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# On Finitary Coding of Topological Markov Chains

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## 1. Introduction

To a non-negative integer matrix $A$ over a finite symbol space $\Sigma$ is associated the topological Markov chain $(X_A, S_A)$. Here $X_A$ is the shift space

$$X_A = \{(\alpha_i, k_i)_{i\in\mathbb{Z}} \in (\Sigma \times \mathbb{N})^{\mathbb{Z}} : 1 \le k_i \le A(\alpha_i, \alpha_{i+1}),\ i \in \mathbb{Z}\}$$

and $S_A$ is the shift on $X_A$,

$$S_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \qquad (x_i)_{i\in\mathbb{Z}} \in X_A .$$

The topological entropy of $(X_A, S_A)$ equals the logarithm of the maximal real eigenvalue $\lambda_A$ of $A$. We consider only irreducible $A$ and put then on $X_A$ the unique $S_A$-invariant probability measure $\mu_{S_A}$ of maximal entropy.

Given two topological Markov chains $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$ we consider codes $\varphi$ from $(X_A, S_A)$ to $(X_{\bar{A}}, S_{\bar{A}})$. These are Borel mappings $\varphi$ whose domain of definition $D_\varphi$ is an $S_A$-invariant Borel subset of $X_A$ of full $\mu_{S_A}$-measure, and that take values in $X_{\bar{A}}$ such that

$$\varphi S_A x = S_{\bar{A}} \varphi x, \qquad x \in X_A .$$

The entropy governs the existence of codes between topological Markov chains that have desirable properties to a considerable extent, as was recently demonstrated by Adler and Marcus [1]. For irreducible topological Markov chains $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$ of equal entropy and equal period they constructed an a.e. one-to-one finitary code $\varphi$ from $(X_A, S_A)$ to $(X_{\bar{A}}, S_{\bar{A}})$ with bounded anticipation and finite expected coding time. That $\varphi$ is finitary with bounded anticipation and finite expected coding time means that one has an $I \in \mathbb{N}$ such that there is for all $x \in D_\varphi$ a (maximal) $i(x) \le 0$ such that the zero-th coordinate of $\varphi x$ is determined by $(x_j)_{i(x)\le j\le I}$, and such that

$$\int_{X_A} i(x)\, d\mu_{S_A}(x) > -\infty .$$

A subclass of the a.e. finite-to-one finitary codes with bounded anticipation and finite expected coding time are the continuous onto codes between irreducible topological Markov chains of equal entropy [3]. The search for invariants of topological Markov chains under continuous one-to-one codes leads to the notion of shift-equivalence [10]. Two matrices $A$ and $\bar{A}$, are called shift equivalent if for some $L \in \mathbb{N}$ there are non-negative integer matrices $Q$ and $R$ such that

$$\bar{A}Q = QA \ , \ AR = R\bar{A} \ , \qquad RQ = A^L \ , \ QR = \bar{A}^L \ .$$

Williams showed that topologically conjugate topological Markov chains $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$ come from shift-equivalent matrices $A$ and $\bar{A}$ and he conjectured the converse to hold [10]. A complete invariant of shift-equivalence is known, namely the (past) dimension group $(K_o, K_o^+)(S_A)$ of the topological Markov chain together with the automorphism that the chain induces on its (past) dimension group. $(K_o, K_o^+)(S_A)$ equals

$$\underset{A}{\underrightarrow{\lim}} \ (\mathbb{Z}^{\Sigma}, \mathbb{Z}_+^{\Sigma})$$

and the alluded to induced automorphism is the one induced by $A$ on this inductive limit [7]. Also for the existence of continuous onto codes necessary conditions are known [4,6,8,9].

Call a finitary code $\varphi$ with bounded anticipation resolvent, if all points $x$ to which a $\varphi x$ can be assigned by continuity, except those in a $\mu_{S_A}$-nullset of the remote past, are uniquely determined by $\varphi x$ together with any of their initial segments $(x_j)_{j \leq i}$, $i \in \mathbb{Z}$. The left resolvent continuous onto codes as considered in [1] are examples of such codes. Our result is that between two irreducible and aperiodic topological Markov chains, whose dimension groups are totally ordered, a.e. finite-to-one resolvent finitary codes with bounded anticipation and finite expected coding time exist if and only if the entropy of the chains are equal and their dimension groups are isomorphic, which, because of the presence of the total order, is here the same as shift equivalence. We shall also see that in the totally ordered situation shift equivalence is a necessary and sufficient condition for the existence of a joint left resolving continuous extension.

In section 2 we are concerned with the necessity of the condition. Finitary codes with the required properties are then constructed in section 4 after some preparations in section 3.

## 2. Necessary Conditions

We continue to consider an irreducible and aperiodic transition matrix $A$ over a finite symbol space $\Sigma$. We introduce more notation, where we view $A$ as a 0-1 matrix, as we shall do in the sequel with all transition matrices without loss of generality. We denote

$$\mathcal{A}_A[i,k] = \{(a_j)_{i\leq j\leq k} \in \Sigma^{[i,k]}: A(a_j,a_{j+1}) = 1, \quad i \leq j < k\}, \quad i,k \in \mathbb{Z}\, , \ i < k.$$

For cylinder sets we use notation like

$$Z(a) = \{(x_j)_{j\in\mathbb{Z}} \in X_A: a = (x_j)_{i\leq j\leq k}\}, \quad a \in \mathcal{A}_A[i,k], \quad k \in \mathbb{N} \ .$$

We also use notation for the spaces of half-infinite A-admissible sequences, e.g.

$$X_A(-\infty,i] = \{(x_j)_{j\leq i} \in \Sigma^{(-\infty,i]}: A(x_j,x_{j+1}) = 1, \ j < i\}, \quad i \in \mathbb{Z}\, .$$

The projection of $X_A(-\infty,k]$ onto $X_A(-\infty,i]$, $i < k$ will be denoted by $p_i$.

$\mathcal{F}_{S_A}$ will denote the group of uniformly finite dimensional homeomorphisms of $(X_A,S_A)$ [7]. That is, $\mathcal{F}_{S_A}$ is the group of all homeomorphisms $U$ of $X_A$ with the property that there is an $I \in \mathbb{N}$ such that

$$(Ux)_i = x_i \ , \qquad |i| > I, \ \ x \in X_A \ .$$

We denote

$$W_{S_A}(x) = \bigcup_{i\in\mathbb{Z}} \{y \in X_A: y_j = x_j, \ j \leq i\}, \quad x \in X_A \ .$$

On the $W_{S_A}(x)$ we use an inductive limit topology that turns them into zero-dimensional $\sigma$-compacta. $\mathcal{C}_{S_A}(x)$ will denote the Boolean ring of compact open subsets of $W_{S_A}(x)$, and $\mathcal{F}_{S_A}(x)$ will denote the group of uniformly finite dimensional homeomorphisms of $W_{S_A}(x)$, that is, $\mathcal{F}_{S_A}(x)$ is the group of all homeomorphisms $U$ of $W_{S_A}(x)$ with the property that there is an $I \in \mathbb{Z}$ such that

$$(Uy)_i = y_i \ , \qquad i \geq I \ , \qquad y \in W_{S_A}(x) \ .$$

$x, x' \in X_A$ are said to be negatively asymptotic if $x' \in W_{S_A}(x)$ , and negatively separated if they are not negatively asymptotic. Use analog terminology also for half-infinite sequences.

By $\tau_{S_A,x}$ we denote the $F_{S_A}(x)$-invariant Borel measure on $W_{S_A}(x)$. Let us at this point also recall the explicit expressions for $\mu_{S_A}$ and $\tau_{S_A,x}$. For this let $a$ be a right eigenvector of $A$ and let $b$ be a left eigenvector of $A$ for $\lambda_A$ and set

$$\mu_{S_A}(Z(\alpha_i)_{0\leq i\leq k}) = \lambda_A^{-k}\Big(\sum_{\alpha\in\Sigma} a_\alpha b_\alpha\Big)^{-1} a_{\alpha_0} b_{\alpha_k}, (\alpha_i)_{0\leq i\leq k} \in A_A[0,k], \ k \in \mathbb{N} \ ,$$

and for an $(\alpha_i)_{i\leq n} \in X_A(-\infty,k]$ that is negatively asymptotic to $x$, have

$$\tau_{S_A,x}(Z(\alpha_i)_{i\leq k}) = \lambda_A^{-k}\, b_{\alpha_k} , \ k \in \mathbb{N} \ .$$

Finally let $q \in \mathbb{N}$ be such that all entries in $A^q$ are positive.

Let now $\bar{A}$ be another irreducible and aperiodic transition matrix. We use similar notation for the objects associated to $\bar{A}$, where we always let a bar appear.

We consider a finitary code $\varphi$ from $(X_A, S_A)$ to $(X_{\bar{A}}, S_{\bar{A}})$. We let $\varphi$ be given by its zero-coordinate mapping $\varphi_0$ whose domain of definition $D_{\varphi_0} \subset X$ , as well as the sets

$$D(\bar{\alpha}) = \{x \in D_{\varphi_0} : \varphi(x) = \bar{\alpha}\}, \qquad \bar{\alpha} \in \bar{\Sigma} \ ,$$

we assume to be open. We want $D_{\varphi_0}$ to be a set of full $\mu_{S_A}$-measure. The domain of definition $D_\varphi$ of $\varphi$ is given by

$$D_\varphi = \bigcap_{i\in\mathbb{Z}} S_A^i D_{\varphi_0} \ ,$$

the code $\varphi$ itself being related to its zero-coordinate mapping $\varphi_0$ by

$$\varphi(x) = (\varphi_0(S^i x))_{i\in\mathbb{Z}} \in X_{\bar{A}} \qquad (x \in D_\varphi) \ .$$

We impose on $\varphi$ the condition that it be finite-to-one. Then

$$\lambda_A = \lambda_{\bar{A}} , \quad \mu_{S_A} = \varphi^{-1} \mu_{S_{\bar{A}}} ,$$

as follows from the theorem of Abramov and Rohlin [1] on the entropy of homomorphic images and the uniqueness of the measure of maximal entropy. We also want $\varphi$ to have bounded anticipation. In fact, since this does not entail any loss of generality, we shall always assume tacitly that $\varphi$ is non-anticipating. Then every one of the open sets $D(\bar{\alpha})$, $\bar{\alpha} \in \bar{\Sigma}$, can be written as a union of cylinder sets of the form $Z(a)$, $a \in A_A [i,0]$, and we define

$$t(x) = \max \left\{ i \leq 0 : Z\left((x_j)_{i \leq j \leq 0}\right) \subset D(\bar{\alpha}) \right. , \qquad x \in D(\bar{\alpha}), \ \bar{\alpha} \in \bar{\Sigma} .$$

The condition of finite expected coding time, that we require $\varphi$ also to satisfy, is now expressed by

$$(1) \qquad \int_{X_A} t(x) \, d\mu_{S_A}(x) > -\infty .$$

We denote by $B^{(1)}$ the set of all $x \in D_\varphi$ such that $\varphi$ when restricted to $D_\varphi \cap W_{S_A}(x)$ becomes one-to-one. Last not least we impose on $\varphi$ the condition that it be resolvent, that is we want that $B^{(1)}$ is a set of full measure.

Let $N$ be the positive integer where

$$\mu_{S_{\bar{A}}}\left(\{\bar{x} \in \varphi(D_\varphi) : |\varphi^{-1}\{\bar{x}\}| = N\}\right) = 1 .$$

We denote by $C^{(1)}$ the set of points $\bar{x}$ in $\varphi(D_\varphi)$ such that

$$|\varphi^{-1}\{\bar{x}\}| = N$$

and such that any two elements in $\varphi^{-1}\{\bar{x}\}$ are negatively separated. $C^{(1)}$ is a set of full $\mu_{S_{\bar{A}}}$-measure. We denote further

$$B^{(2)} = \{x \in D : W_{S_A}(x) \subset D_\varphi\} .$$

(2.1) Lemma. $\mu_{S_A}(B^{(2)}) = 1$ .

Proof. We claim that

$$B^{(2)} = \bigcap_{U \in F_{S_A}} U\, D_\varphi \ .$$

To see this let

$$x \in \bigcap_{U \in F_{S_A}} U\, D_\varphi \ , \qquad y \in W_{S_A}(x) \ .$$

If now $y \notin D_\varphi$ , then by irreducibility and aperiodicity of $A$ one could construct a $y' \notin D_\varphi$ that would differ from $x$ in finitely many coordinates, contradicting the choice of $x$ . Q.e.d.

Denote

$$t(k)(x) = k + t(S^k x), \qquad x \in D_\varphi,\ k \in \mathbb{Z} \ ,$$

and

$$B = \{x \in D_\varphi \colon \lim_{k \to \infty} t(k)(x) = \infty\} \ .$$

(2.2) Lemma. $\mu_{S_A}(B) = 1$ .

Proof. From (1) we have

$$\lim_{k \to \infty} \frac{1}{k}\, t(S^k x) = 0, \qquad \text{f. } \mu_{S_A} - \text{a.a. } x \in D_\varphi \ . \qquad \text{Q.e.d.}$$

That $B$ is a set of full $\mu_{S_A}$-measure can be assured by imposing instead at (1) other conditions on $\varphi$ , e.g. by requiring that

$$\inf_{i \in \mathbb{Z}} t(i)(x) > -\infty \ , \qquad \text{f. } \mu_{S_A} - \text{a.a. } x \in D_\varphi \ .$$

For

$$(\bar{x}_j)_{j \le i} \in X_{\bar{A}}(-\infty, i], \qquad i \in \mathbb{Z} \ ,$$

we denote

$$\varphi^{-1}\left((\bar{x}_j)_{j\leq i}\right) = p_i\left\{x \in D_\varphi : p_i(\varphi x) = (\bar{x}_j)_{j\leq i}\right\} \quad .$$

(2.3) Lemma. Let $\bar{x} \in X_{\bar{A}}$ . Then for every $i \in \mathbb{Z}$ there is a $k \in \mathbb{N}$ such that

$$p_i\varphi^{-1}\{\bar{x}\} = p_i \ \varphi^{-1}\left\{(\bar{x}_j)_{j\leq i+k}\right\} \quad .$$

Proof. We prove first that the set

$$p_i \ \varphi^{-1}\left((\bar{x}_j)_{j\leq i}\right)$$

is finite. For this let

$$(u_j)_{j\geq i+q} \in X_A[i+q,\infty)$$

be such that for some $\ell \in \mathbb{N}$ the expressions

$$t(k)\left((u_j)_{j\geq i+q}\right) , \qquad k \geq i+\ell$$

are meaningful. By irreducibility and aperiodicity of $A$ construct for every

$$(y_j)_{j\leq i} \in p_i \ \varphi^{-1}\left((\bar{x}_j)_{j\leq i}\right)$$

a $\tilde{y} \in X_A$ such that

$$\tilde{y}_j = y_j , \qquad j \leq i$$

$$\tilde{y}_j = u_j , \qquad j \geq i+q$$

and set $\bar{y} = \varphi(\tilde{y})$. One sees that everyone of these $\bar{\tilde{y}}$ can differ from $\bar{x}$ only in the coordinates $\bar{y}_j$, $i \leq j \leq i+\ell$ . Since $\varphi$ is finite-to-one we can conclude that

$$p_i \ \varphi^{-1}\left((\bar{x}_j)_{j\leq i}\right)$$

is a finite set.

Consider then the decreasing family of sets

$$p_i\ \varphi^{-1}\left((\bar{x}_j)_{j\leq i+k}\right), \qquad k\in\mathbb{N},$$

and let

$$(x_j)_{j\leq i}\in\bigcap_{k\in\mathbb{N}} p_i\ \varphi^{-1}\left((\bar{x}_j)_{j\leq i+k}\right).$$

The lemma is proved, once we have shown that there is an $x\in D_\varphi$ such that

$$p_i x=(x_j)_{j\leq i}, \qquad \varphi x=\bar{x},$$

and the continuity of $\varphi$ on $W_{S_A}(x)$ together with a selection argument furnishes such an $x$ . Q.e.d.

Denote for $\bar{x}\in\varphi(D_\varphi)$ by $J(\bar{x})$ the set of $i\in\mathbb{Z}$ such that there is a $k\in\mathbb{N}$ with

$$p_{i-k}\ \varphi^{-1}\{\bar{x}\}=p_{i-k}\ \varphi^{-1}\left((x_j)_{j\leq i}\right),$$

and denote

$$C^{(2)}=\{\bar{x}\in\varphi(D_\varphi): J(\bar{x})=\mathbb{Z}\}.$$

(2.4) <u>Lemma.</u> $\mu_{S_{\bar{A}}}(C^{(2)})=1$ .

<u>Proof.</u> By lemma (2.4)

$$J(\bar{x})\neq\emptyset, \qquad \bar{x}\in C^{(1)}.$$

We set then

$$E = \{\bar{x} \in C^{(1)} : \inf J(\bar{x}) > -\infty\} \quad .$$

Then

$$C^{(2)} \supset C^{(1)} - E \; .$$

Set further

$$E_i = \{\bar{x} \in E: \inf J(\bar{x}) = i\} \; , \qquad i \in \mathbb{Z} \; .$$

Then

$$S_{\bar{A}}^{k} E_i = E_{i-k} \; , \qquad i,k \in \mathbb{Z} \; ,$$

and we conclude that

$$\mu_{S_{\bar{A}}}(E_i) = 0 \; , \qquad i \in \mathbb{Z} \; . \qquad \text{Q.e.d.}$$

Denote

$$C^{(3)} = \quad C^{(2)} \cap \bigcap_{\bar{U} \in F_{S_{\bar{A}}}} \bar{U}\, C^{(1)} \; .$$

(2.5) <u>Lemma.</u> Let $\bar{x} \in C^{(3)}$, $x \in \varphi^{-1}\{\bar{x}\}$. Then $\varphi$ maps $D_\varphi \cap W_{S_A}(x)$ onto $W_{S_{\bar{A}}}(\bar{x})$.

<u>Proof.</u> Let

$$\varphi^{-1}\{\bar{x}\} = \{x^{(n)} : 1 \leq n \leq N\} \; .$$

That

$$\varphi\, W_{S_A}(x) = W_{S_{\bar{A}}}(\bar{x})$$

will be established once we have shown that for all $y \in W_{S_{\bar{A}}}(\bar{x})$

$$\varphi^{-1}\{\bar{y}\} \cap W_{S_A}(x^{(n)}) \neq \emptyset \; , \qquad 1 \leq n \leq N \; .$$

Assume to the contrary that we have a $\bar{y} \in W_{S_{\bar{A}}}(\bar{x})$ and an $n$, $1 \leq n \leq N$, such that

(2) $$\varphi^{-1}\{\bar{y}\} \cap W_{S_A}(x^{(n)}) = \emptyset \; .$$

Let $i \in \mathbb{Z}$ be such that

(3) $$\bar{y}_j = \bar{x}_j \; , \qquad j \leq i \; .$$

There is a $k \in \mathbb{N}$ such that

$$p_{i-k}\,\varphi^{-1}\{\bar{x}\} = p_{i-k}\,\varphi^{-1}((\bar{x})_{j\leq i}) .$$

Hence by (2) and (3), then

$$\varphi^{-1}\{\bar{y}\} \subset \bigcup_{1\leq m\leq N,\, m\neq n} W_{S_A}(x^{(m)}) .$$

By lemma (2.3) there is an $\ell \in \mathbb{N}$ such that also every element of

$$\varphi^{-1}\big((y_i)_{i\leq \ell}\big)$$

is negatively asymptotic to one of the $x^{(m)}$, $1 \leq m \leq N$, $m \neq n$. By irreducibility and aperiodicity of $\bar{A}$ construct an $\bar{x}' \in X_{\bar{A}}$ such that

$$\begin{aligned} \bar{x}'_i &= \bar{x}_i , && i < k , \\ x'_i &= \bar{y}_i , && k \leq i < \ell , \\ x'_i &= \bar{x}_i , && i > \ell + \bar{q} . \end{aligned}$$

Then

$$\varphi^{-1}\{\bar{x}'\} \subset \bigcup_{1\leq m\leq N,\, m\neq n} W_{S_A}(x^{(m)}) ,$$

however by the choice of $\bar{x}$ also $\bar{x}' \in C^{(1)}$ and we have a contradiction.

Q.e.d.

(2.6) <u>Lemma.</u> Let $\bar{x} \in \varphi(B^{(1)}) \cap C^{(3)}$, $x \in B^{(1)} \cap \varphi^{-1}\{\bar{x}\}$. Then the mapping

$$\varphi\colon W_{S_A}(x) \to W_{S_{\bar{A}}}(\bar{x})$$

is a homeomorphism.

Proof. It remains to show that this mapping is open. For this consider a cylinder set

$$Z\big((u_j)_{j\le i}\big) \subset W_{S_A}(x) \ ,$$

where

$$u_j = x_j \ , \qquad\qquad j \le i_o \ .$$

Let

$$\bar{y} \in \varphi\, Z\big((u_j)_{j\le i}\big) \ .$$

Since $\bar{x} \in C^{(2)}$ we have for some $k \in \mathbb{N}$ that

$$\varphi^{-1}\, Z\big((\bar{y}_j)_{j\le i_o}\big) \subset Z\big((x_j)_{j\le i_o - k}\big) \ . \tag{4}$$

Assume now that

$$\big(W_{S_A}(x) - Z\big((u_j)_{j\le i}\big)\big) \cap \varphi^{-1} Z\big((\bar{y}_j)_{j\le i_o+\ell}\big) \neq \emptyset, \qquad \ell \in \mathbb{N} \ . \tag{5}$$

To prove openness it is by lemma (2.5) enough to derive a contradiction to this assumption. By a selection argument we have from (4) that there is a

$$y \in \big(W_{S_A}(x) - Z\big((u_j)_{j\le i}\big)\big) \cap \varphi^{-1}\{\bar{y}\} \ . \tag{6}$$

Since $x \in B^{(1)}$, the mapping

$$\varphi\colon W_{S_A}(x) \to W_{S_A}(\bar{x})$$

is one-to-one. Therefore (6) contradicts the choice of $\bar{y}$ . Q.e.d.

Denote for $\bar{\alpha}_0 \in \bar{\Sigma}$

$$Y(\bar{\alpha}_0) = \big\{(\bar{x}_i)_{i\ge o} \in X_{\bar{A}}[0,\infty) : \bar{x}_0 = \bar{\alpha}_0\big\} \ ,$$

and denote by $\nu(\bar{\alpha}_0)$ the measure on $Y(\bar{\alpha}_0)$ that is given by

$$\nu(\bar{\alpha}_0)\ Z\big((\bar{\alpha}_i)_{0\le i\le k}\big) = \lambda_{\bar{A}}^{-n}\, b_{\bar{\alpha}_0}^{-1}\, b_{\bar{\alpha}_k} \ , \quad (\bar{\alpha}_i)_{0\le i\le k} \in A_{\bar{A}}[0,k], \qquad k \in \mathbb{N} \ .$$

Define then

$$\gamma_k(\bar{x}) = \nu(\bar{x}_{-k})\big(\{\bar{y} \in Y(\bar{x}_{-k}):$$

$$\big((\bar{x}_{i-k})_{i<0},\ (\bar{y}_i)_{i\geq 0}\big) \in C^{(1)},$$

$$\varphi^{-1}\big((\bar{x}_{i-k})_{i<0},\ (\bar{y}_i)_{i\geq 0}\big) \subset B\}\big)\ , \qquad k \in \mathbb{N}$$

and set

$$C^{(4)} = \bigcap_{k\in\mathbb{N}} \{\bar{x} \in X_{\bar{A}} : \gamma_k(\bar{x}) = 1\}$$

(2.7) Lemma. Let $\bar{x} \in \varphi(D_\varphi) \cap C^{(4)}$, $x \in \varphi^{-1}\{\bar{x}\}$. Then $\tau_{S_A,x}$ is a multiple of $\varphi^{-1}\ \tau_{S_{\bar{A}},x}$ .

Proof. The definition of $C^{(4)}$ was designed to ensure that the set

$$W_{S_{\bar{A}}}(\bar{x}) \cap \varphi\big(B \cap W_{S_A}(x)\big)$$

has full $\tau_{S_{\bar{A}},\bar{x}}$-measure. Here the set $B \cap W_{S_A}(x)$ is invariant under the action of $F_{S_A}(x)$, and the images under $\varphi$ of any two points of $B \cap W_{S_A}(x)$ , that are in the same $F_{S_A}$-orbit will be in the same $F_{S_{\bar{A}}}(\bar{x})$-orbit. By transporting the measure $\tau_{S_{\bar{A}}}(\bar{x})$ via $\varphi^{-1}$ to $B \cap W_{S_A}(x)$ we obtain therefore an $F_{S_A}(x)$-invariant Borel measure on $W_{S_A}(x)$ that is finite on compact sets, and hence is a multiple of $\tau_{S_A,x}$ , due to the fact that a Borel measure with this property is unique up to a factor. This uniqueness follows here e.g. from the uniqueness of the trace of AF-algebra that is associated to $F_{S_A}(x)$ [5]. Q.e.d.

(2.8) Theorem. Let $A$ and $\bar{A}$ be irreducible and aperiodic non-negative integer matrices, $\lambda_A = \lambda_{\bar{A}}$ . Let the characteristic polynomial of $A$ be irreducible up to a power and let there exist a continuous left-resolving code of $(X_A,S_A)$ onto $(X_{\bar{A}},S_{\bar{A}})$. Then $A$ and $\bar{A}$ are shift equivalent.

<u>Proof.</u> $\lambda_A = \lambda_{\bar{A}}$ implies that a continuous code of $(X_A, S_A)$ onto $(X_{\bar{A}}, S_{\bar{A}})$ is finite to one [3]. The hypothesis on $A$ is equivalent to the total order of the dimension group of $S_A$ . Hence this dimension group is isomorphic to the range of $\tau_{S_A,x}$ on $C_{S_A}(x)$ for any $x \in X_A$. It follows further that the dimension group at $S_{\bar{A}}$ is also totally ordered [6,9]. Apply lemmas (2.6) and (2.7) together with theorem (4.2) of [7]. Q.e.d.

## 3. Dimension and Synchronisation

We recall how the (future) dimension function $\delta_{S_A}$ of the system $(X_A, S_A)$ arises [7]. Let $x \in X_A$. The group $\mathcal{F}_{S_A}(x)$ acts on $\mathcal{C}_{S_A}(x)$. The dimension function of $(X_A, S_A)$ is the quotient map of $\mathcal{C}_{S_A}(x)$ onto the orbit space of this action. Its range carries an algebraic structure, where we have for $\gamma, \gamma' \in \delta_{S_A}(\mathcal{C}_{S_A}(x))$,

$$\gamma + \gamma' = \delta_{S_A}(C \cup C'), \quad C \in \gamma,\ C' \in \gamma',\ C \cap C' = \emptyset .$$

In this way $\delta_{S_A}(\mathcal{C}_{S_A}(x))$ becomes the positive cone of the future dimension group $(K_o, K_o^+)(S_A^{-1})$ of $(X_A, S_A)$.

Denote for $x' \in X_A$ by $\mathcal{R}(x', x)$ the set of homeomorphisms

$$R: W_{S_A}(x) \to W_{S_A}(x')$$

with the property that there is an $I \in \mathbb{Z}$ such that

$$(Ry)_i = y_i , \qquad i > I,\ y \in W_{S_A}(x) .$$

To define the automorphism $\Phi_{S_A}$ that $S_A$ induces on its future dimension group, take an $R \in \mathcal{R}(S_A x, x)$ and set

$$\Phi_{S_A}\gamma = \delta_{S_A, x}(RS_A C), \qquad C \in \gamma \in \delta_{S_A}\left(\mathcal{C}_{S_A}(x)\right) .$$

These constructions do not depend on the choice of $x \in X_A$. Any $R \in \mathcal{R}(x', x)$ induces an isomorphism of the triples $(K_o, K_o^+, \Phi)(S_A^{-1})$ obtained when employing $x$ and $x'$.

We consider now the situation where one is given irreducible and aperiodic 0-1 transition matrices $A$ and $\bar{A}$ over finite symbol spaces $\Sigma$ and $\bar{\Sigma}$, and where one knows that for some $p \in \mathbb{N}$ the systems $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$ are topologically isomorphic. We choose then a suitably normalized topological conjugacy

$$U: X_A \to X_{\bar{A}}, \qquad S_{\bar{A}}^p U = U S_A^p ,$$

and together with its inverse we describe this conjugacy by mappings, $\Psi_q$, $\bar{\Psi}_q$, $0 \leq q < p$ , where for some $L \in \mathbb{N}$

$$\Psi_q: A_A[1,L] \to \bar{\Sigma}$$

$$\bar{\Psi}_q: A_{\bar{A}}[1,L] \to \Sigma$$

and

$$Ux = (\Psi_{i \,(\mathrm{mod}\, p)}((x_j)_{i \leq j < i+L}))_{i \in \mathbb{Z}}, \qquad x \in X_A ,$$

$$U^{-1}\bar{x} = (\bar{\Psi}_{i \,(\mathrm{mod}\, p)}((\bar{x}_j)_{i-L < j \leq i}))_{i \in \mathbb{Z}}, \qquad \bar{x} \in X_{\bar{A}} .$$

Let $K,N \in \mathbb{N}$, $N \geq K + 3L$. Denote for $0 \leq q < p$ by $S_q[K,N]$ the set of pairs

$$(a,\bar{a}) \in A_A[K+L,N] \times A_{\bar{A}}[1,K+2L]$$

such that

$$\bar{a}_i = \Psi_{i+q\,(\mathrm{mod}\, p)}((a_j)_{i \leq j < i+L}) ,$$

$$a_i = \bar{\Psi}_{i+q\,(\mathrm{mod}\, p)}((\bar{a}_j)_{i-L < j \leq i}), \; K + L \leq i \leq K + 2L .$$

In the sequel $\sigma$ stand for mappings that imitate the action of the shift, where $\sigma$ carries an

$$\big((a,\bar{a}),\alpha\big) \in S_q[K,N] \times \Sigma, \qquad A(a_N,\alpha) = 1 ,$$

into the $(b,\bar{b}) \in S_{q+1\,(\mathrm{mod}\, p)}[K,N]$, where

$$b_i = a_{i+1}\ , \qquad K + 2L \le i < N\ ,$$

$$b_N = \alpha\ ,$$

$$\bar{b}_i = \bar{a}_{i+1}\ , \qquad 1 \le i < K + 2L\ .$$

The notation $\sigma^{-1}$ is to be interpreted similarly.

By a synchronization map (relative to the family $(\Psi_q, \bar{\Psi}_q)_{0 \le q < p}$ ) we mean a one-to-one and onto map

$$\rho:\ S_1[K,N] \to S_0[K,N]$$

with the property that $\rho$ assigns to an $(a,\bar{a}) \in S_1[K,N]$ a $(b,\bar{b}) \in S_0[K,N]$ such that

$$\bar{b}_1 = \bar{a}_1\ , \qquad b = a\ .$$

(3.1) Lemma. Let $A$ and $\bar{A}$ be irreducible and aperiodic with identical Jordan form away from zero. Let for some $p, L \in \mathbb{N}$, where $p$ is such that no ratio of any two eigenvalues is a p-th root of unity,

$$\Psi_q : \mathcal{A}_A\ [1,L] \to \Sigma$$

$$\bar{\Psi}_q : \mathcal{A}_{\bar{A}}\ [1,L] \to \bar{\Sigma}\ , \qquad 0 \le q < p\ ,$$

be mappings that describe a topological conjugacy $U$ between $(X_A, S_A^p)$ and $(X_{\bar{A}}, S_{\bar{A}}^p)$. Then there exists for sufficiently large $K \in \mathbb{N}$, and $N \ge K + 3L$ a synchronization map

$$\rho\ :\ S_1[K,N] \to S_0[K,N]$$

relative to the family $(\Psi_q, \bar{\Psi}_q)_{0 \le q < p}$ .

<u>Proof.</u> Set

$$\bar{S} = U\, S_A\, U^{-1}\ .$$

One has

$$\bar{S}^{\,p} = S_{\bar{A}}^{p} \qquad (7)$$

The Jordan form of $\Phi_{S_A} \otimes 1$ acting on $K_o(S_A) \otimes \mathbb{C}$ is given by the Jordan form of $A$ away from zero. The hypothesis of the lemma, together with (7), implies therefore that

(8) $$\Phi_{\bar{S}} = \Phi_{S_{\bar{A}}} .$$

Choose now an $\bar{x} \in X_{\bar{A}}$ and for every $\bar{\alpha} \in \bar{\Sigma}$ choose a $c^{(\bar{\alpha})} \in X_{\bar{A}}(-\infty, 1]$ that is negatively asymptotic to $\bar{x}$ and where

$$c_1^{(\bar{\alpha})} = \bar{\alpha} .$$

From (8) one has with an $R \in \mathcal{R}(\bar{x}, \bar{S}\, S_{\bar{A}}^{-1}\, \bar{x})$

$$\delta_{S_{\bar{A}}}(R\, \bar{S}\, S_{\bar{A}}^{-1}\, Z(c^{(\bar{\alpha})})) = \delta_{S_{\bar{A}}}(Z(c^{(\bar{\alpha})})), \qquad \bar{\alpha} \in \bar{\Sigma} .$$

From this it follows that there is a $K \in \mathbb{N}$ such that we can find for every $\bar{\alpha} \in \bar{\Sigma}$ an $R_{\bar{\alpha}} \in \mathcal{R}(\bar{x}, \bar{S}\, S_{\bar{A}}^{-1}\, \bar{x})$ such that

$$(R_{\bar{\alpha}}\, \bar{y})_i = \bar{y}_i , \qquad i \geq K,\ \bar{y} \in W_{S_{\bar{A}}}(\bar{x})$$

such that also with $C[K]$ denoting the Boolean subring of $C_{S_{\bar{A}}}(\bar{x})$ with atoms the cylinder sets $Z((u_i)_{i \leq K})$ ,

$$R_{\bar{\alpha}}\, C[K] = C[K] ,$$

and such that

$$R_{\bar{\alpha}}\, \bar{S}\, S_{\bar{A}}^{-1}\, Z(c^{(\bar{\alpha})}) = Z(c^{(\bar{\alpha})}), \qquad \bar{\alpha} \in \bar{\Sigma} .$$

Let then $N \geq K + 3L$ . We can define a mapping

$$\rho\colon S_1[K,N] \to S_0[K,N]$$

with the required properties, setting for $(a,\bar{a}) \in S_1$

$$\bar{a}_i = \Psi_{i+1 \,(\mathrm{mod}\, p)}((a_j)_{i \leq j < j+L}), \qquad K + 2L < i < K + 3L ,$$

by assigning to $(\bar{a},a)$ the $(\bar{b},b) \in S_o[K,N]$ such that $b = a$ and

$$Z((c^{(\bar{\alpha})}, \bar{b})) \subset R_{\bar{\alpha}}\, \bar{S}\, S_{\bar{A}}^{-1}\, Z(c^{(\bar{\alpha})}, (\bar{a}_i)_{1\leq i<K+3L}), \qquad \alpha = \bar{a}_1 .$$

The inverse of $\rho$ can be obtained from $S_{\bar{A}}\, \bar{S}^{-1}\, R_{\bar{\alpha}}^{-1}$ in the same way as $\rho$ was obtained from $R_{\bar{\alpha}}\, \bar{S}\, S_{\bar{A}}^{-1}$ . Q.e.d.

## 4. A coding theorem

(4.1) Proposition. Let $A$ and $\bar{A}$ be shift equivalent irreducible and aperiodic non-negative integer matrices. Then there exists a joint left-resolving continuous extension of $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$, and there exists a uniformly finite-to-one resolving finitary code from $(X_A, S_A)$ to $(X_{\bar{A}}, S_{\bar{A}})$ with bounded anticipation and finite expected coding time.

Proof. For the proof we use for some $p \in \mathbb{N}$, where no ratio of any two eigenvalues is a p-th root of unity, a topological conjugacy of $(X_A, S_A^p)$ and $(X_{\bar{A}}, S_{\bar{A}}^p)$ with lag L, and for some $K, N \in \mathbb{N}$ a resulting synchronization mapping

$$\rho: S_1[K, N+1] \to S_0[K, N+1]$$

as constructed in section 3. Here we can have $N$ a multiple of $p$ and we can also have a block $c \in A_A[1, N]$ such that

$$(c_\ell)_{1 \leq \ell \leq n} \neq (c_{N-n+\ell})_{1 \leq \ell \leq n}, \quad 1 \leq n < N$$

where at least two symbols of $\Sigma$ are admitted as predecessors of $c_1$.

We proceed to construct a topological Markov chain with symbol space $\tilde{\Sigma}$ and transition matrix $\tilde{A}$ that will serve as joint left-resolving continuous extension of $(X_A, S_A)$ and $(X_{\bar{A}}, S_{\bar{A}})$. Here $\tilde{\Sigma}$ is the set of all

$$(b, s) \in A_A[1, 6N-1] \times \left\{ s \in \{0,1\}^{4N} : \sum_{1 \leq j \leq 4N} s_j > 0 \right\}$$

that satisfy the following conditions:

With the notation

$$M(s) = \sum_{1 \leq j \leq 4N} s_j,$$

$$j_1(s) = \min \{ j : s_j = 1 \},$$

$$j_m(s) = \min \{ j > j_{m-1} : s_j = 1 \}, \quad 1 < m \leq M(s),$$

it is required that

$$j_1(s) < 2N, \; N < j_m(s) - j_{m-1}(s) < 2N, \qquad 1 < m \leq M(s) .$$

If for some $j$, $1 \leq j \leq 4N$

$$(b_{j-1+n})_{1 \leq n \leq N} = c ,$$

then

$$s_j = 1 .$$

If $j_1(s) > N$ then

$$(b_{j_1(s)-1+n})_{1 \leq n \leq N} = c .$$

If for some $m$, $1 \leq m \leq M(s)$,

$$N < j_m(s) - j_{m-1}(s),$$

then

$$(b_{j_m(s)-1+n})_{1 \leq n \leq N} = c,$$

and if $j_{M(s)} < 3N$ then there is a $j$, $4N < j < j_{M(s)} + 2N$ , such that

$$(b_{j-1+n})_{1 \leq n \leq N} = c .$$

Now set for $(b,s)$, $(b',s') \in \tilde{\Sigma}$

$$\tilde{A}\;(b,s),\;(b',s') = 1$$

if

$$b'_j = b_{j-1} , \qquad 1 \leq j < 6N - 1$$

and

$$s'_j = s_{j-1} , \qquad 1 \leq j < 4N$$

and where

$$s'_{4N} = 1$$

if either

$$(b_{4N+n})_{1\leq n\leq N} = c$$

or

$$S_{3N+1} = 1, \quad (b_{4N+\ell+n})_{1\leq n\leq N} \neq c, \quad 0 \leq \ell < N .$$

We have in this way constructed an irreducible and aperiodic topological Markov chain $(X_{\tilde{A}}, S_{\tilde{A}})$. This construction is such that the mapping that carries a sequence

$$\left(b^{(i)}, s^{(i)}\right)_{i\in\mathbb{Z}} \in X_{\tilde{A}}$$

into the sequence

$$\left(b^{(i)}_{6N-1}\right)_{i\in\mathbb{Z}} \in X_A$$

is a left-resolving homomorphism.

We define a homomorphism

$$\psi: \left(X_{\tilde{A}}, S_{\tilde{A}}\right) \to \left(X_{\bar{A}}, S_{\bar{A}}\right)$$

by first assigning to a sequence

$$\left(b^{(i)}, s^{(i)}\right)_{i\in\mathbb{Z}} \in X_{\tilde{A}}$$

the sequence

$$\left(a^{(i)}, \bar{a}^{(i)}\right)_{i\in\mathbb{Z}} \in \left(S_0[K,N+1] \cup S_1[K,N+1]\right)^{\mathbb{Z}}$$

where

$$a^{(i)} = \left(b^{(i)}_{5N-2+j}\right)_{K+L\leq j\leq N+1}, \quad i \in \mathbb{Z},$$

and where

$$(9) \quad \left(a^{(i)}, \bar{a}^{(i)}\right) = \begin{cases} \sigma((a^{(i-1)}, \bar{a}^{(i-1)}), b^{(i)}_{6N-1}), & \text{if} \quad \sum_{1\leq j\leq N} s^{(i)}_j = 1, \\ \rho\sigma((a^{(i-1)}, \bar{a}^{(i-1)}), b^{(i)}_{6N-1}), & \text{if} \quad \sum_{1\leq j\leq N} s^{(i)}_j = 0, \; i \in \mathbb{Z}, \end{cases}$$

and setting

$$\psi(b^{(i)},s^{(i)})_{i\in\mathbb{Z}} = (a_1^{(i)})_{i\in\mathbb{Z}} .$$

To see that $\psi$ is actually a homomorphism we observe that it is left-resolving. For this let $I \in \mathbb{Z}$. We reverse (9) to get

$$(10)\quad \left(a^{(i-1)},\bar{a}^{(i-1)}\right) = \begin{cases} \sigma^{-1}((a^{(i)},\bar{a}^{(i)}),\bar{a}_1^{(i-1)}), & \text{if } \sum\limits_{1\le j\le N} s_j^{(i)} = 1, \\ \sigma^{-1}(\rho^{-1}(a^{(i)},\bar{a}^{(i)}),\bar{a}_1^{(i-1)}), & \text{if } \sum\limits_{1\le j\le N} s_j^{(i)} = 0, \end{cases} \quad i \in \mathbb{Z} .$$

Then note that

$$s_j^{(I+\ell)} = s_{j+\ell}^{(I)}, \qquad 1 \le j \le N,\ 1 \le \ell \le 3N .$$

Thus the

$$\left(s_j^{(I+\ell)}\right)_{1\le j\le N}, \qquad 1 \le \ell \le 3N ,$$

can be read off from $s^{(I)}$. Also at least one of the

$$s_j^{(I)}, \qquad 2N < j \le 4N ,$$

equals one. It follows that one can use (10) to compute $a^{(I+1)},\bar{a}^{(I+1)}$, and therefore also $b^{(I+1)}$ from the $a_1^{(i)}$, $i \le I + 3N$ .

To present now a finitary coding $\varphi$ from $(X_A,S_A)$ to $(X_{\bar{A}},S_{\bar{A}})$ with the stated properties, define first a suitable finitary code $\eta$ from $(X_A,S_A)$ to $(X_{\tilde{A}},S_{\tilde{A}})$ and then set $\varphi = \psi\eta$ . The domain of definition $D_{\eta_o}$ at the zero-coordinate mapping $\eta_o$ of $\eta$ is

$$D_{\eta_o} = \bigcup_{\ell\in\mathbb{N}} S_A^{\ell}\, Z(c) ,$$

and

$$\eta_o(x) = (b(x),\, s(x)), \qquad x \in D_{\eta_o} ,$$

where

$$b(x) = (x_j)_{1\le j<6N} ,$$

and where $s_j(x)$ is given by the indicator function of

$$\Big(X_A - \bigcup_{1\le m<2N} S_A^{-m-j}Z(c)\Big) \cap \Big(\bigcup_{\ell\in\mathbb{N}} S_A^{\ell N-j}Z(c)\Big), \qquad 1 \le j \le 4N .$$

Q.e.d.

(4.2) Theorem. Let $A$ and $\bar{A}$ be irreducible and aperiodic non-negative integer matrices with the same maximal real eigenvalue $\lambda$. Let the characteristic polynomials of $A$ and $\bar{A}$ be equal, up to powers, to the minimum polynomial of $\lambda$. Then the following are equivalent:

1. There exists an a.e. finite-to-one resolving finitary code with bounded anticipation and finite expected coding time from $(X_A,S_A)$ to $(X_{\bar{A}},S_{\bar{A}})$.

2. There exists a joint left-resolving continuous extension of $(X_A,S_A)$ and $(X_{\bar{A}},S_{\bar{A}})$.

3. $A$ and $\bar{A}$ are shift equivalent.

4. $(K_o,K_o^+)(S_A) = (K_o,K_o^+(S_{\bar{A}})$.

Proof. The condition on the characteristic polynomials of $A$ and $\bar{A}$ means that the (future) dimension groups of $(X_A,S_A)$ and $(X_{\bar{A}},S_{\bar{A}})$ are totally ordered. If the (future) dimension group of $(X_A,S_A)$ is totally ordered then it is isomorphic to the range of measure $\tau_{S_A,x}$ on $C_{S_A}(x)$, $x \in X_A$. We apply this remark to $A$ and $\bar{A}$ and infer, that the existence of a code from $(X_A,S_A)$ to $(X_{\bar{A}},S_{\bar{A}})$ with the stated properties as well as the existence of a joint left-resolving continuous extension of $(X_A,S_A)$ and $(X_{\bar{A}},S_{\bar{A}})$, implies by lemmas (2.6) and (2.7), that up to a factor these ranges for $(X_A,S_A)$ and $(X_{\bar{A}},S_{\bar{A}})$ are identical. It follows then from theorem (4.2) of [7] that $A$ and $\bar{A}$ are shift-equivalent. To go in the other direction use proposition (4.1). Q.e.d.

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## References

1. L.M. Abramov and V.A. Rohlin: The entropy of a skew product of measure preserving transformations. Amer.Math.Soc.Translations 48 (1965) 255-265.

2. R. Adler and B. Marcus: Topological entropy and equivalence of dynamical systems. Memoirs of the Amer.Math.Soc. Nr. 219 (1979).

3. E.M. Coven and M.E. Paul: Endomorphisms of irreducible subshifts of finite type. Math.Systems Theory 8 (1974), 167-175.

4. J. Cuntz and W. Krieger: Topological Markov chains with dicyclic dimension groups. Journal für die reine und angewandte Mathematik 320 (1980), 44-51.

5. E. Effros and C.L. Shen: Dimension groups and finite difference equations. J. of Operator Theory 2 (1979), 215-231.

6. B. Kitchen: preprint.

7. W. Krieger: On dimension functions and topological Markov chains. Inventiones Math. 56 (1980), 239-250.

8. B. Marcus: Factors and extensions of full shifts. Monatshefte f. Mathematik 88 (1979), 239-247.

9. M. Nasu: Uniformly finite-to-one and onto extensions of homomorphisms between strongly connected graphs. Preprint, Research Institute of Electrical Communication, Tohoku University, Sendai, Japan.

10. R.F. Williams: Classification of subshifts of finite type. Ann. of Math. 98 (1973), 120-153. Errata: Ann. of Math. 99 (1974), 380-381.