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INSTANTON SINGULARITIES IN EUCLIDEAN Φ^4 QUANTUM FIELD THEORIES

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Abstract We discuss some rigorous results regarding the large order behaviour of the renormalized perturbation expansion in euclidean Φ^4_d models.

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"Perturbation theory", in the context of quantum field theory, usually refers the use of a power series expansion of some object of interest in powers g^n of some parameter g of the theory. Consider, for example, the pressure p(g) in a Φ^4_d model. (d is the dimension of space-time.)

$$p(g) = \lim_{\Lambda\to\mathbb{R}^d} \lim_{U\to\infty} (1/|\Lambda|)\log Z(\Lambda,U) \tag{1}$$

where Z(Λ,U) is the partition function

$$Z(\Lambda,U) = \int \exp\left[\int_\Lambda d^d x\{-(g+\delta g)\varphi^4(x)+\delta\zeta(\nabla\varphi)^2(x)+\delta m^2\varphi^2(x)+\delta E\}\right] d\mu_{C^U}(\varphi). \tag{2}$$

Here dμ is the gaussian measure of mean zero and covariance C^U. C^U is a regularized version of the operator $(-\Delta+1)^{-1}$, like for example

$$C^U = (-\Delta+1)^{-1}e^{\Delta/U}. \tag{3}$$

When d=2,3 and the counterterms $\delta g, \delta\zeta, \delta m^2$ and δE are carefully chosen polynomials in g (whose coefficients are functions of U,Λ) each term in the formal power series expansion of $1/|\Lambda|\log Z(\Lambda,U)$ in powers of g converges as $U\to\infty$ and $\Lambda\to\mathbb{R}^d$. In addition p(g) exists as a C^∞ function at least for g small and positive and the formal power series is its asymptotic expansion.

$$p(g) \sim \sum_{n\geq 1} a_n^r(-g)^n \tag{4}$$

The superscript r is not a power. It stands for "renormalized".

When d=4 the situation is not so nice. Each term in the formal power series still converges provided $\delta g, \delta\zeta, \delta m^2$ and δE are themselves carefully chosen formal power series in g. This is what is meant by the statement "Φ^4_4 is renormalizeable". δE is normally chosen so that p(g) is exactly zero. Hence we should consider objects like the euclidean green's functions rather than p(g). That is not important. What is important is that it is quite possible that there does not exist a quantum field theory having the Φ^4_4 formal power series as asymptotic expansions. In fact that is the current conventional wisdom.

Formal power series like these have played an extremely important role in quantum field theory, both in numerical calculations and in rigorous analysis. In this seminar we shall look at the large n behaviour of the a_n^r's. The technique we shall use is a Laplace expansion and is known as the Lipatov method [Li,BGZ]. The rigorous application of this method to Φ^4 models has been carried out for lattice models by Spencer [Sp], for d=2 by Breen [B] and for d=3 in [MR,FR] . For work on d=4 see [P,MNRS].

To demonstrate this technique we shall first consider an artificial model that lives in a world, W, containing only finitely many points:

$$Z(g) = \frac{\int \exp\left[-g\sum_{x\in W}\varphi(x)^4\right]\exp\left[-1/2\sum_{y,z\in W}\varphi(y)C(y,z)^{-1}\varphi(z)\right]\prod_{x\in W}d\varphi(x)}{\int \exp\left[-1/2\sum_{y,z\in W}\varphi(y)C(y,z)^{-1}\varphi(z)\right]\prod_{x\in W}d\varphi(x)} \tag{5}$$

Z(g) has the formal power series expansion $Z(g) \sim \sum a_n(-g)^n$ with

$$a_n = \frac{1}{n!}\,\frac{\int\left[\sum_{x\in W}\varphi(x)^4\right]^n\exp\left[-1/2\sum_{y,z\in W}\varphi(y)C(y,z)^{-1}\varphi(z)\right]\prod_{x\in W}d\varphi(x)}{\int \exp\left[-1/2\sum_{y,z\in W}\varphi(y)C(y,z)^{-1}\varphi(z)\right]\prod_{x\in W}d\varphi(x)} \tag{6}$$

We will calculate in particular the limit as $n\to\infty$ of $(a_n/n!)^{1/n}$. This is precisely the radius of convergence of the Borel transform of the above series. To perform this calculation make the change of variables $\varphi(x) = n^{1/2}\psi(x)$.

$$(a_n/n!)^{1/n} = \left[\frac{n^{2n}}{n!^2}\,\frac{\int\left[\sum_{x\in W}\psi(x)^4\right]^n\exp\left[-n/2\sum_{y,z\in W}\psi(y)C(y,z)^{-1}\psi(z)\right]\prod_{x\in W}d\psi(x)}{\int \exp\left[-n/2\sum_{y,z\in W}\psi(y)C(y,z)^{-1}\psi(z)\right]\prod_{x\in W}d\psi(x)}\right]^{1/n}$$

$$= \left[\frac{n^{2n}}{n!^2}\right]^{1/n} \frac{\|\exp-S(\psi)\|_n}{\|\exp-S_F(\psi)\|_n}$$

where
$$S(\psi) = (1/2)\sum_{y,z\in W} \psi(y)C(y,z)^{-1}\psi(z) - \log \sum_{x\in W} \psi(x)^4 \quad (7)$$
and
$$S_F(\psi) = (1/2)\sum_{y,z\in W} \psi(y)C(y,z)^{-1}\psi(z).$$

Stirling's formula and the fact that $\lim_{n\to\infty}\|f\|_n=\|f\|_\infty$ now implies

$$\lim_{n\to\infty} (a_n/n!)^{1/n} = \exp\{-\inf S(\psi) + 2\} \equiv R_C \quad (8)$$

This conclusion with the appropriate choice of C applies to the $\Phi^4{}_d$ model when d=2,3:

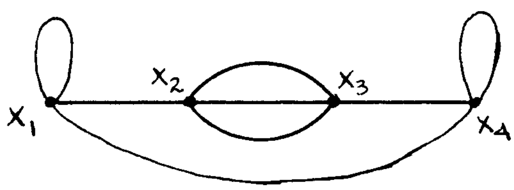
THEOREM L [B,MR,FR] Let B(t) be the Borel transform of the pressure in the $\Phi^4{}_d$ model with d=2,3. B(t) is analytic in the disc $|t| < R_C$ with $C=(-\Delta+1)^{-1}$. Furthermore B(t) has a singularity at $t=-R_C$.

This singularity is harmless. It implies that the perturbation series for p(g) diverges. But that has been known for years and had been anticipated for even more years. The singularity is not an obstruction to Borel summability and indeed the perturbation series is still Borel summable. On the other hand the conclusion of this theorem is probably false when d=4. There is a result [MNRS] which says that if you take an appropriate B(t) for the $\Phi^4{}_4$ model and throw away by hand what we call the "useless parts of the counterterms" (this does not affect the finiteness of the theory but it destroys the locality of the interaction) the result is analytic in $\{|t|<R_C\}$. But there is good reason to believe [P] that the useless parts of the counterterms are responsible for the presence of a "renormalon" singularity in B(t) on the positive t-axis. This would mean that it is impossible for a Borel summable $\Phi^4{}_4$ theory to exist and would complement results suggesting that it is impossible to constuct $\Phi^4{}_4$ as a limit of lattice approximations [A,F].

We shall now outline what is involved in proving the above theorem. In particular we shall now restrict to d=2,3 but we shall return to d=4 later. The manipulations involved in relating the pressure (1) to an object susceptible to a direct application of the argument (6)-(8) are best explained in terms of Feynman graphs. The integral

$$\int \left[\int_\Lambda d^dx \;\varphi^4(x)\right]^n d\mu_C(\varphi) \tag{9}$$

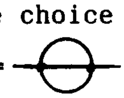
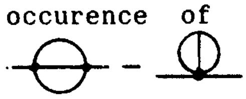
is an integral of a polynomial against a gaussian measure and can be evaluated exactly in terms of C. It is a sum of (4n-1)!! terms each of which is the value of a Feynman diagram. One example of a Feynman diagram which contributes to (9) when n=4 is



which has the value

$$\int_\Lambda \prod_{i=1}^{4} d^dx_i \quad C(x_1,x_1)C(x_1,x_2)C(x_1,x_4)C(x_2,x_3)^3C(x_3,x_4)C(x_4,x_4) \tag{10}$$

where C(x,y) is the kernel of C viewed as an integral operator. Of course the integral in (10) may or may not converge depending on C. In the case of Φ^4_d it converges for d=1 and diverges for all d>1. This is where renormalization comes in i.e. the introduction of a cutoff covariance C^U, the introduction of counterterms δg etc. and the removal of the cutoff. For example it is possible to choose the counterterms in such a way that they simply forbid the presence of Feynman diagrams containing tadpoles  . This is called Wick ordering and renders finite each term in the perturbation series when d=2. When d=3 this does not suffice. To render the theory finite it is also necessary to forbid the occurence of the subgraphs

(which can be arranged by a suitable choice of δE) and to replace each occurence of the "blob" subgraph B=  by the "renormalized blob"  (which can be arranged by a suitable choice of δm^2).

Theorem L is proven by replacing the series $B(t) \sim \Sigma a_n{}^r(-t)^n/n!$ by another which has the same large n behaviour. In fact this is done several times ending with a series whose coefficients can be analyzed using arguments like those in (6)-(8). In particular in (6) we have an integral of the n^{th} power of a polynomial against a Gaussian measure. Because of renormalization and the logarithm in (1), $a_n{}^r$ is not of this form. To "correct" this, one can

(i) (not needed in d=2) observe that simply dropping all graphs containing blob subgraphs cannot affect the validity of Theorem L. This is done by proving an upper bound on (a) the effect that the introduction of renormalized blobs into a blob-free graph can have on the value of the graph and (b) the number of ways it is possible to introduce blobs into a graph. Using these upper bounds it is possible to show that a blob-free graph gives a more important contribution than all graphs gotten by inserting blobs into it.

(ii) replace the covariance $(-\Delta+1)^{-1}$ in all remaining graphs of order n by an ultraviolet cutoff covariance $C^{(n)}$ whose cutoff grows with the order n at an appropriate rate. Again this can be done without affecting the validity of Theorem L. This is proven using bounds that say in effect that when a graph does not contain any divergent subgraphs the contributions to the value of the graph coming from high momenta are relatively unimportant.

(iii) (again not needed in d=2) put back the blob subgraphs that were deleted in the first step. This time the blobs are not renormalized. This is o.k. since we now have an ultraviolet cutoff.

(iv) introduce an order dependent volume cutoff. With one choice of boundary conditions the introduction of a volume cutoff increases the value of Feynman graphs. With a different choice it decreases them.

(v) drop the logarithm. The logarithm of the formal power series $1 + \Sigma_{n\geq 1} b_n(-g)^n$ is $\Sigma_{n\geq 1} a_n(-g)^n$ with

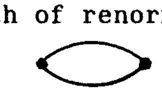
$$a_n = b_n + \sum_{2\leq m\leq n} (-1)^{m-1} B(m,n)/m$$

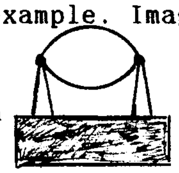
$$B(m,n) = \sum_{i_1+\dots i_m=n} b_{i_1} \dots b_{i_m}$$

The b_n term dominates the $B(m,n)$ terms because b_j grows very rapidly with j (roughly (j!)).

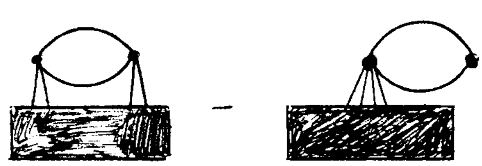
By turning the rough ideas of (i) to (v) into detailed bounds one can replace the series in (4) by a series $\Sigma b_n(-g)^n$ where each b_n, like (9) (but with the covariance C, volume Λ and even the function that is raised to the n^{th} power depending on n), can be analyzed by using a steepest descent argument.

That brings us to d=4. Φ^4_4 is a strictly renormalizable model while $\Phi^4_{2,3}$ are superrenormalizable. Hence, by definition, the counterterms $\delta m^2(U,g)$ etc. are polynomials in g in the latter case while they are formal power series in the former. From our point of view the fundamental difference between Φ^4_d with d<4 and d=4 is not simply one of complexity. It is the following. In both cases the n^{th} order of perturbation theory is given by the sum of (4n-1)!! Feynman graphs. When d<4 there is a constant K such that all Feynman graphs of order n are bounded by K^n. But when d=4 nesting of renormalization subtractions can result in some graphs having values of the order of n!. This happens to a relatively small fraction of the graphs so that the Borel transform still has a nonzero radius of convergence [dCR]. But these renormalon factorials are probably still dominant in controlling the radius of convergence.

The phenomenon that is responsible for the birth of renormalon n!'s may be seen in the following example. Imagine that 

appears as a subgraph in some graph . Upon renormalization we

get



$$= \int dxdy\; C(x,y)^2[f(x,y) - f(x,x)] \qquad (11)$$

where f represents the rest of the graph. Now decompose the covariance $C = \Sigma_{i\geq 0}C^{(i)}$ with $C^{(i)}$ being the part of C having energy about M^i and M is just some constant bigger than one. For example we could choose

$$C^{(i)}(x,y) = \frac{e^{\Delta/M^{2i}} - e^{\Delta/M^{2i+2}}}{-\Delta + 1}(x,y) \approx M^{2i} e^{-M^{2i}(x-y)^2} \qquad (12)$$

Then (11) decomposes into a huge multiple sum. For the purposes of illustration we shall just consider

$$\sum_j \int dxdy\; C^{(j)}(x,y)^2[f^{(i)}(x,y) - f^{(i)}(x,x)]. \qquad (13)$$

We are just considering the case in which the two lines of the subgraph have the same scale j and all the lines of f ending at x or y have the same scale i. Furthermore, at first we shall hold i fixed and consider the sum over j.

Case j>i: We first consider separately the two terms in the integral (13). Since j>i we have that $\exp[-M^{2j}(x-y)^2] < \exp[-M^{2i}(x-y)^2]$ so that it pays to use the distance decay in the $C^{(j)}$'s rather than that in $f^{(i)}$ to perform the integral over y. This integral gives

$$\int d^4y\; \exp[-M^{2j}(x-y)^2] \sim M^{-4j}. \qquad (14)$$

For the integral over x we must use the decay (of scale i) from $f^{(i)}$ and so this integral gives M^{-4i}. Finally the explicit factors of M^{2j} in the two $C^{(j)}$'s give M^{4j} and the result is

$$\int dxdy\; C^{(j)}(x,y)^2 f^{(i)}(x,y) \sim (M^{2j})^2 M^{-4j} M^{-4i} F = M^{-4i} F \qquad (15)$$

$$\int dxdy\; C^{(j)}(x,y)^2 f^{(i)}(x,x) \sim (M^{2j})^2 M^{-4j} M^{-4i} F = M^{-4i} F \qquad (16)$$

Here F represents genericly all the remaining contributions to our integrals from f. It may depend on i and may even represent many different values. The conclusion is that if we keep the subgraph and its counterterm separate the sum over j diverges in both cases. However if we combine the two

$$\int dxdy\ C^{(j)}(x,y)^2[f^{(i)}(x,y) - f^{(i)}(x,x)].$$

$$= \int dxdy\ C^{(j)}(x,y)^2(x-y)\partial_y f^{(i)}(x,y'). \tag{17}$$

This time the integral of $|x-y|\exp[-M^{2j}(x-y)^2]$ over y gives M^{-5j} and the gradient ∂ applied to some $C^{(i)}$ inside $f^{(i)}$ gives an extra M^i and we get

$$\sum_{j=i+1}^{\infty} \int dxdy\ C^{(j)}(x,y)^2[f^{(i)}(x,y) - f^{(i)}(x,x)]$$
$$\sim \sum_{j=i+1}^{\infty} (M^{2j})^2 M^{-5j}M^{i}M^{-4i}F = \sum_{j=i+1}^{\infty} M^{-4i}M^{i-j}F \sim M^{-4i}F. \tag{18}$$

So the renormalization cancellation has caused the sum to converge.

Case $j\leq i$: This time we would prefer if possible to use covariances $C^{(i)}$ from $f^{(i)}$ to perform the integrals over x and y. For the subgraph this is possible for x and y. For the counterterm it is possible only for x. Hence

$$\sum_{j=0}^{i} \int dxdy\ C^{(j)}(x,y)^2 f^{(i)}(x,y) \sim \sum_{j=0}^{i} (M^{2j})^2 M^{-4i}M^{-4i}F$$
$$= M^{-4i}\sum_{j=0}^{i} M^{-4(i-j)}F \sim M^{-4i}F \tag{19}$$

$$\sum_{j=0}^{i} \int dxdy\ C^{(j)}(x,y)^2 f^{(i)}(x,x) \sim \sum_{j=0}^{i} (M^{2j})^2 M^{-4j}M^{-4i}F$$
$$= (i+1)M^{-4i}F. \tag{20}$$

The i-dependence hidden inside F may reduce the M^{-4i} to M^{-2i}. Even so when the sum over i is ultimately performed the subgraph and the counterterm converge individually and we see that we gain nothing by performing a cancellation between (19) and (20). (In a sense the difference between the two cases $j>i$ and $j\leq i$ is that in the former case the subgraph "looks pointlike" because of the very strong decay of $C^{(j)}$ and hence we get a good cancellation between the subgraph and the purely local counterterm.) In fact we see that (20), the "useless" part of the counterterm, i.e. the part of the counterterm that is not used for a renormalization cancellation (but which must be there none-the-less to preserve the locality of the interaction) gives a larger contribution than either (18) or (19).

A fundamental difference between superrenormalizeable and strictly renormalizeable models is that in the former we always have a "bounded density" of counterterms. In other words there is a bounded power of $(i+1)$ per decay factor M^{-2i}. In the latter this is not so. It is possible to have, in a graph of n vertices, roughly n powers of $(i+1)$ and only one M^{-2i}. This gives rise to a renormalon factorial by

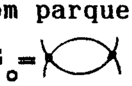
$$\sum_{i=0}^{\infty} (i+1)^n M^{-2i} \sim K^n n! \tag{21}$$

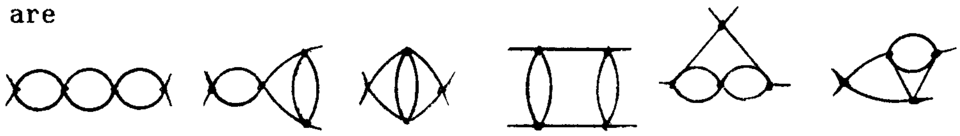
There is a reorganized version of the perturbation expansion that allows one to separate renormalon effects from instanton effects. One takes the ordinary perturbation theory expansion, decomposes covariances into their energy scales (12) and resums the "useless" parts of the counterterms. The result is an expansion similar to ordinary perturbation theory except that

(a) a subdiagram is renormalized only if it is superficially divergent and its internal lines are of higher scale than its external lines (as in the case $j>i$ above).

(b) each vertex is equipped with a "running" coupling constant. This means that, for example, the factor of g that is normally associated with the vertex is replaced by a factor g_i that depends on the energy scales of the lines ending at that vertex. This g_i is g plus contributions from the resummed useless counterterms and is essentially the coupling constant of scale i of the renormalization group.

In [MNRS] it is shown that if one replaces the "running coupling constant" by the real coupling constant g in the above expansion, i.e. one simply drops the useless counterterms, the Borel transform of the result is indeed analytic in the disc predicted by the Lipatov method.

One can get a rough idea of the effect of the running coupling constant as follows. The dominant contributions to g_i come from parquet graphs. These are graphs formed by taking the graph $G_0=$ , replacing in all possible ways one vertex by another G_o and repeating any number of times. For example all possible parquet graphs of order 4 are



The contribution to g_i of the sum of all such graphs is a geometric series

$$g_i \sim g[1 + g\beta_2 i + \ldots + (g\beta_2 i)^n + \ldots]. \qquad (22)$$

Here β_2 is a numerical constant that is well known to renormalization group people. Under our conventions its value is $9/(2\pi^2)$. We remark in passing that this series sums to $g/(1-g\beta_2 i)$ so that if g is negative g_i decays like 1/i for large i. This is the asymptotic freedom of the negative coupling Φ^4_4 model. Taking the Borel transform of (22) gives

$$B\{g_i\}(t) \sim t + t^2\beta_2 i/2 + \ldots + t(t\beta_2 i)^n/(n+1)! + \ldots . \qquad (23)$$
$$= [e^{t\beta_2 i} - 1]/\beta_2 i$$

Hence we have the introduction of exponentially growing factors $\exp[t\beta_2 i]$ into an expansion which, without the running coupling constants, has only exponentially decaying factors. (Recall (18).) The renormalon singularity occurs when the Borel parameter t gets big enough that the growth rate equals the decay rate i.e. when $t=2/\beta_2=4\pi^2/9$. The "extra" factor of 2 is there because symmetry under interchange of x and y improves the M^{i-j} of (18) to $M^{2(i-j)}$. By way of comparison the instanton singularity occurs at $t=-(3/2)(4\pi^2/9)$ in Φ^4_4.

Of course we have left out many details. It is not too hard to show that the 6-point euclidean green's function (for example) in Φ^4_4 is analytic at least to the position of the expected first renormalon singularity [FR2]. On the other hand it appears rather difficult to show that all the infinitely many terms containing the singularity do not cancel out. At the present time about the best that we can say is that it appears feasible to prove the existence of a renormalon singularity in a vector Φ^4_4 model when the number N of components of the vector is very large. In this case there is a small parameter, 1/N, which suppresses all but a finite number of the singularity bearing terms. In addition the Lipatov singularity moves very far from the origin and does not need to be treated carefully.

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