charge + 2 , Bose statistics ; J^i, ξ^α, $\bar{\xi}^\alpha$ are sources of the basic fields : J^i of the Bose type, ξ^α, $\bar{\xi}^\alpha$ of the Fermi type.

Performing the Slavnov transformation Eq.(26) , and using the quantum action principle yield in terms of the Green functional [1] $Z_c(J,\eta)$:

$$-\delta Z_c(\mathcal{J},\eta) \equiv \int dx \left(J^i \delta_{\gamma^i} - \xi^\alpha \delta_{\zeta^\alpha} - \bar{\xi}^\alpha g^i_\alpha \delta_{J^i}\right)(x) \cdot\cdot$$

$$\text{(31)} \quad \cdot\cdot\, Z_c(\mathcal{J},\eta) = \int dx\, \Delta(x)\, Z_c(\mathcal{J},\eta)$$

where $\mathcal{J}$ stands for the sources and η for the external fields :

$$\text{(32)} \quad \mathcal{J} = \left\{ J^i,\ \xi^\alpha,\ \bar{\xi}^\alpha \right\} \qquad \eta = \left\{ \gamma^i,\ \zeta^\alpha \right\}$$

$\int \Delta(x)\, dx$ is the most general dimension five insertion carrying [illegible] $\Phi\Pi$ charge -1, which is of the form :

$$\text{(33)} \quad \int dx\, \Delta(x) = \int dx\, N_5 \left[-s^P \left(\mathcal{L}_{eff} + \gamma^i P_i + \zeta^\alpha P_\alpha \right) + \hbar Q \right](x)$$

where s^P is the naïve transformation Eq.(11) corresponding to Eq.(26), and $\hbar Q$ lumps together all radiative corrections.

Making explicit the external field dependence, we shall write

$$\text{(34)} \quad \begin{aligned} \Delta &= \Delta_0 + \gamma^i \Delta_i + \zeta^\alpha \Delta_\alpha \\ Q &= Q_0 + \gamma^i Q_i + \zeta^\alpha Q_\alpha \end{aligned}$$

Introducing a new classical field β carrying $\Phi\Pi$ charge $+1$ and linearly coupled to Δ, the Lagrangian becomes :

$$\text{(35)} \quad \begin{aligned} \mathcal{L}_{eff}^{(\eta,\beta)} &= \mathcal{L}_{eff} + \gamma^i P_i + \zeta^\alpha P_\alpha + \beta \Delta \\ &= \mathcal{L}_{eff}^{(\eta)} + \beta \Delta \end{aligned}$$

Performing now the quantum variation of the fields :

$$\delta\Phi_i(x) = \delta\lambda\, N_2\left[\delta_{\gamma i}\int \mathcal{L}^{(\eta,\beta)}_{eff}(y)\,dy\right](x)$$

(36) $$\delta\bar{C}_\alpha(x) = \delta\lambda\, N_2\left[\delta_{\xi^\alpha}\int \mathcal{L}^{(\eta,\beta)}_{eff}(y)\,dy\right](x)$$

$$\delta C_\alpha(x) = \delta\lambda\left(\mathcal{G}^i_\alpha\,\Phi_i\right)(x)$$

the quantum action principle yields :

$$\mathcal{S}\, Z_c(J,\eta,\beta) = \int dx\, \delta_{\beta(x)}\, Z_c(J,\eta,\beta) +$$

(37) $$+\int dx\left[\beta\left\{\mathcal{S}^\Delta\left(\mathcal{L}_{eff} + \gamma^i \mathring{P}_i + \xi^\alpha \mathring{P}_\alpha\right) + \mathcal{S}\Delta\right\} + O(\hbar\Delta,\beta)\right](x)$$

$$\cdot\ Z_c(J,\eta,\beta)$$

where $\mathcal{S}$ is the tree Slavnov transformation ; $\mathcal{S}^\Delta$ generates the variation corresponding to

$$\delta\Phi_i = \delta\lambda\, N_2\left(-\mathcal{S} P_i + \hbar Q_i\right)$$

(38) $$\delta\bar{C}_\alpha = \delta\lambda\, N_2\left(\mathcal{S} P_\alpha - \hbar Q_\alpha\right)$$

$$\delta C_\alpha = 0$$

and $Z_c(J,\eta,\beta)$ is the Green functional corresponding to $\mathcal{L}^{(\eta\beta)}_{eff}$

Thus, computing

(39) $$\mathcal{S}^2 Z_c(J,\eta) = -\int dx\left[\bar{\xi}^\alpha \mathcal{G}^i_\alpha\, \delta_{\gamma i}\right](x)\, Z_c(J,\eta)$$

where we have used the abbreviation

$$Z_c(\mathcal{J},\eta) = Z_c(\mathcal{J},\eta,0)$$

we get

$$\begin{aligned} & -\int dx \left[\bar{\xi}^{\alpha} \mathcal{G}_{\alpha}^{i} \delta_{\gamma i} \right](x)\, Z_c(\mathcal{J},\eta) = \\ (40) \quad & = \mathcal{S} \int dx\, \delta_{\beta(x)}\, Z_c(\mathcal{J},\eta,\beta)\Big|_{\beta=0} \\ & = -\int dx \left[s^{\Delta}\left(\mathcal{L}_{eff} + \gamma^{i}\mathring{P}_i + \zeta^{\alpha}\mathring{P}_{\alpha}\right) + s\Delta + O(\hbar\Delta) \right](x) \ldots \\ & \ldots Z_c(\mathcal{J},\eta) \end{aligned}$$

since

$$(41) \quad \int dx\, dy\, \delta_{\beta(x)}\, \delta_{\beta(y)}\, Z_c(\mathcal{J},\eta,\beta) = 0$$

In terms of the vertex functional $\Gamma(\Psi,\eta) = \Gamma(\phi_i, c_{\alpha}, \bar{c}_{\alpha}, \eta)$ which is the Legendre transform of $Z_c(\mathcal{J},\eta)$ with respect to $\mathcal{J}$. Eq.(40) reads :

$$\begin{aligned} & \int dx \left[\delta_{c_{\alpha}}\Gamma\, \mathcal{G}_{\alpha}^{i}\, \delta_{\gamma i}\Gamma \right](x) = \\ (42) \quad & = \int dx \left[s^{\Delta}\left(\mathcal{L}_{eff} + \gamma^{i}P_i + \zeta^{\alpha}P_{\alpha}\right) + s\Delta + O(\hbar\Delta) \right](x) \\ & = \int dx\, \Delta'(x) + O(\hbar\Delta) \end{aligned}$$

Eq.(42) leads to a consistency condition for Δ. Indeed Eq.(42) is a perturbed version of the equation

(43) $$\int dx \left[\delta_{c_\alpha} \Gamma \; \mathcal{G}_\alpha^i \; \delta_{\gamma_i} \Gamma \right](x) = 0$$

with a perturbation of order Δ' . Eq.(43) is itself a quantum extension of :

(44) $$\int dx \left[\delta_{c_\alpha} \left\{ \int dy \; \mathcal{L}_{eff}^{(\eta)}(y) \right\} \mathcal{G}_\alpha^i \; \delta_{\gamma_i} \left\{ \int dz \; \mathcal{L}_{eff}^{(\eta)}(z) \right\} \right](x) = 0$$

since for any field ω

(45) $$\delta_{\omega(x)} \Gamma(\Psi, \eta) = \delta_{\omega(x)} \int dy \; \mathcal{L}_{eff}^{(\eta)}(y) + O(\hbar)$$

We have seen in the previous section that Eq.(44) , as an equation for $\delta_{c_\alpha} \mathcal{L}_{eff}$ possesses the general solution

(46) $$\delta_{c_\alpha} \int \mathcal{L}_{eff}^{(\eta)}(x) dx = \Gamma^{\alpha\alpha'} \mathcal{G}_{\alpha'}^i \; \delta_{\gamma_i} \int \mathcal{L}_{eff}^{(\eta)}(x)\, dx$$

where $\Gamma^{\alpha\alpha'}$ is a numerical symmetrical matrix.
This provides a solution to Eq.(43):

(47) $$\delta_{c_\alpha} \Gamma = \Gamma^{\alpha\alpha'} \mathcal{G}_i^{\alpha'} \delta_{\gamma_i} \Gamma$$

This is the general solution of Eq.(43) because Eq.(43) is a quantum perturbation of Eq.(44).
Hence the general solution of Eq.(42) is given by

(48) $$\delta_{c_\alpha} \int \mathcal{L}_{eff}^{(\eta)}(x) dx = \Gamma^{\alpha\alpha'} \mathcal{G}_{\alpha'}^i \; \delta_{\gamma_i} \int \mathcal{L}_{eff}^{(\eta)}(x) dx + O(\Delta')$$

which, taking into account the structure of the γ couplings can be written in the form

(49) $$\delta_{\xi_\alpha}\int \mathcal{L}^{(\eta)}_{eff}(x)\,dx = \Gamma^{\alpha\alpha'\,i}\, \mathcal{G}_{\alpha'}\,\delta_{\gamma^i}\int \mathcal{L}^{(\eta)}_{eff}(x)\,dx + \delta_{\xi_\alpha}\int \mathcal{R}^{(\Delta')}_{(x)}\,dx$$

where $\mathcal{R}^{(\Delta')}$ is of the order of Δ'.

Substituting Eq.(49) into Eq.(42) leads to

(50) $$\int dx\ \delta_{C_\alpha(x)}\int dy\ \mathcal{R}^{(\Delta')}(y)\ \mathcal{G}^{i}_{\alpha}\,\delta_{\gamma^i(x)}\int dz\ \mathcal{L}^{(\eta)}_{eff}(z)$$
$$= \int dx\ \Delta'(x) + O(\hbar\Delta)$$

Recalling Eqs.(28),(29), one has

(51) $$\mathcal{G}^{i}_{\alpha}\,P_i = \mathcal{G}^{i}_{\alpha}\,\mathring{P}_i + O(\hbar)$$

Hence, using Eq.(26),

(52) $$\mathcal{G}^{i}_{\alpha}\,\delta_{\gamma^i(x)}\int dy\,\mathcal{L}^{(\eta)}_{eff}(y) = s\,\mathcal{G}^{i}_{\alpha}\,\Phi_i + O(\hbar)$$
$$= s^2 C_\alpha + O(\hbar)$$

We have thus obtained the consistency condition :

(53) $$s^{\Delta}\int dx\left(\mathcal{L}_{eff} + \gamma^i\,\mathring{P}_i + \xi^\alpha\,\mathring{P}_\alpha\right)(x) + s\int dx\ \Delta(x)$$
$$= -\,s^2\int \mathcal{R}^{(\Delta')}_{(x)}\,dx + O(\hbar\Delta)$$

Now, the most general form of Δ is

$$\Delta(x) = \left(\Delta^{\alpha}\bar{C}_{\alpha}\right)(x) + \left(C_{\alpha}\,\Delta^{\alpha\beta\gamma}_{yz}\,\bar{C}_{\beta}(y)\bar{C}_{\gamma}(z)\right)(x) +$$
$$+ \left(C_{\alpha}\,C_{\beta}\,\bar{C}_{\gamma}\,\bar{C}_{\delta}\,\bar{C}_{\eta}\right)(x)\,\Delta^{\alpha\beta,\gamma\delta\eta}$$
$$+ \left(\gamma^{i}\Delta_{i} + \zeta^{\alpha}\Delta_{\alpha}\right)(x) \tag{54}$$

where dimensions and quantum numbers are taken into account by

$$\Delta_{i}(x) = \frac{1}{2}\left[\Theta^{\alpha\beta}_{ij}\,\phi_{j}\,\bar{c}_{\alpha}\,\bar{c}_{\beta} + \bar{c}_{\alpha}\,\Sigma^{\alpha\beta}_{i}\,\bar{c}_{\beta}\right](x) \tag{55}$$

with

$$\Sigma^{\alpha\beta}{}_{\gamma\mu} = \Sigma^{\alpha\beta}{}_{\gamma}\,\partial_{\mu} \tag{56}$$

and

$$\Delta_{\alpha}(x) = -\frac{1}{6}\,\Gamma_{\alpha}{}^{\beta\gamma\delta}\left(\bar{c}_{\beta}\,\bar{c}_{\gamma}\,\bar{c}_{\delta}\right)(x) \tag{57}$$

for some c-number coefficients $\Theta^{\alpha\beta}_{ij}$, $\Sigma^{\alpha\beta}{}_{\gamma}$, $\Gamma_{\alpha}{}^{\beta\gamma\delta}$
Since, due to power counting $\mathcal{R}^{(\Delta)}$ cannot depend on the external fields (γ^{i}, ζ^{α}) , the external field dependent part of the left hand side of Eq.(53) , must be $O(\hbar\Delta)$, namely,

$$s^{\Delta}\int dx\left(\gamma^{i}\,\mathring{P}_{i} + \zeta^{\alpha}\,\mathring{P}_{\alpha}\right)(x) + s\int dx\,\Delta(x) = O(\hbar\Delta) \tag{58}$$

It is shown in Appendix B , by cohomological methods, that, as a result, the external field dependent part of Δ is of the form

$$\int\left(\gamma^{i}\Delta_{i} + \zeta^{\alpha}\Delta_{\alpha}\right)(x)\,dx = -s^{\mathring{P}+\Pi}\int\left[\gamma^{i}(\mathring{P}+\Pi)_{i} + \zeta^{\alpha}(\mathring{P}+\Pi)_{\alpha}\right](x)\,dx$$
$$+ O(\hbar\Delta) \tag{59}$$

where Π_i, Π_α can be parametrized in terms of numerical coefficients $\hat{\Theta}^\alpha_{ij}$, $\hat{\Sigma}^\alpha_i$, $\hat{\Sigma}^\alpha_\beta$ according to

$$\Pi_i = \hat{\Theta}^\alpha_{ij}\,\phi_j\,\bar{c}_\alpha + \hat{\Sigma}^\alpha_i\,\bar{c}_\alpha \tag{60}$$

$$\Pi_\alpha = \hat{\Gamma}_\alpha^{\ \beta\gamma}\,\bar{c}_\beta\,\bar{c}_\gamma$$

with

$$\hat{\Sigma}^\alpha_{\beta,\mu} = \hat{\Sigma}^\alpha{}_\beta\,\partial_\mu \tag{61}$$

The coefficients $\hat{\Theta}$, $\hat{\Sigma}$, $\hat{\Gamma}$ can be chosen and will be chosen to be linear in Θ, Σ, Γ, as shown in Appendix B.

Now, we know from the quantum action principle that the external field dependent part of Δ is of the form (cf. Eqs.(33),34):

$$\int dx\left(\gamma^i\Delta_i + \zeta^\alpha\Delta_\alpha\right)(x) = -\mathcal{S}^P\int dx\left(\gamma^i P_i + \zeta^\alpha P_\alpha\right)(x) + \hbar\int\left(\gamma^i Q_i + \zeta^\alpha Q_\alpha\right)(x)\,dx \tag{62}$$

where Q_i, Q_α are formal power series in $\hbar$ and in the coefficients of $\mathcal{L}^{(P)int}_{eff}$. From the previous observation that Π_i, Π_α are linear in Δ_i, Δ_α, it follows that Π_i, Π_α are formal power series of the same type as Q_i, Q_α.

Finally the system

$$\Pi_i = \Pi_\alpha = 0 \tag{63}$$

which insures that (cf. Eq.(59)

$$\int dx\left(\gamma^i\Delta_i + \zeta^\alpha\Delta_\alpha\right)(x) = O(\hbar\Delta) \tag{64}$$

is soluble for P_i, P_α as formal power series in $\hbar$ considering Π_i, Π_α

as formal power series in $\hbar$, $\mathcal{P}_i - \mathring{\mathcal{P}}_i$, $\mathcal{P}_\alpha - \mathring{\mathcal{P}}_\alpha$, and comparing (64) with (62) we see that, to lowest non vanishing order in $\hbar$, $\mathcal{P}_i - \mathring{\mathcal{P}}_i$, $\mathcal{P}_\alpha - \mathring{\mathcal{P}}_\alpha$,

(65)
$$\Pi_i \simeq \mathcal{P}_i - \mathring{\mathcal{P}}_i$$
$$\Pi_\alpha \simeq \mathcal{P}_\alpha - \mathring{\mathcal{P}}_\alpha$$

Hence system (63) assumes the form

(66)
$$\mathcal{P}_i - \mathring{\mathcal{P}}_i = \Theta_i(\hbar, \mathcal{P} - \mathring{\mathcal{P}})$$
$$\mathcal{P}_\alpha - \mathring{\mathcal{P}}_\alpha = \Theta_\alpha(\hbar, \mathcal{P} - \mathring{\mathcal{P}})$$

where both Θ_i, Θ_α are $O(\hbar, (\mathcal{P} - \mathring{\mathcal{P}})^2)$. This system has a unique solution $O(\hbar)$ for $\mathcal{P}_i - \mathring{\mathcal{P}}_i$, $\mathcal{P}_\alpha - \mathring{\mathcal{P}}_\alpha$.

Let us now look at the external field independent terms in Δ, which is $O(\hbar\Delta)$ in the external fields. The consistency condition now reads

(67)
$$\mathcal{J}^2 \int R^{(\Delta')}_{(x)} dx = -\mathcal{J} \int dx \Delta_0(x) + O(\hbar\Delta)$$

i.e.

(68)
$$\mathcal{J} \int dx \left(\mathcal{J} R^{(\Delta')} + \Delta_0 \right)(x) = O(\hbar\Delta)$$

where

(69)
$$\Delta_0 = \Delta \Big|_{\gamma = \zeta = 0}$$

If we can prove that $\int \Delta_0(x)\, dx$ is of the form

(70)
$$\int dx \Delta_0(x) = \mathcal{J} \int dx \hat{\Delta}_0(x) + O(\hbar\Delta)$$

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comparing with Eq.(33)(34) which can be cast into the form

$$\Delta_0 = -s\,\mathcal{L}_{eff} + \hbar\,\tilde{Q}_0\left(\hbar, \mathcal{L}_{eff}^{int}\right) \tag{71}$$

where $\hbar\tilde{Q}_0 = \hbar Q_0 + (s - s^P)\mathcal{L}_{eff}$ is a formal power series in the indicated arguments, it follows that $\tilde{Q}_0$ is of the form

$$\tilde{Q}_0\left(\hbar, \mathcal{L}_{eff}^{int}\right) = s\,\hat{Q}_0\left(\hbar, \mathcal{L}_{eff}^{int}\right) + O(\hbar\Delta) \tag{72}$$

Since the equation

$$\mathcal{L}_{eff} - \mathcal{L} - \hbar\,\hat{Q}_0\left(\hbar, \mathcal{L}_{eff}^{int}\right) = 0 \tag{73}$$

is soluble for

$$\mathcal{L}_{eff} - \mathcal{L} = O(\hbar) \tag{74}$$

its solution leaves us with

$$\Delta_0 = O(\hbar\Delta) \tag{75}$$

Recalling that also (cf.Eq.(64))

$$\Delta_i = O(\hbar\Delta), \quad \Delta_\alpha = O(\hbar\Delta) \tag{76}$$

we conclude that

$$\Delta = O(\hbar\Delta) \tag{77}$$

hence

$$\Delta = 0 \tag{78}$$

which we want to achieve.

Now the consistency condition shows that

$$\int dx\,\Delta_0(x) = -\mathscr{s}\int R^{(\Delta')}_{(x)}dx + \Delta_4 + O(\hbar\Delta) \tag{79}$$

where

$$\mathscr{s}\,\Delta_4 = 0 \tag{80}$$

Thus, there remains to prove that any Δ_4 can be put in the form

$$\Delta_4 = \mathscr{s}\,\widehat{\Delta}_4 \tag{81}$$

Using for Δ_4 the expansion (cf.Eq.(54))

$$\begin{aligned}\Delta_4(x) = {} & \Delta_4^{\alpha}(x)\,\bar{c}_\alpha(x) + c_\alpha(x)\,\Delta^{\alpha\beta\gamma}_{4\,xyz}\,\bar{c}_\beta(y)\,\bar{c}_\gamma(z) + \\ & + c_\alpha(x)\,c_\beta(x)\,\bar{c}_\gamma(x)\,\bar{c}_\delta(x)\,\bar{c}_\eta(x)\,\Delta_4^{\alpha\beta,\gamma\delta\eta}\end{aligned} \tag{82}$$

since (80) implies

$$\mathscr{s}^2\Delta_4 = 0 \tag{83}$$

a discussion similar to that found at the end of section A (cf.Eqs.(17),(18)) shows that

$$\begin{aligned}&\Delta_4^{\alpha\beta,\gamma\delta\eta} = 0 \\ &\Delta^{\alpha\beta\gamma}_{4\,xyz} = \Gamma^{\alpha\delta\gamma}\,\delta(x-z)\,\mathcal{M}_\delta{}^{\beta}(x,y)\end{aligned} \tag{84}$$

where the numerical tensor $\Gamma^{\alpha\delta\gamma}$ is symmetric in α and δ.
Using now

$$\mathscr{s}\,\Delta_4 = 0$$

yields

(85) $$\Gamma^{\widetilde{\alpha\delta}\,\gamma} = 0$$

and

(86) $$\delta_{\omega_\alpha(x)} \Delta_4^\beta(y) - \delta_{\omega_\beta(y)} \Delta_4^\alpha(x) - f^{\alpha\beta}{}_\gamma\, \delta(x-y)\Delta_4^\gamma(y) = 0$$

One can show [20] that the general solution of Eq.(86), the first cohomology condition for the gauge Lie algebra, which is nothing else than the Wess Zumino [9] consistency condition, is

(87) $$\Delta_4^\alpha(x) = \delta_{\omega_\alpha(x)} \hat{\Delta}_4 + g_\alpha(x)$$

where $g_\alpha(x)$, the Bardeen [10] anomaly has the form

(88) $$g^\alpha(x) = \partial^\mu \epsilon_{\mu\nu\rho\sigma}\left[D^{\alpha\beta\gamma}\partial^\nu a_\beta^\rho a_\gamma^\sigma + F^{\alpha\beta\gamma\delta} a_\beta^\nu a_\gamma^\rho a_\delta^\sigma\right]$$

where $D^{\alpha\beta\gamma}$ is a totally symmetric invariant tensor with indices in the adjoint representation of $\mathfrak{h}$ and

(89) $$F^{\alpha\beta\gamma\delta} = \frac{1}{12}\left[D^{\alpha\beta\lambda} f_\lambda{}^{\gamma\delta} + D^{\alpha\gamma\lambda} f_\lambda{}^{\delta\beta} + D^{\alpha\delta\lambda} f_\lambda{}^{\beta\gamma}\right]$$

Such an anomaly can only arise if the tree Lagrangian contains $\epsilon_{\mu\nu\rho\sigma}$ or γ_5 symbols and if there is a non trivial D tensor. In the absence of such an anomaly, one has :

(90) $$\bar{C}_\alpha(x)\Delta_4^\alpha(x) = \bar{C}_\alpha(x)\, \delta_{\omega_\alpha(x)} \hat{\Delta}_4 = \mathcal{S}\, \hat{\Delta}_4$$

which completes the proof of the Slavnov identity in such cases.

C. The Faddeev Popov ghost equation of motion

It will be of interest in the following to write down the Faddeev Popov ghost equation of motion in terms of the Slavnov identity. It follows from Eq.(42) that once the Slavnov identity has been proved,

$$\int \left[\delta_{C_\alpha} \Gamma \;\; \mathcal{G}_i^{\alpha} \, \delta_{\gamma_i} \Gamma \right](x) \, dx = 0 \tag{91}$$

and we know that the general solution of this equation is

$$\delta_{C_\alpha(x)} \Gamma = \Gamma^{\alpha}{}_{\alpha'} \, \mathcal{G}_i^{\alpha'} \, \delta_{\gamma_i(x)} \Gamma \tag{92}$$

where $\Gamma^{\alpha}{}_{\alpha'}$ is a symmetrical matrix. Taking the Legendre transform of Eq.(92) yields

$$\Gamma^{\alpha}{}_{\alpha'} \, \mathcal{G}_i^{\alpha'} \, \delta_{\gamma_i(x)} Z_c(\mathcal{J}, \eta) = \bar{\xi}^{\alpha}(x) \tag{93}$$

which is therefore the $\Phi\Pi$ equation of motion. One should also note that $\Gamma^{\alpha}{}_{\alpha'}$ is invertible in the tree approximation and therefore to all orders so that Eq.(93) may be written in the form

$$\mathcal{G}_i^{\alpha} \, \delta_{\gamma_i(x)} Z_c(\mathcal{J}, \eta) = (\Gamma^{-1})^{\alpha}{}_{\alpha'} \, \bar{\xi}^{\alpha'}(x) \tag{94}$$

3 - Physical Interpretation : An Example [16]

Given a gauge theory which has been renormalized in such a way that a Slavnov identity holds, there remains to show that one can interpret it in physical terms, which is not obvious since many ghost fields are involved (∂a, c, $\bar{c}$ ...). First one should specify the connection between the parameters left arbitrary in the Lagrangian, and physical parameters, (masses, coupling constants) through the fulfillment of suitable normalization conditions. Then, once the theory has been set up within the framework of a fixed Fock space, one has to specify a physical subspace within which the theory is reasonable, e.g. the S operator is unitary and independent from unphysical parameters among which the g_α^i cf. Eq.(26).

In order to make this program explicit we shall treat in some details the SU2 Higgs Kibble model [16] in a way which parallels our treatment of the abelian Higgs Kibble model [1].

A. The Classical Theory.

The basic fields are $\underline{\Psi} = \{\sigma, \pi_\alpha, a_{\alpha\mu}, c_\alpha, \bar{c}_\alpha\}, \alpha = 1,2,3$

At the classical level, the Lagrangian is invariant under the Slavnov transformation :

$$\delta\pi_\alpha = \delta\lambda\left[-\frac{e}{2}\,\varepsilon_\alpha{}^{\beta\gamma}\,\pi_\beta\bar{c}_\gamma + \frac{e}{2}\,(\sigma+F)\,\bar{c}_\alpha\right] \equiv \delta\lambda\,\mathcal{L}_\alpha$$

$$\delta\sigma = \delta\lambda\left[-\frac{e}{2}\,\pi^\alpha\bar{c}_\alpha\right] \equiv \delta\lambda\,\mathcal{L}_0$$

(95)
$$\delta a_{\alpha\mu} = \delta\lambda\left[\partial_\mu\bar{c}_\alpha - e\,\varepsilon_\alpha{}^{\beta\gamma}a_{\beta\mu}\bar{c}_\gamma\right] \equiv \delta\lambda\,\mathcal{L}_{\alpha\mu}$$

$$\delta\bar{c}_\alpha = \delta\lambda\left[\frac{e}{2}\,\varepsilon_\alpha{}^{\beta\gamma}\,\bar{c}_\beta\,\bar{c}_\gamma\right] \equiv \delta\lambda\,\bar{\mathcal{L}}_\alpha$$

$$\delta c_\alpha = \delta\lambda\left[\partial^\mu a_{\alpha\mu} + \rho\,\pi_\alpha\right] \equiv \delta\lambda\,\mathcal{G}_\alpha$$

where $\varepsilon^{\alpha\beta\gamma}$ are the SU(2) structure constants, e plays the role of a coupling constant, and F is the σ field translation parameter. This particular choice of transformation laws implies the invariance under a global SU(2) symmetry transforming $\underline{\pi}$, $\underline{a}_\mu$, $\underline{C}$, $\underline{\bar{C}}$ as vectors and leaving σ invariant. Even if this symmetry is to be preserved Eq.(95) is not the most general transformation law fulfilling the compatibility conditions Eqs.(14,15), which depends on four parameters besides those which label the gauge function : e, F, and wave function renormalizations for the σ and $\bar{C}$ fields. Given these parameters and introducing external fields $\{\eta\} = \{\gamma^\alpha, \gamma^0, \gamma^{\alpha\mu}, \zeta^\alpha\}$ coupled to $\{\mathcal{L}\} = \{\mathcal{L}_\alpha, \mathcal{L}_0, \mathcal{L}_{\alpha\mu}, \bar{\mathcal{L}}_\alpha\}$, and sources $\{J\} = \{J^\alpha, J^0, J^{\alpha\mu}, \xi^\alpha, \bar{\xi}^\alpha\}$ coupled to $\{\Psi\} = \{\pi_\alpha, \sigma, a_{\alpha\mu}, \bar{c}_\alpha, c_\alpha\}$ the Slavnov identity reads :

$$
\begin{aligned}
(96)\quad \mathcal{S} Z_c(J,\eta) = \int dx \Big[& J^\alpha(x)\, \delta_{\gamma^\alpha(x)} + J^0(x)\, \delta_{\gamma^0(x)} + J^{\alpha\mu}(x)\, \delta_{\gamma^{\alpha\mu}(x)} \\
& - \xi^\alpha(x)\, \delta_{\zeta^\alpha(x)} - \bar{\xi}^\alpha(x) \left(\partial^\mu \delta_{J^{\mu\alpha}(x)} + \rho\, \delta_{J^\alpha(x)} \right) \Big] Z_c(J,\eta) = \\
& = 0
\end{aligned}
$$

We have seen that the most general action compatible with Eq.(96) is of the form

$$
\begin{aligned}
(97)\quad \int \mathcal{L}^{cl}(x)\, dx = \int dx \Big\{ & \mathcal{L}_{inv.}(x) - \frac{1}{K}\Big[G^\alpha(x) G_\alpha(x) - \int dy\, C_\alpha(x)\, \mathcal{M}^{\alpha\beta}_{xy}\, \bar{C}_\beta(y) \Big] \\
& + \gamma^\alpha(x)\, \mathcal{L}_\alpha(x) + \gamma^0(x)\, \mathcal{L}_0(x) + \gamma^{\alpha\mu}(x)\, \mathcal{L}_{\alpha\mu}(x) + \zeta^\alpha(x)\, \bar{\mathcal{L}}_\alpha(x) \Big\}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{L}_{inv.} = & - \frac{Z_a}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha + \frac{Z_1}{2} D_\mu \underline{\Phi}\, D^\mu \underline{\Phi} \\
(98)\qquad & + \frac{\mu^2}{2} \underline{\Phi}\cdot\underline{\Phi} - \frac{\lambda^2}{4!} \left(\underline{\Phi}\cdot\underline{\Phi}\right)^2 - \left[\frac{\mu^2}{2} F^2 - \frac{\lambda^2}{4!} F^4 \right]
\end{aligned}
$$

where

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$$G_{\alpha\mu\nu} = \partial_\mu a_{\alpha\nu} - \partial_\nu a_{\alpha\mu} - e\,\varepsilon_\alpha{}^{\beta\gamma} a_{\beta\mu} a_{\gamma\nu}$$

$$\underline{\Phi} = \begin{pmatrix} \{\pi_\alpha\} \\ \sigma + F \end{pmatrix} = \begin{pmatrix} \vec{\pi} \\ \sigma + F \end{pmatrix} ; \quad \underline{\tilde{\Phi}} \cdot \underline{\Phi} = \vec{\pi}^2 + (\sigma + F)^2$$

(99)
$$D_\mu = \partial_\mu \mathbb{1} - \frac{e}{2} \left(\begin{array}{c:c} \vec{\varepsilon}.\vec{a}_\mu & \vec{a}_\mu \\ \hdashline -\vec{a}_\mu & 0 \end{array} \right)$$

$$(\vec{\varepsilon}.\vec{a}_\mu)_\alpha{}^\beta = -\varepsilon_\alpha{}^{\beta\gamma} a_{\gamma\mu}$$

$$\mathcal{M}^{\alpha\beta}_{xy} = \partial^\mu \left(\partial^x_\mu \delta(x-y)\, \delta^{\alpha\beta} + e\, \varepsilon^{\alpha\beta\gamma} a_{\gamma\mu}(x) \delta(x-y) \right)$$
$$+ \frac{\rho e}{2} \left(\varepsilon^{\alpha\beta\gamma} \pi_\gamma(x) + \delta^{\alpha\beta} \left(\sigma(x) + F \right) \right) \delta(x-y)$$

and where the coefficients are so adjusted that the coefficient of the term linear in σ vanishes :

(100)
$$\mu^2 - \frac{F^2 \lambda^2}{3!} = 0$$

The theory thus depends on ten parameters, four specifying the transformation law, one related with the field vacuum expectation value, five specifying the external field independent part of the Lagrangian, constrained by condition (14) , (15) . One can alternatively specify the following physical parameters : m_a , M , $m_{\phi\pi}$, which give the positions of the poles in the transverse photon, σ , $C, \bar{C}$ propagators respectively :

(101)
(a) $$\Gamma_{a^T a^T}(m_a^2) = 0$$
(b) $$\Gamma_{\sigma\sigma}(M^2) = 0$$
(c) $$\Gamma_{c\bar{c}}(m^2_{\phi\pi}) = 0$$

the residues of these poles :

(102)
(a) $$\Gamma'_{a^T a^T}(m_a^2) = Z_a$$
(b) $$\Gamma'_{\sigma\sigma}(M^2) = Z_1$$

(c) $$\Gamma'_{c\bar{c}}(m^2_{\phi\pi}) = \frac{1}{\kappa}$$

the value of the coupling constant

(103) $$\Gamma_{a^\tau a^\tau \sigma}(m^2_a, m^2_a, M^2) = \varepsilon$$

These normalization conditions together with Eq.(100) which is equivalent to

(104) $$\langle\sigma\rangle = 0$$

fix the values of Z_a , Z_1 , μ^2, λ^2, $\varkappa$, ρ , e , F , leaving free two parameters in the definition of the transformation laws. For simplicity we shall of course choose

(105) $$Z_a = Z_1 = 1$$

These normalization conditions together with the Slavnov symmetry actually imply that the masses associated with the $\partial\vec{a}$, $\vec{\pi}$ channel are pairwise degenerate with those of the $c\,\bar{c}$ channel. Thus, in view of the residual SU2 symmetry, all the ghost masses are degenerate.

Within the Fock space defined by the quadratic part of the Lagrangian, we shall define the bare physical subspace generated by application on the vacuum of the asymptotic fields $a_\mu^{T\,in}$, σ^{in} which explains that c , $\bar{c}$, $\partial\vec{a}$, $\vec{\pi}$ are considered as "ghosts".

According to this definition, it is easy to see [21] that the matrix elements of the S operator between bare physical states do not depend on the gauge parameters κ , $m_{\phi\pi}$.

8. Radiative Corrections - Slavnov Identities, Normalization Conditions.

Now, according to the analysis of section 2 , it is possible to find an effective Lagrangian such that the Slavnov identity (96) holds to all orders (where now ρ is to be determined as a formal power series in $\hbar$). We also know that the Faddeev Popov equation of motion is : (cf. Eq.(94))

$$\left[\partial^{\mu}\delta_{\gamma^{\alpha\mu}(x)}+\rho\,\delta_{\gamma^{\alpha}(x)}\right]Z_c(J,\eta)=\bar{\kappa}\,\xi_{\alpha}(x) \tag{106}$$

where $\bar{\kappa}$ is some formal power series in $\hbar$.

The theory depends on ten formal power series whose lowest order terms specify the tree approximation Lagrangian. Eight of them can be fixed by imposing the normalization conditions (101, 102, 103, 104). These normalization conditions are enough to interpret the theory in the initial Fock space, because, as a consequence of the Slavnov identity, the ghost mass degeneracy still holds :

Expressing the Slavnov identity in terms of the vertex functional Γ yields :

$$\begin{aligned}\int dx\Big\{&\delta_{\pi_{\alpha}(x)}\Gamma\,\delta_{\gamma^{\alpha}(x)}\Gamma+\delta_{\sigma(x)}\Gamma\,\delta_{\gamma^{o}(x)}\Gamma+\delta_{a_{\alpha\mu}(x)}\Gamma\,\delta_{\gamma^{\alpha\mu}(x)}\Gamma+\\&+\delta_{\bar{c}_{\alpha}(x)}\Gamma\,\delta_{\zeta^{\alpha}(x)}\Gamma+\delta_{c_{\alpha}(x)}\Gamma\left[\partial^{\mu}a_{\alpha\mu}(x)+\rho\,\pi_{\alpha}(x)\right]\Big\}=0\end{aligned} \tag{107}$$

In particular, one gets the following information on the two point functions :

$$\begin{aligned}&\tilde{\Gamma}_{\pi^2}(p^2)\,\tilde{\Gamma}_{\gamma}(p^2)+\tilde{\Gamma}_{\pi a_L}(p^2)\,\tilde{\Gamma}_{\gamma L}(p^2)+\rho\,\tilde{\Gamma}_{\bar{c}c}(p^2)=0\\&\tilde{\Gamma}_{\pi a_L}(p^2)\,\tilde{\Gamma}_{\gamma}(p^2)+\tilde{\Gamma}_{(a_L)^2}(p^2)\,\tilde{\Gamma}_{\gamma L}(p^2)-p^2\,\tilde{\Gamma}_{\bar{c}c}(p^2)=0\end{aligned} \tag{108}$$

where

$$\begin{aligned}&\tilde{\Gamma}_{\pi^2}(p^2)=\tilde{\Gamma}_{\pi_{\alpha}\pi^{\alpha}}(p^2)\\&\tilde{\Gamma}_{\pi a_L}(p^2)=i\,p_{\mu}\,\tilde{\Gamma}_{\pi^{\alpha}a_{\alpha\mu}}(p)\\&\tilde{\Gamma}_{(a_L)^2}(p^2)=-p_{\mu}\,p_{\nu}\,\tilde{\Gamma}_{a_{\alpha\mu}a_{\nu}^{\alpha}}(p)\\&\tilde{\Gamma}_{\gamma}(p^2)=\tilde{\Gamma}_{\bar{c}_{\alpha}\gamma^{\alpha}}(p)\\&\tilde{\Gamma}_{\gamma L}(p^2)=-\frac{i\,p^{\mu}}{p^2}\,\tilde{\Gamma}_{\bar{c}_{\alpha}\gamma^{\alpha\mu}}(p)\end{aligned} \tag{109}$$

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On the other hand the $\Phi\pi$ equation of action (108) yields :

(110) $$\rho \tilde{\Gamma}_{\gamma}(p^2) - p^2 \tilde{\Gamma}_{\gamma L}(p^2) = \bar{\kappa}\, \tilde{\Gamma}_{\bar{c}c}(p^2)$$

It follows that :

(111) $$\det \begin{pmatrix} \tilde{\Gamma}_{\gamma^2} & \tilde{\Gamma}_{\pi a_L} \\ \tilde{\Gamma}_{\pi a_L} & \tilde{\Gamma}_{a_L^2} \end{pmatrix}(p) = \left[\tilde{\Gamma}_{\gamma}(p^2)\, \tilde{\Gamma}_{\gamma L}(p^2)\right]^{-1} \left[\bar{\kappa}\, \tilde{\Gamma}_{a_L \pi}(p^2) - \rho p^2\right] \tilde{\Gamma}^{2}_{\bar{c}c}(p^2)$$

which shows that this determinant has a double zero at

$$p^2 = m^2_{\Phi\pi}$$

3. The Physical S Operator : Gauge Invariance.

According to the LSZ asymptotic theory, the physical S operator is given in the perturbative sense in terms of the Green's functional $Z = \exp \frac{i}{\hbar} Z_c$, by :

(112) $$S_{phys} = S_{phys}(J)\Big|_{J=0}$$

where

(113) $$\begin{aligned} S_{phys}(J) &= :\exp \int dx \left\{ \sigma^{in}(x)\, K^{\sigma}_{xy}\, \delta_{J^{\sigma}(y)} + \right. \\ &\qquad \left. + a^{T\,in}_{\alpha\mu}(x)\, K^{\alpha\mu\,\beta\nu}_{xy}\, \delta_{J^{\beta\nu}(y)} \right\} : Z(J,\eta)\Big|_{\eta=0} \\ &\stackrel{\text{Def.}}{=} \sum_{phys} Z(J,\eta) \end{aligned}$$

In the above formula, σ^{in} , K^{σ} ; $a^{\tau\, in}_{\alpha\mu}$, $K^{\alpha\mu\beta\nu}$ are the canonically quantized asymptotic fields and differential operators involved in their equations of motion, derived from the asymptotic Lagrangian determined by Γ, which in the present case, can be read off from the tree approximation.

We want to prove :

$$\text{(114)} \qquad \frac{\partial S_{phys}}{\partial m^2_{\phi\pi}} = \frac{\partial S_{phys}}{\partial \varkappa} = 0$$

Owing to Lowenstein's action principle, [3] , one has

$$\text{(115)} \qquad \frac{\hbar}{i}\, \partial_\lambda Z(J,\eta) = \Delta_\lambda Z(J,\eta)$$

where λ is one of the parameters $\varkappa$, $m^2_{\phi\pi}$ and Δ_λ is a dimension four insertion obtained by differenciating $\int dx\, \mathcal{L}_{eff}(x)$ with respect to λ . Using the Slavnov identity, we are going to show that Δ_λ can be written as :

$$\text{(116)} \quad \Delta_\lambda = \sum_{i=1}^{5} C_\lambda^{o,i}\, \Delta_i^o + \sum_{i=1}^{5} C_\lambda^{s,i}\, \Delta_i^s$$

where the Δ_i^o's are such that

$$\text{(117)} \qquad \sum\nolimits_{phys} \Delta_i^o\, Z(J,\eta) = 0\ , \quad i=1,\dots 5$$

and therefore leave unaltered the physical normalization conditions (101 a,b , 102 a,b , 103) . In the following these insertions will be called non physical. On the other hand the Δ_i^s's are symmetric insertions, namely

$$\text{(118)} \qquad \mathcal{S}\ \Delta_i^s\, Z(J,\eta) = 0 \qquad i=1,\dots,5$$

Thus applying Eqs.(116,117) to the physical normalization conditions (101 a,b , 102 a,b , 103) , yields a linear homogeneous system of equations of the form

$$\sum_{i=1}^{5} C_\lambda^{s,i} \Delta_i^{s,j} = 0 \qquad j = 1, \dots 5 \tag{119}$$

for which we are going to see that [11,1]

$$\det \| \Delta_i^{s,j} \| \neq 0 \tag{120}$$

since this will prove to be true in the tree approximation. Hence it follows that

$$C_\lambda^{s,i} = 0 \qquad i = 1, \dots 5 \tag{121}$$

and the gauge invariance of the S operator follows from Eqs.(112,113,115, 116, 117).

The decomposition of Δ_λ given in Eq.(116) follows first from the Slavnov identity :

$$0 = \partial_\lambda \mathcal{S} Z = (\partial_\lambda \mathcal{S}) Z + \frac{i}{\hbar} \mathcal{S} \Delta_\lambda Z \tag{122}$$

Hence

$$\begin{aligned} (\partial_\lambda \mathcal{S}) Z &\equiv - \frac{\partial \rho}{\partial \lambda} \int \bar{\xi}^\alpha(x) \, \delta_{J^\alpha(x)} dx \, Z \\ &= \frac{i}{\hbar} [\Delta_\lambda, \mathcal{S}] Z \end{aligned} \tag{123}$$

Noticing that

$$\int dx \, \bar{\xi}^\alpha(x) \delta_{J^\alpha(x)} = - \left[\int dx \left(J^\alpha(x) \delta_{J^\alpha(x)} + \gamma^\alpha(x) \delta_{\gamma^\alpha(x)} \right), \mathcal{S} \right] \tag{124}$$

we define Δ^{o}_{λ} by

(125) $$\Delta^{o}_{\lambda} Z = \frac{\hbar}{i} \int dx \left[J^{\alpha}(x)\, \delta_{J^{\alpha}(x)} + \gamma^{\alpha}(x)\, \delta_{\gamma^{\alpha}(x)} \right] Z$$

which is a dimension four insertion as follows from Lowenstein's action principle [3] .
Hence

(126) $$\left[\Delta_{\lambda} - \frac{\partial \rho}{\partial \lambda}\, \Delta^{o}_{\lambda}\, ,\ \mathcal{S} \right] = 0$$

thus

(127) $$\Delta_{\lambda} = \frac{\partial \rho}{\partial \lambda}\, \Delta^{o}_{\lambda} + \Delta^{S}_{\lambda}$$

where Δ^{S}_{λ} is a dimension four symmetric insertion. We are thus left with finding a basis of symmetrical insertions. Since we know that, given the Slavnov identity $\mathcal{L}_{eff}$ depends on nine parameters, namely four to specify the couplings with the external fields and five to specify the remaining part of the Lagrangian, there are nine independent symmetrical insertions.

We shall first construct the four missing unphysical insertions. The method consists in constructing insertions which are realized by differential operators as a consequence of the action principle, and study their commutators with $\mathcal{S}$.

First consider [11] ,[1]

(128) $$Q^{\varepsilon}_{\pi} = \frac{\hbar^{2} \rho}{\bar{\kappa}} \int dx\ \delta_{\bar{\xi}^{\alpha}(x+\varepsilon)}\, \delta_{J_{\alpha}(x)}$$

$$Q^{\varepsilon}_{a_L} = \frac{\hbar^{2}}{\bar{\kappa}} \int dx\ \delta_{\bar{\xi}^{\alpha}(x+\varepsilon)}\, \partial_{\mu}\, \delta_{I_{\alpha\mu}(x)}$$

and define Δ^{ε}_{i} , $i = (\pi, a_L)$, by

$$\Delta_i^{\varepsilon} Z = : \mathcal{S} Q_i^{\varepsilon} : Z \tag{129}$$

where as usual the dots indicate the substraction of the disconnected part. One has :

$$[\Delta_i^{\varepsilon}, \mathcal{S}] Z = - \mathcal{S}^2 Q_i^{\varepsilon} Z = - [\mathcal{S}^2, Q_i^{\varepsilon}] Z \tag{130}$$

$$= \begin{cases} \rho \int dx \; \bar{\xi}^{\alpha}(x+\varepsilon)\, \delta_{J^{\alpha}(x)} Z & i = \pi \\ \int dx \; \bar{\xi}^{\alpha}(x+\varepsilon)\, \partial^{\mu} \delta_{J^{\alpha}_{\mu}(x)} Z & i = a_L \end{cases}$$

These commutators have obviously finite limits as $\varepsilon \to 0$. Consequently the infinite part of Δ_i^{ε} is symmetric. Substracting the infinite parts, we get in the limit $\varepsilon \to 0$ some Δ_i's ($i = \pi$, a_L) such that

$$[\Delta_i, \mathcal{S}] Z = \begin{cases} \rho \int dx\, \bar{\xi}^{\alpha}(x)\, \delta_{J^{\alpha}(x)} Z & i = \pi \\ \int dx\, \bar{\xi}^{\alpha}(x)\, \partial^{\mu} \delta_{J^{\alpha}_{\mu}(x)} Z & i = a_L \end{cases} \tag{131}$$

Furthermore

$$\begin{aligned} \sum_{phys} \Delta_i^{\varepsilon} Z \Big|_{J=0} &= \sum_{phys} : \mathcal{S} Q_i^{\varepsilon} : Z \Big|_{J=0} = \\ &= \sum_{phys} \int dx \left[J^0(x)\, \delta_{J^0(x)} + J^{\alpha\mu}(x)\, \delta_{J^{\alpha\mu}(x)} \right] Q_i^{\varepsilon} Z \Big|_{J=0} \\ &= \sum_{phys} \int dp \left[\tilde{J}^0(p)\, \tilde{\Gamma}^{\varepsilon}_{0,i}(p)\, \delta_{\tilde{J}^0(p)} + \tilde{J}^{\alpha\mu}(p)\, \tilde{\Gamma}^{\varepsilon}_{a,i}(p)\, \delta_{\tilde{J}^{\alpha\mu}(p)} \right] Z \Big|_{J=0} \\ &= \sum_{phys} \int dx \left[\tilde{\Gamma}^{\varepsilon}_{0,i}(M^2)\, J^0(x)\, \delta_{J^0(x)} + \tilde{\Gamma}^{\varepsilon}_{a,i}(m_a^2)\, J^{\alpha\mu}(x)\, \delta_{J^{\alpha\mu}(x)} \right] Z \Big|_{J=0} \end{aligned} \tag{132}$$

where

(133) $$\tilde{\Gamma}^{\varepsilon}_{o,i}(p^2) = \delta_{\tilde{\sigma}(p)}\,\delta_{\tilde{\gamma}^o(-p)}\,Q^{\varepsilon}_i\,\Gamma\Big|_{\psi=\eta=0}$$

$$\tilde{\Gamma}^{\varepsilon}_{a,i}(p^2) = \frac{1}{3}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta_{\tilde{a}^{\mu}_{\alpha}(p)}\,\delta_{\tilde{\gamma}^{\alpha\nu}(-p)}\,Q^{\varepsilon}_i\,\Gamma\Big|_{\psi=\eta=0}$$

Let now

(134) $$\Delta^{o\varepsilon}_i = \Delta^{\varepsilon}_i - \tilde{\Gamma}^{\varepsilon}_{oi}(M^2)\Delta^{\varepsilon}_{\sigma} - \tilde{\Gamma}^{\varepsilon}_{ai}(m^2_a)\,\Delta^{S}_{a}$$

where the symmetrical insertions

$$\Delta^{S}_{\sigma} = \int dx\left[J^o(x)\,\delta_{J^o(x)} + \gamma^o(x)\,\delta_{\gamma^o(x)}\right]$$

(135) $$\Delta^{S}_{a} = \int dx\left[J^{\alpha\mu}(x)\,\delta_{J^{\alpha\mu}(x)} + \gamma^{\alpha\mu}(x)\,\delta_{\gamma^{\alpha\mu}(x)} + \xi^{\alpha}(x)\,\delta_{\xi^{\alpha}(x)} + \right.$$
$$\left. + J^{\alpha}(x)\,\delta_{J^{\alpha}(x)} + \gamma^{\alpha}(x)\,\delta_{\gamma^{\alpha}(x)}\right]$$

respectively coincide with $\int dx\; J^o(x)\,\delta_{J^o(x)}$ and $\int dx\; J^{\alpha\mu}(x)\,\delta_{J^{\alpha\mu}(x)}$ under application of $\sum_{\text{phys.}}$ and restriction at $\mathcal{J}=0$.

After what was seen before the finite part Δ^o_i of $\Delta^{o\varepsilon}_i$ is non physical and satisfies Eq. (131). From these commutation relations one can check that

(136) $$\Delta^{o,S}_1 = \int dx\left[\bar{\xi}^{\alpha}(x)\,\delta_{\bar{\xi}^{\alpha}(x)}\right] - \Delta^o_{\pi} - \Delta^o_a$$

$$\Delta^{o,S}_2 = \int dx\left[\gamma^{\alpha}(x)\,\delta_{\gamma^{\alpha}(x)} + J^{\alpha}(x)\,\delta_{J^{\alpha}(x)}\right] + \Delta^o_{\pi}$$

are symmetrical, (and obviously non physical).

In the same way one can check that

$$(137) \quad \Delta_3^{os} = \int dx \left[\xi^\alpha(x) \delta_{\xi^\alpha(x)} + \zeta^\alpha(x) \delta_{\zeta^\alpha(x)} \right]$$

is symmetrical (obviously non physical). Finally Δ_4^{os} defined by

$$(138) \quad \Delta_4^{os} Z = \int dx \, J^o(x) \, Z$$

which can be realized by performing a σ field translation, is certainly symmetrical and non physical.

These four symmetrical non physical insertions are independent as can be checked in the tree approximation (Δ_4^{os} has a σ^3 term which the others do not have ; Δ_2^{os} has a π^4 term, not contained in the others ; $\Delta_1^{o,s}$ has a $\zeta^\alpha \zeta_\alpha$ term which is not in $\Delta_3^{o,s}$).

To complete the basis of symmetrical insertions there remains to find five others which are independent on the physical normalization points. According to the general argument i.e. the implicit function theorem for formal power series [1], we know that there are four of them whose tree approximations are the four terms of $\mathcal{L}_{inv.}$ (98). A fifth one is Δ_α^s . Their independence on the normalization points, i.e. Eq.(120) can be decided at the tree level and can indeed be verified. This completes the proof.

D. Radiative Corrections : Unitarity of the Physical S Operator.

Usually, the unitarity of the S operator follows from the relationship between time ordered and antitime ordered products, together with the hermiticity of the Lagrangian. Here we are investigating the unitarity of the physical S operator so that we have to show the cancellation of ghosts in intermediate states. Furthermore the Lagrangian is not hermitian. We shall however show that the unitarity of the S operator can be derived from the usual relation between time ordered and anti-time ordered products, thanks to an additional symmetry property of the Lagrangian, and the Slavnov identity. Together with

$$(139) \quad Z(J) = \left\langle T \exp\left\{ \frac{i}{\hbar} \int dx \left[\mathcal{L}_{eff}^{int}(x) + \underline{J}(x) \underline{\Psi}(x) \right] \right\} \right\rangle$$

we introduce

$$(140) \quad \bar{Z}(J) = \langle \bar{T} \exp\left\{-\frac{i}{\hbar}\int dx \left[\mathcal{L}_{eff}^{int}(x) + J(x)\Psi(x)\right]\right\}\rangle$$

where T and $\bar{T}$ respectively denote time ordered and anti-time ordered products.

We first define the S operator in the full Fock space (including the ghosts) through the LSZ formula :

$$(141) \quad S = \Sigma \; Z(J)\big|_{J=0}$$

where $\Sigma = : \exp \int dx dy \Psi_i^{in}(x) K_{xy}^{ij} \delta_{J_j(x)} :$

which is the straightforward extension of $\mathcal{S}_{phys}$.

One has :

$$(142) \quad \Sigma \; Z(J) \cdot \Sigma \; \bar{Z}(J) = \mathbb{1}$$

and if the Lagrangian is hermitian,

$$(143) \quad \Sigma \bar{Z}(J) = \left(\Sigma Z(J)\right)^{\dagger}$$

Let now E_0 be the projector on the bare physical subspace. By application of Wick's theorem one obtains :

$$(144) \quad \Sigma_{phys} Z(J) \exp \mathcal{A} \; \Sigma_{phys} \bar{Z}(J) = E_0$$

where

$$(145) \quad \mathcal{A} = i\hbar \int dx \left[\overleftarrow{\delta}_{J_g} \; \overleftarrow{K}^g S_+^g * K^g \delta_{J_g}\right](x)$$

the index g indicating that the summation is restricted to the ghost fields, and $i\hbar S_+^g$ being the positive frequency part of the asymptotic ghost field commutators.

We are now first going to see that

$$[Z(\mathcal{J})]^\dagger = \tilde{\mathcal{C}}\,\bar{Z}(\mathcal{J}) \tag{146}$$

where $\mathcal{J}^{i\dagger}$ is the coefficient of $\psi_i^\dagger$ in $\underline{\mathcal{J}}\cdot\underline{\psi}$.

$$\begin{aligned} &\tilde{\mathcal{C}}\,\bar{\xi} = \xi \\ &\tilde{\mathcal{C}}\,\xi = -\bar{\xi} \\ &\tilde{\mathcal{C}}\,\mathcal{J} = \mathcal{J}^\dagger \quad \text{otherwise.} \end{aligned} \tag{147}$$

This is a consequence of the corresponding property at the Lagrangian level :

$$\mathcal{L}_{eff}^\dagger = \mathcal{C}\,\mathcal{L}_{eff} \tag{148}$$

where

$$\begin{aligned} &\mathcal{C}\,c = \bar{c} \\ &\mathcal{C}\,\bar{c} = -c \\ &\mathcal{C}\,\eta = \eta^\dagger \quad \text{(all classical fields)} \\ &\mathcal{C}\,\psi = \psi^\dagger \quad \text{otherwise.} \end{aligned} \tag{149}$$

This property is due to the fact that the $\mathcal{C}$ operation and the hermitian conjugation transform the Slavnov identity in the same way, leave the normalization conditions unchanged (the $\mathcal{C}$ and $\dagger$ operations are defined in a natural way on Γ) , and that the theory is uniquely defined by the normalization conditions and the Slavnov identity. (101, 102, 103, 104, 105, 96).

Thus Σ_{phys} is hermitian since the asymptotic Lagrangian is hermitian, which follows from (146) , hence (142) can be rewritten

$$\Sigma_{phys} Z(\mathcal{J})\left[\exp \mathcal{A}^{*}\right]\left[\Sigma_{phys} Z(\mathcal{J})\right]^\dagger = E_0 \tag{150}$$

where

(151) $$\mathcal{A}^c = i\hbar \int dx \left[\overleftarrow{\delta}_{J^g} \overleftarrow{K}^g S_+^g * \overrightarrow{K}^g \delta_{\tilde{J}^g} \right](x)$$

with

(152) $$\tilde{J}^g = \tilde{\mathcal{C}} J^g$$

Let us consider

(153) $$\mathcal{U}(\lambda) = \exp \lambda \mathcal{A}^{\mathcal{C}}$$

Then

(154) $$\partial_\lambda S_{phys}(J)\, \mathcal{U}(\lambda)\, S^\dagger_{phys}(J)\Big|_{J=0} = \\ = S_{phys}(J)\, \mathcal{A}^c\, \mathcal{U}(\lambda)\, S^\dagger_{phys}(J)\Big|_{J=0}$$

Introducing

(155) $$\bar{G}_\alpha = \frac{\rho \pi_\alpha - \partial^\mu a_{\alpha\mu}}{2\rho\, T_\gamma (m^2_{\Phi\pi})}$$

whose variation under a Slavnov transformation reduces on mass shell to $\mathcal{C}_\alpha$, and using the restricted 't Hooft gauge [22] defined by

(156) $$\Gamma_{a_L \pi}(m^2_{\Phi\pi}) = 0$$

in which the ghost propagators have only simple poles [23] , allows to rewrite

(157) $$\mathcal{A}^{\mathcal{C}} = i\hbar \int dx \Big[\overleftarrow{\delta}_{\bar{g}} (\overleftarrow{K} S_+ * \overrightarrow{K})_{\bar{g}g}\, \overrightarrow{\delta}_g + \\ + \overleftarrow{\delta}_g (\overleftarrow{K} S_+ * \overrightarrow{K})_{g\bar{g}}\, \overrightarrow{\delta}_{\bar{g}} + \overleftarrow{\delta}_g (\overleftarrow{K} S_+ * \overrightarrow{K})_{gg}\, \overrightarrow{\delta}_g + \\ + \overleftarrow{\delta}_{\bar{\xi}} (\overleftarrow{K} S_+ * \overrightarrow{K})_{c\bar{c}}\, \overrightarrow{\delta}_\xi + \overleftarrow{\delta}_\xi (\overleftarrow{K} S_+ * \overrightarrow{K})_{\bar{c}c}\, \delta_{\bar{\xi}} \Big](x)$$

Now making use of the Slavnov identity on the ghost mass shell and of the vanishing source restriction allows [1] to reduce (154, 157) to

$$\partial_\lambda S_{phys}(J)\,\mathcal{U}(\lambda)\,S^{+}_{phys}(J)\Big|_{J=0} =$$

(158)
$$= S_{phys}(J)\Big[i\hbar\int dx\,\Big\{\overleftarrow{\delta}_g\,(\overleftarrow{K}\,S_{+} * \overrightarrow{K})_{gg}\,\overrightarrow{\delta}_g\Big\}(x)\Big]\,\mathcal{U}(\lambda)\,S^{+}_{phys}(J)\Big|_{J=0}$$

which upon integration with respect to λ leads to :

$$S_{phys}(J)\,\mathcal{U}(1)\,S^{+}_{phys}(J)\Big|_{J=0} =$$

(159)
$$= S_{phys}(J)\,\exp\Big\{i\hbar\int dx\,\Big[\overleftarrow{\delta}_g\,(\overleftarrow{K}\,S_{+} * \overrightarrow{K})_{gg}\,\overrightarrow{\delta}_g\Big](x)\Big\}\,S^{+}_{phys}(J)\Big|_{J=0}$$

Since the expectation value between physical states of the time ordered product of an arbitrary number of gauge operators is disconnected [21]

(160)
$$S_{phys}\,S^{+}_{phys} = E_0$$

i.e.

(161)
$$E_0\,S\,E_0\,S^{+}E_0 = E_0$$

Similarly, one can prove that :

(162)
$$E_0\,S^{+}E_0\,S\,E_0 = E_0$$

which shows that the physical S operator $E_0\,S\,E_0$ obeys perturbative unitarity.

Conclusion

Gauge theories can be characterized by the fulfillment of Slavnov identities when the underlying Lie algebra is semi-simple, i.e. is sufficiently rigid against perturbations. Then, simple power counting arguments are sufficient to prove that, indeed Slavnov identities can be fulfilled, in the absence of Adler Bardeen anomalies.

In particular, we have shown, on the SU2 Higgs Kibble model whose particle interpretation can be completely analyzed that the gauge independence and unitarity of the physical S-operator follow.

It is believed that both the lack of rigidity of the underlying Lie algebra and the possible occurrence of Adler Bardeen anomalies can only be mastered by more sophisticated tools based on a closer analysis of the consistency conditions involving the behaviour of the theory under dilatations [7],[20].

The analysis of gauge independent local operators, although not touched upon here [1] should also be tractable in terms of the methods used here.

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APPENDIX A : Cohomology of Lie Algebras

This appendix is a brief summary of definitions and results needed here which are not easily found in classical text books [24].

Definition : Let $\mathfrak{h}$ be a Lie algebra with structure constants $f^{\alpha\beta}{}_{\gamma}$, which is the sum of a semi-simple algebra $\mathcal{S}$ and an abelian algebra $\mathcal{A}$. A cochain of order n with value in a representation space V on which $\mathfrak{h}$ acts through a completely reduced representation :

$$\mathfrak{h} \ni h^{\alpha} \to t^{\alpha}$$

is a totally antisymmetric tensor built on $\mathfrak{h}$ whose components

$$\Gamma^{\alpha_1 \dots \alpha_n}$$

are elements of V . The set of such cochains is called $C^n(V)$.
We define the coboundary operator d^n :

$$C^n(V) \xrightarrow{d^n} C^{n+1}(V) \tag{A1}$$

by :

$$(d^n\Gamma)^{\alpha_1 \dots \alpha_{n+1}} = \sum_{k=1}^{n+1} (-)^{k-1} t^{\alpha_k} \Gamma^{\alpha_1 \dots \hat{\alpha}_k \dots \alpha_{n+1}} + \sum_{k<l=1}^{n+1} (-)^{k+l} f^{\alpha_k \alpha_l}{}_{\lambda} \Gamma^{\lambda \alpha_1 \dots \hat{\alpha}_k \dots \hat{\alpha}_l \dots \alpha_{n+1}} \tag{A2}$$

in which capped indices are to be omitted.
The fundamental property is :

$$d^{n+1} \circ d^n = 0 \tag{A3}$$

a consequence of the commutation relations

(A4) $$\left[t^\alpha , t^\beta \right] = f^{\alpha\beta}{}_\gamma \, t^\gamma$$

and the Jacobi identities.

An element Γ of $C^n(V)$ is called a cocycle if

(A5) $$d^n \Gamma = 0$$

The set of cocycles is denoted $Z^n(V)$.

An element Γ of $C^n(V)$ is called a coboundary if

(A6) $$\Gamma = d^{n-1} \hat{\Gamma}$$

for some $\hat{\Gamma} \in C^{n-1}(V)$

Obviously every coboundary is a cocycle (cf. Eq.(A3)).

The converse is not always true.

However in the present case where the representation is fully reduced, the parametrization of all coboundaries can be found as follows :

We first split Γ into an invariant and a non invariant part :

(A7) $$\Gamma = \Gamma_{\natural} + \Gamma_{\flat}$$

such that

$$t^2 \Gamma_{\natural} = 0 \ , \ t^2 \Gamma_{\flat} \neq 0$$

Here we have defined

(A8) $$t^2 = t^\alpha t_\alpha$$

where indices are raised and lowered by means of a non degenerate invariant symmetrical tensor.

The restriction of Eq.(A5) to Γ_b yields through multiplication by t_{α_1} and summation over α_1 :

$$\text{(A9)} \qquad \Gamma_b = d^{n-1}\,\hat{\Gamma}$$

where

$$\text{(A10)} \qquad (\hat{\Gamma})^{\alpha_1\ldots\alpha_{n-1}} = \frac{t_\alpha}{t^2}\,\Gamma_b^{\alpha\alpha_1\ldots\alpha_{n-1}}$$

(Use has been made of the commutation rules between the t^{α}'s.)

Next we look at $\Gamma_{\mathfrak{h}}$. The cocycle condition reduces to :

$$\text{(A11)} \qquad \sum_{k<l=1}^{n+1} (-)^{k+l} f^{\alpha_k\alpha_l}{}_\lambda\, \Gamma_{\mathfrak{h}}^{\lambda\alpha_1\ldots\hat{\alpha}_k\ldots\hat{\alpha}_l\ldots\alpha_{n+1}} = 0$$

We define the two operations

$$\text{(A12)} \qquad {}^n I^\rho : C^n(V) \rightarrow C^{n-1}(V) : \quad ({}^n I^\rho \Gamma)^{\alpha_1\ldots\alpha_{n-1}} = \Gamma^{\rho\alpha_1\ldots\alpha_{n-1}}$$

and

$$\text{(A13)} \qquad {}^n\theta^\rho : C^n(V) \rightarrow C^n(V) \quad ({}^n\theta^\rho\Gamma)^{\alpha_1\ldots\alpha_n} = t^\rho\Gamma^{\alpha_1\ldots\alpha_n} + \sum_{k=1}^{n}(-)^{k-1} f^{\rho\alpha_k}{}_\lambda\,\Gamma^{\lambda\alpha_1\ldots\hat{\alpha}_k\ldots\alpha_n}$$

(θ^ρ transforms Γ according to the sum of n adjoint representations of $\mathfrak{h}$ and t^ρ).

One can check that

$$ {}^{n+1}I^{\rho} \circ d^{n} + d^{n-1} \circ {}^{n}I^{\rho} = {}^{n}\theta^{\rho} \tag{A14} $$

and furthermore

$$ d^{n} \circ {}^{n}\theta^{\rho} = {}^{n+1}\theta^{\rho} \circ d^{n} \tag{A15} $$

Thus if $\quad d^{n}\,\Gamma_{\mathfrak{h}} = 0$

$$ {}^{n}\theta^{\rho}\,\Gamma_{\mathfrak{h}} = d^{n-1} \circ {}^{n}I^{\rho}\,\Gamma_{\mathfrak{h}} \tag{A16} $$

Hence

$$ ({}^{n}\theta)^{2}\,\Gamma_{\mathfrak{h}} = d^{n-1} \circ {}^{n-1}\theta_{\rho} \circ {}^{n}I^{\rho}\,\Gamma_{\mathfrak{h}} \tag{A17} $$

Reducing the antisymmetrized product of n adjoint representations according to

$$ \Gamma_{\mathfrak{h}} = \Gamma_{\mathfrak{h}}^{\mathfrak{h}} + \Gamma_{\mathfrak{h}}^{b} \tag{A18} $$

with

$$ ({}^{n}\theta)^{2}\,\Gamma_{\mathfrak{h}}^{\mathfrak{h}} = 0 \;, \quad ({}^{n}\theta)^{2}\,\Gamma_{\mathfrak{h}}^{b} \neq 0 $$

we find that

$$ \Gamma_{\mathfrak{h}}^{b} = \frac{1}{({}^{n-1}\theta)^{2}}\, d^{n-1} \circ {}^{n-1}\theta_{\rho} \circ {}^{n}I^{\rho}\;\Gamma_{\mathfrak{h}} \tag{A19} $$

and $\Gamma_{\mathfrak{h}}^{\mathfrak{h}}$ is arbitrary.

To sum up, in the case of a completely reduced representation and a Lie algebra which is the sum of a semi-simple and an abelian algebra, every cocycle is a coboundary up to totally invariant cochains. When $\mathfrak{h}$ is semi-simple however there is no such cochain for $n = 1, 2$. (In this last case, the invariance which implies the cocycle condition and the non degeneracy of the Killing form yield the result).

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APPENDIX B

We want to show here that the equation :

$$s^{\Delta}\left(\gamma^i(x)\,\overset{\bullet}{P}_i(x)+\xi^{\alpha}(x)\,\overset{\bullet}{P}_{\alpha}(x)\right)+s\left(\gamma^i(x)\,\Delta_i(x)+\xi^{\alpha}(x)\,\Delta_{\alpha}(x)\right)\equiv$$

(B1)
$$\equiv \gamma^i(x)\Big[\left(\Delta_j(y)\,\delta_{\phi_j(y)}-\Delta_{\alpha}(y)\,\delta_{\bar{c}_{\alpha}(y)}\right)\overset{\circ}{P}_i(x)-$$
$$-\left(\overset{\bullet}{P}_j(y)\,\delta_{\phi_j(y)}+\overset{\circ}{P}_{\alpha}(y)\,\delta_{\bar{c}_{\alpha}(y)}\right)\Delta_i(x)\Big]+$$
$$+\xi^{\alpha}(x)\Big[-\Delta_{\beta}(y)\,\delta_{\bar{c}_{\beta}(y)}\,\overset{\bullet}{P}_{\alpha}(x)+\overset{\circ}{P}_{\beta}(y)\,\delta_{\bar{c}_{\beta}(y)}\,\Delta_{\alpha}(x)\Big]=0$$

implies the structure :

$$\gamma^i(x)\,\Delta_i(x)+\xi^{\alpha}(x)\,\Delta_{\alpha}(x)=-\Big[s^{\pi}\left(\gamma^i(x)\overset{\circ}{P}_i(x)+\xi^{\alpha}(x)\,\overset{\circ}{P}_{\alpha}(x)\right)+$$
$$+s\left(\gamma^i(x)\,\Pi_i(x)+\xi^{\alpha}(x)\,\Pi_{\alpha}(x)\right)\Big]\equiv$$

(B2)
$$\equiv\gamma^i(x)\Big[\left(\Pi_j(y)\,\delta_{\phi_j(y)}+\Pi_{\alpha}(y)\,\delta_{\bar{c}_{\alpha}(y)}\right)\overset{\circ}{P}_i(x)+\left(\overset{\bullet}{P}_j(y)\,\delta_{\phi_j(y)}+\overset{\circ}{P}_{\alpha}(y)\,\delta_{\bar{c}_{\alpha}(y)}\right)\Pi_i(x)\Big]$$
$$-\xi^{\alpha}(x)\left(\Pi_{\beta}(y)\,\delta_{\bar{c}_{\beta}(y)}\,\overset{\bullet}{P}_{\alpha}(x)+\overset{\bullet}{P}_{\beta}(y)\,\delta_{\bar{c}_{\beta}(y)}\,\Pi_{\alpha}(x)\right)$$

From this result it follows that Eq.(59) which is a perturbation of order $\hbar\Delta$ of Eq.(B2) is a consequence of the consistency condition Eq.(58) which is a perturbation of the same order of Eq.(B1).

Let us recall the notations

(B3)
$$\overset{\circ}{P}_i(x)=t^{\alpha}_{ij}\left[\Phi_j\,\bar{c}_{\alpha}\right](x)+q^{\alpha}_i\,\bar{c}_{\alpha}(x)$$

where :

$$q^{\alpha}_a=t^{\alpha}_{ab}\,F_b$$

(B4)
$$q^{\alpha}_{\beta,\mu}=q^{\alpha}_{\beta}\,\partial_{\mu}$$

Since q^{α}_{β} is an invariant tensor it can always be chosen of the form :

(B5) $$q^{\alpha}_{\beta} = q\,\delta^{\alpha}_{\beta}$$

Also :

(B6) $$\mathring{P}_{\alpha}(x) = \frac{1}{2}\, f_{\alpha}{}^{\beta\gamma}(\bar{c}_{\beta}\,\bar{c}_{\gamma})(x)$$

Δ_i , Δ_α , Π_i , Π_α are given by :

$$\Delta_i(x) = \frac{1}{2}\left[\Theta^{\alpha\beta}_{ij}\,\Phi_j\,\bar{c}_\alpha\,\bar{c}_\beta + \bar{c}_\alpha\,\Sigma^{\alpha\beta}_i\,\bar{c}_\beta\right](x)$$

$$\Delta_\alpha(x) = -\frac{1}{6}\,\Gamma_\alpha{}^{\beta\gamma\delta}(\bar{c}_\beta\,\bar{c}_\gamma\,\bar{c}_\delta)(x)$$

(B7) $$\Pi_i(x) = \left[\hat{\Theta}^{\alpha}_{ij}\,\Phi_j\,\bar{c}_\alpha + \hat{\Sigma}^{\alpha}_i\,\bar{c}_\alpha\right](x)$$

$$\Pi_\alpha(x) = \frac{1}{2}\,\hat{\Gamma}_\alpha{}^{\beta\gamma}(\bar{c}_\beta\,\bar{c}_\gamma)(x)$$

with

$$\Sigma^{\alpha\beta}{}_{\gamma,\mu} = \Sigma^{\alpha,\beta}{}_{\gamma}\,\partial_\mu$$

(B8)

$$\Sigma^{\alpha}{}_{\gamma,\mu} = \Sigma^{\alpha}_{\gamma}\,\partial_\mu$$

Following these definitions, we shall reduce (B1) and (B2) to c numbers. First, substituting Eqs.(B3), (B6), and Eq.(B7) into Eq.(B1) yields :

$$\frac{1}{2}\gamma^{i}(x)\left[t^{\alpha}_{ik}\,\Theta^{\beta\gamma}_{kj} - \Theta^{\alpha\beta}_{ik}\,t^{\gamma}_{kj} + \frac{1}{3}\,\Gamma_\lambda{}^{\alpha\beta\gamma}t^{\lambda}_{ij} - f_\lambda{}^{\alpha\beta}\Theta^{\lambda\gamma}_{ij}\right](\Phi_j\,\bar{c}_\alpha\,\bar{c}_\beta\,\bar{c}_\gamma)(x)$$
$$+\,\frac{1}{2}\gamma^{a}(x)\left[t^{\alpha}_{ab}\,\Sigma^{\beta\gamma}_{b} - \Theta^{\alpha\beta}_{ab}\,q^{\gamma}_{b} + \frac{1}{3}\,\Gamma_\lambda{}^{\alpha\beta\gamma}q^{\lambda}_{a} - f_\lambda{}^{\alpha\beta}\Sigma^{\lambda\gamma}_{a}\right](\bar{c}_\alpha\,\bar{c}_\beta\,\bar{c}_\gamma)(x)$$

(B9) $$+\;\cdots\cdots$$

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$$+\frac{1}{2}(\gamma^{\nu\mu}\bar{c}_\alpha\bar{c}_\beta)(x)\Big[t^{\alpha}_{\eta\xi}\Sigma^{\beta,\gamma}_{\xi}-\Theta^{\alpha\beta}_{\eta\xi}q^{\gamma}_{\xi}+\Gamma_\lambda^{\alpha\beta\gamma}q^{\lambda}_{\eta}-$$

(B9)

$$-\frac{1}{2}f_\lambda^{\alpha\beta}\Sigma^{\lambda,\gamma}_{\eta}+\Sigma^{\lambda,\gamma}_{\eta}f_\lambda^{\beta\gamma}\Big]\partial_\mu\bar{c}_\gamma(x)$$

$$+\zeta^{\eta}(x)\Big[-\frac{1}{6}\Gamma_\lambda^{\alpha\beta\gamma}f_\eta^{\lambda\delta}+\frac{1}{4}f_\lambda^{\alpha\beta}\Gamma_\eta^{\lambda\gamma\delta}\Big](\bar{c}_\alpha\bar{c}_\beta\bar{c}_\gamma\bar{c}_\delta)(x)=0$$

which in terms of the coefficients writes :

(B10) $$-\frac{1}{24}\Big[\sum_{i=1}^{4}(-)^{i+1}f_\eta^{\alpha_i\lambda}\Gamma_\lambda^{\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_4}+\sum_{l<k=1}^{4}(-)^{l+k}f_\lambda^{\alpha_l\alpha_k}\Gamma_\eta^{\lambda\alpha_1\ldots\hat{\alpha}_l\ldots\hat{\alpha}_k\ldots\alpha_4}\Big]=0$$

(B11) $$\frac{1}{6}\Big[\sum_{i=1}^{3}(-)^{i+1}\big[t^{\alpha_i},\Theta^{\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_3}\big]_{ij}+\sum_{l<k=1}^{3}(-)^{l+k}f_\lambda^{\alpha_l\alpha_k}\Theta_{ij}^{\alpha_1\ldots\hat{\alpha}_l\ldots\hat{\alpha}_k\ldots\alpha_3}+$$

$$+\Gamma_\lambda^{\alpha_1\alpha_2\alpha_3}t^{\lambda}_{ij}\Big]=0$$

(B12) $$\frac{1}{6}\Big[\sum_{i=1}^{3}(-)^{i+1}t^{\alpha_i}_{ab}\Sigma_b^{\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_3}+\sum_{l<k=1}^{3}(-)^{l+k}f_\lambda^{\alpha_l\alpha_k}\Sigma_a^{\lambda\alpha_1\ldots\hat{\alpha}_l\ldots\hat{\alpha}_k\ldots\alpha_3}-$$

$$-\sum_{i=1}^{3}(-)^{i+1}\Theta_{ab}^{\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_3}t^{\alpha_i}_{bc}F_c+\Gamma_\lambda^{\alpha_1\alpha_2\alpha_3}t^{\lambda}_{ab}F_b\Big]=0$$

where, for convenience, we have replaced $\alpha,\beta,\gamma,\delta$ of Eq.(B9) by α_1, α_2, α_3, α_4.

Finally :

$$\frac{1}{4}\Big[(\tau^{\alpha}\Sigma^{\beta})^{\gamma}_{\eta}-(\tau^{\beta}\Sigma^{\alpha})^{\gamma}_{\eta}-f_\lambda^{\alpha\beta}\Sigma^{\lambda,\gamma}_{\eta}-$$

(B13)

$$-2\,\Theta^{\alpha\beta}_{\eta\xi}q^{\gamma}_{\xi}+2\,\Gamma_\lambda^{\alpha\beta\gamma}q^{\lambda}_{\eta}\Big]=0$$

where τ^{α} is defined by :

$$(B14)\quad \left(\tau^{\alpha}\Sigma^{\beta}\right)_{\eta}^{\gamma} = t^{\alpha}_{\eta\xi}\Sigma^{\beta,\gamma}_{\xi} + f_{\lambda}{}^{\gamma\alpha}\Sigma^{\beta,\lambda}_{\eta}$$

In Eqs.(B10), (B11), (B12) we have followed the conventions introduced in Appendix A . Thus, taking into account the definition of the cocycle condition given in Appendix A and putting :

$$(B15)\quad \begin{aligned} \left(t^{\rho}\Gamma^{\alpha\beta\gamma}\right)_{\lambda} &= f_{\lambda}{}^{\rho\xi}\Gamma_{\xi}{}^{\alpha\beta\gamma} \\ \left(t^{\rho}\Theta^{\alpha\beta}\right)_{ij} &= \left[t^{\rho},\Theta^{\alpha\beta}\right]_{ij} \\ \left(t^{\rho}\Sigma^{\alpha\beta}\right)_{a} &= t^{\rho}_{ab}\Sigma^{\alpha\beta}_{b} \\ \left(t^{\rho}\Sigma^{\alpha}\right)_{\eta}^{\gamma} &= \left(\tau^{\rho}\Sigma^{\alpha}\right)_{\eta}^{\gamma} \end{aligned}$$

we see that Eq.(B10), (B11), (B12) can be written as :

$$(B16)\quad \left(d^{3}\Gamma\right)_{\lambda}^{\alpha_1\ldots\alpha_4} = 0$$

$$(B17)\quad \left(d^{2}\Theta\right)_{ij}^{\alpha_1\alpha_2\alpha_3} + \Gamma_{\lambda}{}^{\alpha_1\alpha_2\alpha_3}t^{\lambda}_{ij} = 0$$

$$(B18)\quad \left(d^{2}\Sigma\right)_{a}^{\alpha_1\alpha_2\alpha_3} - \sum_{i=1}^{3}(-)^{i+1}\Theta_{ab}^{\alpha_1..\hat{\alpha}_i..\alpha_3}t^{\alpha_i}_{bc}F_c + \Gamma_{\lambda}{}^{\alpha_1\alpha_2\alpha_3}t^{\lambda}_{ab}F_b = 0$$

Comparing the coboundary operators which operate on $\Theta^{\alpha\beta}_{ab}$ and $\Sigma^{\alpha\beta}_{a}$ we see that :

$$(B19)\quad \left(d^{2}\Theta\right)_{ab}^{\alpha_1\alpha_2\alpha_3}F_b = \left[d^{2}(\Theta F)\right]^{\alpha_1\alpha_2\alpha_3} - \sum_{i=1}^{3}(-)^{i+1}\Theta_{ab}^{\alpha_1..\hat{\alpha}_i..\alpha_3}t^{\alpha_i}_{bc}F_c$$

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Hence, taking into account Eq.(B17), Eq.(B18) assumes the form :

(B20) $$\left[d^2\left(\Sigma - \Theta F\right)\right]_a^{\alpha_1\alpha_2\alpha_3} = 0$$

Finally Eq.(B13) writes

(B21) $$\left(d^1\Sigma\right)_\eta^{\alpha\beta,\gamma} - 2q\left(\Theta_\eta^{\alpha\beta,\gamma} - \Gamma_\eta^{\alpha\beta\gamma}\right) = 0$$

In much the same way as for Eq.(B1) expressing Eq.(B2) in terms of its coefficients yields :

(B22) $$\Gamma_\eta^{\alpha_1\alpha_2\alpha_3} = -\left(d^2\hat{\Gamma}\right)_\eta^{\alpha_1\alpha_2\alpha_3}$$

(B23) $$\Theta_{ij}^{\alpha_1\alpha_2} = -\left[\left(d^1\hat{\Theta}\right)_{ij}^{\alpha_1\alpha_2} - \hat{\Gamma}_\lambda^{\alpha_1\alpha_2} t_{ij}^{\lambda}\right]$$

(B24) $$\begin{aligned}\Sigma_a^{\alpha_1\alpha_2} &= -\left[\left(d^1\{\hat{\Sigma} - \hat{\Theta}F\}\right)_a^{\alpha_1\alpha_2} + \left(d^1\hat{\Theta}\right)_{ab}^{\alpha_1\alpha_2} F_b - \hat{\Gamma}_\lambda^{\alpha_1\alpha_2} t_{ab}^{\lambda} F_b\right] \\ &= -\left[\left(d^1\{\hat{\Sigma} - \hat{\Theta}F\}\right)_a^{\alpha_1\alpha_2} - \Theta_{ab}^{\alpha_1\alpha_2} F_b\right]\end{aligned}$$

(B25) $$\Sigma_\eta^{\alpha,\beta} = -2\left[\left(\tau^\alpha\hat{\Sigma}\right)_\eta^{\beta} + q\left(\hat{\Theta}_\eta^{\alpha,\beta} - \hat{\Gamma}_\eta^{\alpha\beta}\right)\right]$$

We are now going to show that Eqs.(B22), (B23), (B24), (B25) are consequences of the results of Appendix A , and that consequently $\hat{\Gamma}, \hat{\Theta}, \hat{\Sigma}$ can be chosen linear in Γ, Θ, Σ .

First, Eq.(B22) is indeed a consequence of Eq.(B16) since the cochain Γ takes values in the adjoint representation of $\mathfrak{h}$ which is semi-simple so that $\Gamma_{\mathfrak{h}}$. Then, owing to the relation :

$$\text{(B26)}\quad \left[d^2\left(\hat{\Gamma}_\lambda\, t^\lambda\right)\right]_{ij}^{\alpha_1\alpha_2\alpha_3} \equiv \left(d^2\hat{\Gamma}\right)_\lambda^{\alpha_1\alpha_2\alpha_3} t_{ij}^\lambda = -\Gamma_\lambda^{\alpha_1\alpha_2\alpha_3} t_{ij}^\lambda$$

Eq.(B17) becomes :

$$\text{(B27)}\quad \left[d^2\left(\Theta - \hat{\Gamma}_\lambda\, t^\lambda\right)\right]_{ij}^{\alpha_1\alpha_2\alpha_3} = 0$$

Since no antisymmetric invariant tensor of rank two on $\mathfrak{h}$ exists if $\mathfrak{h}$ is semi-simple we see that Eq.(B27) and Eq.(B20) imply Eq .(B23) and Eq.(B24) respectively.

Finally substituting Eq.(B23) into Eq.(B21) we get :

$$\text{(B28)}\quad \left(d^1\Sigma\right)_\eta^{\alpha\beta,\gamma} + 2q\left(d^1\hat{\Theta}\right)_\eta^{\alpha\beta,\gamma} + 2q\left(\Gamma_\eta^{\alpha\beta\gamma} - \hat{\Gamma}_\lambda^{\alpha\beta} f_\eta^{\lambda\gamma}\right) = 0$$

Now considering $\hat{\Gamma}_\eta^{\alpha\gamma}$ as a cochain of order one with values in the tensor product of the adjoint representation with itself and applying the coboundary operator d we get :

$$\text{(B29)}\quad \left(d^1\hat{\Gamma}\right)_\eta^{\alpha\beta,\gamma} = f_\eta^{\alpha\lambda}\hat{\Gamma}_\lambda^{\beta\gamma} - f_\eta^{\beta\lambda}\hat{\Gamma}_\lambda^{\alpha\gamma} + f_\lambda^{\gamma\alpha}\hat{\Gamma}_\eta^{\beta\lambda} - f_\lambda^{\gamma\beta}\hat{\Gamma}_\eta^{\alpha\lambda} - f_\lambda^{\alpha\beta}\hat{\Gamma}_\eta^{\lambda\gamma}$$

Comparing with Eq.(B22) :

$$\text{(B30)}\quad -\Gamma_\eta^{\alpha\beta\gamma} = f_\eta^{\alpha\lambda}\hat{\Gamma}_\lambda^{\beta\gamma} + f_\eta^{\beta\lambda}\hat{\Gamma}_\lambda^{\gamma\alpha} + f_\eta^{\gamma\lambda}\hat{\Gamma}_\lambda^{\alpha\beta} - f_\lambda^{\alpha\beta}\hat{\Gamma}_\eta^{\lambda\gamma} - f_\lambda^{\beta\gamma}\hat{\Gamma}_\eta^{\lambda\alpha} - f_\lambda^{\gamma\alpha}\hat{\Gamma}_\eta^{\beta\gamma}$$

we see that Eq.(B28) has the form :

$$\text{(B31)}\quad \left[d^1\left(\Sigma + 2q\left\{\hat{\Theta} - \hat{\Gamma}\right\}\right)\right]_\eta^{\alpha\beta,\gamma} = 0$$

Since no invariant tensor of rank one on $\mathfrak{h}$ exists if $\mathfrak{h}$ is semi-simple, Eq.(B31) can be solved according to

$$\text{(B32)} \quad \Sigma_{\eta}^{\alpha,\beta} = -2\left(\tau^{\alpha}\hat{\Sigma}\right)_{\eta}^{\beta} - 2g\left(\hat{\Theta}_{\eta}^{\alpha,\beta} - \hat{\Gamma}_{\eta}^{\alpha\beta}\right)$$

The possibility of choosing $\hat{\Gamma}$, $\hat{\Sigma}$, $\hat{\Theta}$ linear in Γ, Θ, Σ stems from the explicit construction given in Appendix A .

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