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BRAIDED GROUPS (*)

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ABSTRACT We prove a highly generalized Tannaka-Krein type reconstruction theorem for a monoidal category $\mathcal{C}$ functored by $F : \mathcal{C} \to \mathcal{V}$ to a suitably cocomplete rigid quasitensor category $\mathcal{V}$. The generalized theorem associates to this a bialgebra or Hopf algebra $\mathrm{Aut}\,(\mathcal{C}, F, \mathcal{V})$ in the category $\mathcal{V}$. As a corollary, to every cocompleted rigid quasitensor category $\mathcal{C}$ is associated $\mathrm{Aut}\,(\mathcal{C}) = \mathrm{Aut}\,(\mathcal{C}, \mathrm{id}, \bar{\mathcal{C}})$. It is braided-commutative in a certain sense and hence analogous to the ring of "co-ordinate functions" on a group or supergroup, i.e., a "braided group". We derive the formulae for the transmutation of an ordinary dual quasitriangular Hopf algebra into such a braided group. More generally, we obtain a Hopf algebra $B(A_1, f, A_2)$ (in a braided category) associated to an ordinary Hopf algebra map $f : A_1 \to A_2$ between ordinary dual quasitriangular Hopf algebras A_1, A_2.

1 INTRODUCTION

In [17, Sec. 4] we proved a highly generalized Tannaka-Krein type reconstruction theorem of the following form. If $F : \mathcal{C} \to \mathcal{V}$ is any monoidal functor from a rigid monoidal category $\mathcal{C}$ to a rigid quasitensor category $\mathcal{V}$, and assuming certain representability conditions are satisfied, then there is a Hopf algebra in the category $\mathcal{V}$ such that F factors through the forgetful functor from the modules of the Hopf algebra in $\mathcal{V}$. We gave a construction for the universal object with this property, denoted $\mathrm{Aut}\,(\mathcal{C}, F, \mathcal{V})$. The theorem has many applications, particularly in physics where the representability assumptions tend to be

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satisfied. In this paper we want to give a more precise mathematical version of [17, Sec. 4] in a dual form with comodules rather than modules. In this form the representability assumptions can be satisfied by cocompleteness requirements on $\mathcal{V}$. In fact, the situation here is quite analogous to the case over Vec. This dual formulation is announced in [16] and applied in [14]. Nevertheless, it is necessary to give the details of the proofs of [17] in this dual form and we do so here.

The definitions are recalled in the Preliminaries. Briefly, a braided monoidal or "quasitensor" category means a monoidal category which is commutative up to isomorphism. The commutativity isomorphism or quasisymmetry, Ψ, need not have square one. Such categories occur in a wide variety of contexts. They were first formally introduced into category theory in [8]. In the case $\mathcal{V} = Vec$ we have the familiar Tannaka-Krein type reconstructions as in [27][3] and (in the Hopf algebra case) [30]. See also [31]. It is clear that this would have no problem generalizing to $\mathcal{V}$ any suitable rigid symmetric monoidal or "tensor" category (because only general facts about Vec are used): the novel ingredient of [17, Sec. 4] was to take this further to the braided monoidal case where we must choose carefully between Ψ and Ψ^{-1} in $\mathcal{V}$. With the correct choices it was found that the construction does not get "tangled up" in $\mathcal{V}$. If $\mathcal{C}$ is also braided monoidal then $\mathrm{Aut}\,(\mathcal{C}, F, \mathcal{V})$ is quasitriangular in a sense that generalizes the definition of an ordinary quasitriangular Hopf algebra or "quantum group" in Vec defined by Drinfeld[4]. We will find the same below: we can allow $\mathcal{V}$ to be braided monoidal and if $\mathcal{C}$ is also braided monoidal the resulting object is a dual quasitriangular Hopf algebra in the category $\mathcal{V}$.

This is the origin of the Tannaka-Krein type theorem that we obtain. It is quite different in motivation and goal from the existing Tannaka-Krein reconstruction theorem of Pareigis[25]. In the framework there the author has rather strong conditions on $\mathcal{C}$ (it is a $\mathcal{V}$-category) sufficient to ensure that $\mathcal{C}$ is identified as the comodules of the reconstructed Hopf algebra in $\mathcal{V}$. This is a more classical Tannaka-Krein theorem. By contrast in our setting $\mathcal{C}$ is *not* typically identified with the comodules of $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$; the latter is usually much bigger. Instead our generalization slightly abuses Tannaka-Krein ideas to obtain a certain universal Hopf algebra for a pair $\mathcal{C} \to \mathcal{V}$ even when $\mathcal{C}$ is not at all the comodules of any Hopf algebra in $\mathcal{V}$. The situation is something like that of the isometry group of a Riemannian manifold: this is defined for all Riemannian manifolds though it contains most of the information about the metric only in the case when the Riemannian manifold is a group or homogeneous space. More generally, it could be trivial. In the same way $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ is a natural and useful object associated to the pair though it could be trivial if $\mathcal{C}$ is far from the representations of a Hopf algebra.

On the other hand it would certainly be an interesting task to generalize the more classical Tannaka-Krein theorem of Pareigis to the case when $\mathcal{V}$ is braided (rather than symmetric as in [25]) and $\mathcal{C}$ is of the special type that could arise as the comodules of a Hopf algebra in $\mathcal{V}$. Such a theorem would be useful if many such Hopf algebras in braided categories were known. We leave this for further work. In the present paper we are concerned with something else, namely a construction that generates Hopf algebras in braided categories in the first place. For example, the braided monoidal categories $\mathcal{C}$ which arise in general low-dimensional algebraic quantum field theories are very general and cannot be expected to be the comodules or modules of any kind of Hopf algebra except in very special cases. Nevertheless, our theorem does provide such an object, which plays the role of "internal symmetry group" of the quantum field theory[12], though it cannot in general capture all the structure of the quantum field theory. This is the physical motivation behind our results[12].

A main achievement of our resulting theorem in [17] and below is that it is now general enough to be applied in the case when we are given a rigid braided monoidal category $\mathcal{C}$ with no functor at all! Basically, just take $\mathcal{V} = \mathcal{C}$ with the identity functor. More precisely, we take here $\mathcal{V} = \bar{\mathcal{C}}$ a cocompletion over $\mathcal{C}$. This therefore associates to any rigid braided monoidal category $\mathcal{C}$ a Hopf algebra $\mathrm{Aut}\,(\mathcal{C})$. In this case $\mathrm{Aut}\,(\mathcal{C})$ is dual quasitriangular with a trivial dual quasitriangular structure: it is therefore "commutative" in a certain sense. Such a situation can be called *braided commutative* and in this case the Hopf algebra $\mathrm{Aut}\,(\mathcal{C})$ is like the ring of functions on a "group", i.e. a *braided group.* This is the kind of structure explored (in a dual form) in [12] in connection with quantum field theory in low dimensions. It should be mentioned that the precise notion of "braided-commutativity" that arises here as a property of $\mathrm{Aut}(\mathcal{C})$ is not however, described intrinsically in terms of the Hopf algebra itself but rather in terms of a pair consisting of a Hopf algebra in the braided monoidal category and a subcategory of comodules with respect to which the Hopf algebra behaves as a commutative one. This is a useful notion even for ordinary Hopf algebras (weaker than the usual one) and is so far the only one that extends well to a general braided monoidal category where $\Psi^2 \neq \mathrm{id}$. We describe it in detail in Section 3.

As a purely algebraic application, every dual quasitriangular Hopf algebra A gives such a braided group $\underline{A}$ in the category ${}^A\mathcal{M}$. Here ${}^A\mathcal{M}$ denotes comodules. This is the main result of Section 4. Similarly if $f: A_1 \to A_2$ is a Hopf algebra map between dual quasitriangular Hopf algebras, there is an induced functor $F: {}^{A_1}\mathcal{M} \to {}^{A_2}\mathcal{M}$. Then there is an associated dual quasitriangular Hopf algebra $B(A_1, f, A_2)$ in ${}^{A_2}\mathcal{M}$, i.e., a braided quantum group. This application is also explained in Section 4. We conclude with a simple example.

The results in this last section have important consequences for the theory of ordinary (dual) quasitriangular Hopf algebras, independently of our original context in category theory. Roughly speaking, $\underline{A}$ is like a "linearized" version of A in which the original (dual) quantum group is viewed in new "co-ordinates" (new category) where it has better properties. This is something like choosing geodesic co-ordinates for a metric: by moving to co-ordinates determined by the metric itself we make the metric locally linear. In the same way, by moving to a category determined by A itself we make it more commutative with better algebraic properties. Many theorems familiar to Hopf algebraist for ordinary commutative Hopf algebras now hold directly (in the braided category) for $\underline{A}$. After working them out, we can then change "co-ordinates" back to obtain results about ordinary Hopf algebras. An application of such ideas to the theory of ordinary Hopf algebras is in [20]. The results of Section 4 thus provide a new tool for algebraists and it is hoped that they will be of interest even for readers with no interest in category-theory for its own sake. For such algebraically-minded readers we have included an appendix that derives the most important of the formulae for $\underline{A}$ by purely algebraic (category-theory free) methods. Nothing in the main body of the paper depends on this appendix.

Finally, I note that Hopf algebras in braided monoidal categories of ribbon type, and some of their properties have recently been studied in [11]. Some relations between this and the present work will be explored elsewhere.

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PRELIMINARIES

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and functorial isomorphisms $\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ for all objects X, Y, Z, and a unit object $\underline{1}$ with functorial isomorphisms $l_X : X \to \underline{1} \otimes X, r_X : X \to X \otimes \underline{1}$ for all objects X. The Φ should obey a well-known pentagon coherence identity while the l and r obey triangle identities of compatibility with Φ. See [10].

A monoidal category $\mathcal{C}$ is *rigid* if for each object X, there is an object X^* and functorial morphisms $\mathrm{ev}_X : X^* \otimes X \to \underline{1}$, $\pi_X : \underline{1} \to X \otimes X^*$ such that

$$X \cong \underline{1} \otimes X \xrightarrow{\pi \otimes \mathrm{id}} (X \otimes X^*) \otimes X \cong X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} X \otimes \underline{1} \cong X \tag{1}$$

$$X^* \cong X^* \otimes \underline{1} \xrightarrow{\mathrm{id} \otimes \pi} X^* \otimes (X \otimes X^*) \cong (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \underline{1} \otimes X^* \cong X^* \tag{2}$$

compose to id_X and id_{X^*} respectively. We shall also sometimes have recourse to a *quasisymmetry* or "braiding" Ψ. This is a collection of functorial isomorphisms $\Psi_{X,Y} : X \otimes Y \to Y \otimes X$ obeying two hexagon coherence identities with Φ. If we suppress writing Φ, l, r explicitly then these take the form

$$\Psi_{X,Y \otimes Z} = \Psi_{X,Z} \circ \Psi_{X,Y}, \quad \Psi_{X \otimes Y,Z} = \Psi_{X,Z} \circ \Psi_{Y,Z},$$

while identities such as $\Psi_{X,\underline{1}} = \mathrm{id} = \Psi_{\underline{1},X}$ can be deduced. A monoidal category with such a structure is called *braided monoidal.* These were introduced in a purely category-theoretic context in [8], and occur naturally in connection with low dimension topology[5], as well as more recently in the context of quantum groups [19, Sec. 7] where they were called "quasitensor" categories. If $\Psi^2 = \mathrm{id}$ then one of the hexagons is superfluous and we have an ordinary symmetric monoidal category or "tensor" category. Our main results below arose as a generalization of classical Tannaka-Krein work such as in [3] and our proofs use similar techniques. Likewise, our notations $\otimes, \Phi, \underline{1}, l, r, \Psi$ follow the notations for the symmetric monoidal or tensor case as used there. Finally, a monoidal functor between rigid monoidal categories is one that respects $\otimes$ in the form $c_{X,Y} : F(X) \otimes F(Y) \cong F(X \otimes Y)$ for functorial isomorphisms $c_{X,Y}$, as well as Φ, l, r. It then also respects $*$ up to isomorphism.

Let k denote a field. Most of what we say also works over a commutative ring (by using projective modules). We denote by ${}_k\mathcal{M}$ the category of modules over k. This is a symmetric monoidal or tensor category with the usual tensor product of k-modules (e.g. of vector spaces). For another example, ${}_k\mathrm{Super}\mathcal{M}$ is the category of $\mathbb{Z}_2$-graded k-modules. These form a symmetric monoidal or tensor category with the usual tensor product of modules, the usual associativity of modules, but a new symmetry $\Psi_{V,W} : V \otimes W \to W \otimes V$ defined by $\Psi(v \otimes w) = (-1)^{|v||w|} w \otimes v$ on elements v, w homogeneous of degree $|v|, |w|$. The finite-dimensional versions $\mathcal{C} = {}_k\mathcal{M}^{f.d.}, \mathcal{C} = {}_k\mathrm{Super}\mathcal{M}^{f.d.}$ are examples of rigid symmetric monoidal or tensor categories (i.e. have $\Psi^2 = \mathrm{id}$).

Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra over k. We use the axioms of Drinfeld[4]. It is well known that the category of H-modules (representations of H) form a braided monoidal category, ${}_H\mathcal{M}$. See [19, Sec. 7] for an early treatment. If we limit attention to finite-dimensional modules, $\mathcal{C} = {}_H\mathcal{M}^{f.d.}$ is a rigid braided monoidal category. The H-module structure on the tensor product is determined by the comultiplication or "coproduct" $\Delta : H \to H \otimes H$. Writing this as $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ the action of h on $(v \otimes w) \in V \otimes W$ is $h \triangleright (v \otimes w) = \sum h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w$. Here $\triangleright$ denotes the actions. The quasisymmetry is given by $\Psi_{V,W}(v \otimes w) = \sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v$ where we write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. The dual object V^* is given by V^* as a linear space and action $(h \triangleright f)(v) = f((Sh) \triangleright v)$ for all $v \in V, f \in V^*$. Here S is the antipode of H.

We now describe the less familiar dual setting. A dual quasitriangular Hopf algebra

$(A,\mathcal{R})$ is a Hopf algebra A and $\mathcal{R}\in(A\otimes A)^*$ obeying some obvious axioms obtained by dualizing Drinfeld's. This was explained probably for the first time in [15, Sec. 4] and has since then also been used by various other authors. For the record, the axioms are, explicitly,

$$\mathcal{R}(ab\otimes c)=\sum\mathcal{R}(a\otimes c_{(1)})\mathcal{R}(b\otimes c_{(2)}) \tag{3}$$

$$\mathcal{R}(a\otimes bc)=\sum\mathcal{R}(a_{(1)}\otimes c)\mathcal{R}(a_{(2)}\otimes b) \tag{4}$$

$$\sum b_{(1)}a_{(1)}\mathcal{R}(a_{(2)}\otimes b_{(2)})=\sum\mathcal{R}(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)} \tag{5}$$

for all $a,b,c\in A$. $\mathcal{R}$ should be invertible in the convolution algebra.

A right A-comodule V is a vector space V and a linear structure map $\beta_V:V\to V\otimes A$ required to obey $(\beta_V\otimes\mathrm{id})\circ\beta_V=(\mathrm{id}\otimes\Delta)\circ\beta_V$ and $(\mathrm{id}\otimes\epsilon)\circ\beta_V=\mathrm{id}$. We often write such comodules in the standard notation $\beta_V(v)=\sum v^{(\bar{1})}\otimes v^{(\bar{2})}$. The category of right A-comodules is denoted ${}^A\mathcal{M}$ and is a braided monoidal one. The finite-dimensional comodules $\mathcal{C}={}^A\mathcal{M}^{f.d.}$ form a rigid braided monoidal category. Explicitly, the tensor product of comodules is determined by the algebra structure. Here

$$\beta_{V\otimes W}=(\mathrm{id}\otimes\cdot)\circ\Psi^{Vec}_{A,W}\circ(\beta_V\otimes\beta_W). \tag{6}$$

Here Ψ^{Vec} is the symmetry or usual twist map in ${}_k\mathcal{M}$. The quasisymmetry is defined by

$$\Psi_{V,W}=(\Psi^{Vec}_{V,W}\otimes\mathcal{R})\circ\Psi^{Vec}_{A,W}\circ(\beta_V\otimes\beta_W). \tag{7}$$

The duals require the existence of an antipode: β_{V^*} is defined by $\beta_{V^*}(f)=(f\otimes S)\circ\beta_V$. Here the left hand side in $V^*\otimes A$ is viewed as a map $V\to A$.

This completes the description of various examples of rigid symmetric monoidal and braided monoidal categories $\mathcal{C}$. Algebras can of course be defined in any monoidal category: an algebra in $\mathcal{C}$ is an object A of $\mathcal{C}$ with morphisms $A\otimes A\to A$, $\eta:\underline{1}\to A$ obeying axioms of associativity and unity (η plays the role of unit). Algebra maps are assumed unital. If $\mathcal{C}$ has direct sums, then these maps should respect them. If $\mathcal{C}$ has direct sums, is k-linear and the multiplication and unit are k-linear, we say that A is an algebra in $\mathcal{C}$ over k. We use the term "algebra" loosely to cover all these cases. Note that the quasisymmetry of $\mathcal{C}$ is relevant only if we want a tensor product algebra structure $A_1 \otimes A_2$. Namely,

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{\Psi_{A_2,A_1}} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{\cdot \otimes \cdot} A_1 \otimes A_2. \tag{8}$$

Likewise, coalgebras make sense in any monoidal category: a coalgebra in $\mathcal{C}$ is an object C of $\mathcal{C}$ with morphisms $\Delta : C \to C \otimes C$, $\epsilon : C \to \underline{1}$ obeying axioms of coassociativity and counity. Indeed, a monoidal category with reversed arrows is also a monoidal category, and a coalgebra is just an algebra in this. We write the comultiplication formally as $\Delta c = \sum c_{(1)} \otimes c_{(2)}$ in the usual way as in [29]. Coalgebra maps are assumed counital. Again, $\mathcal{C}$ needs to have a quasisymmetry if we want to have a tensor product coalgebra $C_1 \otimes C_2$ defined like (8) with arrows reversed,

$$C_1 \otimes C_1 \xrightarrow{\Delta \otimes \Delta} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow{\Psi_{C_1,C_2}} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2).$$

Hopf algebras make sense in any symmetric monoidal or braided monoidal category. The symmetric monoidal case is well known. For the braided monoidal case see [17][18]: A Hopf algebra in the category $\mathcal{C}$ is an object A of $\mathcal{C}$ which is both an algebra and a coalgebra in $\mathcal{C}$ and for which these are compatible in the sense (omitting Φ for brevity) of commutativity of

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{\cdot} & A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow \Delta \otimes \Delta & & & & \uparrow \cdot \otimes \cdot \\ A \otimes A \otimes A \otimes A & & \xrightarrow{\mathrm{id} \otimes \Psi_{A,A} \otimes \mathrm{id}} & & A \otimes A \otimes A \otimes A. \end{array} \tag{9}$$

The axioms for the antipode do not involve Ψ and are the same as the usual ones (as morphisms in $\mathcal{C}$). The idea of working over a general symmetric monoidal category $\mathcal{C}$ is nothing new, e.g.[25][24][28] as well as [23]. What is new in [17][18][12] is to generalize further to $\mathcal{C}$ rigid braided monoidal.

For any coalgebra or Hopf algebra A in a monoidal category $\mathcal{C}$, we define a (right) comodule in $\mathcal{C}$ as an object V in $\mathcal{C}$ and morphism β_V obeying the same axioms $(\beta_V \otimes \mathrm{id}) \circ \beta_V = (\mathrm{id} \otimes \Delta) \circ \beta_V$ and $(\mathrm{id} \otimes \epsilon) \circ \beta_V = \mathrm{id}$ (as morphisms and suppressing Φ). Instead of

$$
\begin{tikzcd}
V \arrow[r, "\beta^L_V"] \arrow[d, "\beta^L_V"'] & A\otimes V \arrow[r, "\Psi_{A,V}"] \arrow[d, "\beta^L_V"] & V\otimes A \arrow[r, "S"] \arrow[dr, "\beta^L_V"] & V\otimes A \arrow[r, "\beta^L_V"] & A\otimes V\otimes A \arrow[ddd, "\Psi_{A,V}"] \\
A\otimes V \arrow[r, "\Delta"] \arrow[dd, "\Psi_{A,V}"'] & A\otimes A\otimes V \arrow[rr, "\Psi_{A,A\otimes V}"] \arrow[d, "\Psi_{A\otimes A,V}"] & & A\otimes V\otimes A \arrow[ur, "\mathrm{id}\otimes S"] \arrow[dl, "\Psi_{A,V}"] & \\
& V\otimes A\otimes A \arrow[r, "\Psi_{A,A}"] & V\otimes A\otimes A \arrow[drr, "\mathrm{id}\otimes S"] & & \\
V\otimes A \arrow[ur, "\Delta"] \arrow[rr, "\Delta\circ S"'] & & V\otimes A\otimes A & & V\otimes A\otimes A \arrow[ll, "S\otimes\mathrm{id}"]
\end{tikzcd}
$$

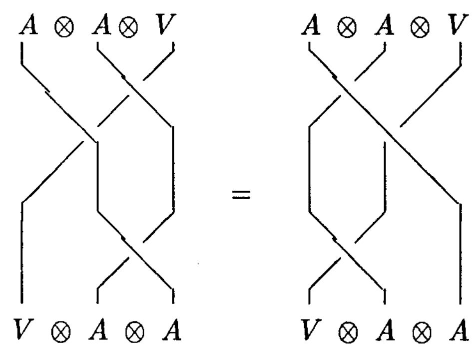
Figure 1: Diagram showing $(\mathrm{id}\otimes S)\circ\Psi\circ\beta^L_V$ is a right comodule. The middle cell is verified by (10)

Ψ^{Vec} in (6) we use now a quasisymmetry in $\mathcal{C}$. This makes the category of comodules of a Hopf algebra in a braided monoidal category $\mathcal{C}$ into a monoidal one. Likewise,

Lemma 1.1 *Let A be a Hopf algebra in a rigid braided monoidal category $\mathcal{C}$. Then the category of A-comodules in $\mathcal{C}$ is also rigid monoidal. We denote it ${}^A\mathcal{C}$.*

Proof The proof is entirely elementary. The dual comodule β_{V^*} is obtained by dualizing $\beta_V : V \to V\otimes A$ to a map $(\beta_V)^* : V^* \to A\otimes V^*$ (a left A-comodule) followed by Ψ_{A,V^*} and S. Explicitly, $(\beta_V)^* = (\mathrm{ev}_V\otimes\mathrm{id})\circ(\mathrm{id}\otimes\beta_V\otimes\mathrm{id})\circ\pi_V$ and $\beta_{V^*} = (\mathrm{id}\otimes S)\circ\Psi_{A,V^*}\circ(\beta_V)^*$. The verification of the comodule structures is straightforward. For example, that β_{V^*} as defined becomes a right comodule is part of a general fact which we spell out in Figure 1: any left comodule (V,β^L_V) is converted by $(\mathrm{id}\otimes S)\circ\Psi$ to a right comodule (we use this applied to $(V^*,(\beta_V)^*)$). The top left cell commutes by β^L a left comodule, while the bottom cell is an elementary identity $(S\otimes S)\circ\Psi_{A,A}\circ\Delta = \Delta\circ S$ that holds for any Hopf algebra A in a braided category. The central cell uses the hexagon coherence identities for Ψ while the lower left and upper central cells use functoriality of Ψ under the morphisms Δ,β^L respectively. Such braid relations and use of functoriality are general feature of almost all the proofs in Section 2 below, and will often be left to the reader. The π,ev in the category of comodules are π,ev in $\mathcal{C}$ viewed as intertwiners. The proof is equally straightforward.

Although we will usually leave the checking of braid relations and functoriality to the reader, it is worth mentioning some useful shorthand notation that the reader might like to use. This consists in writing all morphisms pointing downwards with $\Psi =$, $\Psi^{-1} =$ as braids and $\mathrm{ev} =$, $\pi =$. The identity object $\underline{1}$ along with l, r (and Φ) can be suppressed. We can also write morphisms corresponding to unary operators as marked vertical lines while binary ones (such as the multiplication for a Hopf algebra in the category) can be written as $\cup$-vertices, and cobinary ones (such as the comultiplication) as $\cap$-vertices. The coherence theorem for braided monoidal categories means that compositions of Ψ, Ψ^{-1} corresponding to the same braid compose to the same morphism. Functoriality of Ψ means that the braid crossings (i.e. Ψ, Ψ^{-1}) commute with any unary, binary, cobinary or other morphisms. As an example of the use of such notation, commutativity of the central cell in Figure 1 is easily checked from



(10)

This diagrammatic short-hand notation has been used by many people working with braided monoidal categories, including the author. It can be used quite systematically to present proofs, as for example in [21][20]. We note also some recent work of [9] for results leading to an ambient topological interpretation of such notation. In the present paper we try to keep such unfamiliar notation to a minimum in order to maintain continuity with more conventional techniques.

Finally, a dual quasitriangular structure $\mathcal{R}$ on a Hopf algebra A in $\mathcal{C}$ is a morphism $A \otimes A \to \underline{1}$ obeying axioms similar to the unbraided case (3)-(5). Twist maps implicit there must be replaced by Ψ, Ψ^{-1}. More subtle however, is the role of the opposite multiplication in the left-hand side of (4)-(5). The obvious $\cdot^{op} = \cdot \circ \Psi^{-1}_{A,A}$ does not work (it yields instead a Hopf algebra in $\mathcal{C}$ with the opposite braiding, $\tilde{\Psi}_{V,W} = \Psi^{-1}_{W,V}$). The correct notion of opposite multiplication is found below and is described in Theorem 2.8. Moreover, the condition $\cdot = \cdot^{op}$ is the braided-commutativity condition stated in Definition 3.1 below.

2 RECONSTRUCTION THEOREM

In this section we prove the main theorem.

Lemma 2.1 *Let U, W be any objects in a rigid braided monoidal category $\mathcal{V}$. Then there are isomorphisms*

$$\theta_V : \mathrm{Hom}(U^* \otimes W, V) \cong \mathrm{Hom}(W, U \otimes V)$$

for all objects V, functorial in V.

Proof If $f : U^* \otimes W \to V$ then consider $\theta_V(f) = (\mathrm{id} \otimes f) \circ \pi_U : W \to U \otimes U^* \otimes W \to U \otimes V$. Similarly if $g : W \to U \otimes V$ consider $\theta_V^{-1}(g) = (\mathrm{ev}_U \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) : U^* \otimes W \to U^* \otimes U \otimes V \to V$. These operations are inverse due to the rigidity axioms (1)-(2).

Theorem 2.2 *Let $\mathcal{C}$ be a monoidal category and $\mathcal{V}$ a rigid braided monoidal category cocomplete over $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{V}$ be a monoidal functor. Then there exists a $\mathcal{V}$-Hopf algebra, $A = \mathrm{Aut}\,(\mathcal{C}, F, \mathcal{V})$, such that F factorizes monoidally as $\mathcal{C} \to {}^A\mathcal{V} \to \mathcal{V}$. Here ${}^A\mathcal{V}$ is the category of A-comodules in $\mathcal{V}$ and the second factor is the forgetful functor. A is universal with this property. That is, if A' is any other such $\mathcal{V}$-Hopf algebra then there exists a unique $\mathcal{V}$-Hopf algebra map $A \to A'$ inducing a functor ${}^A\mathcal{V} \to {}^{A'}\mathcal{V}$ such that*

$$\begin{array}{ccc} & \mathcal{C} & \\ \swarrow & & \searrow \\ {}^A\mathcal{V} & \rightarrow & {}^{A'}\mathcal{V} \end{array}$$

commutes.

Proof The key step is to consider the functor $\widetilde{F} : \mathcal{V} \to Ens : V \mapsto \mathrm{Nat}(F, F_V)$. Here Nat denotes natural transformations and $F_V : \mathcal{C} \to \mathcal{V}$ is defined by $F_V(X) = F(X) \otimes V$. We first show that $\widetilde{F}$ is representable. I.e. there exists an object A in $\mathcal{V}$ and functorial isomorphisms

$$\theta_V : \mathrm{Hom}(A, V) \cong \mathrm{Nat}(F, F_V). \tag{11}$$

In view of Lemma 2.1 we need to take $A = \coprod_X F(X)^* \otimes F(X)/\sim$ where we must quotient by an equivalence relation $\sim$ to ensure that the collection $\mathrm{Hom}(F(X), F(X) \otimes V)$ corresponding to $\mathrm{Hom}(A, V)$ is natural. If $\mathcal{V}$ is concrete then the desired equivalence relation is therefore just

$$F(\phi)^* y^* \otimes x \sim y^* \otimes F(\phi) x$$

for all $\phi : X \to Y, x \in F(X), y^* \in F(Y)^*$. This is just the dual of the "coherent matrix-valued functions on $\mathcal{C}$" in [17, Secs 2,4]. There is a fancy way to say this for abstract categories: Let $A = \overset{\lim}{\leftarrow} A_{X,Y,\phi}$ be the colimit over $X, Y, \phi : X \to Y$ of $A_{X,Y,\phi}$ obtained as coequalizer of the diagram

$$\coprod_{X',Y'} F(Y')^* \otimes F(X') \underset{g_{X,Y,\phi}}{\overset{f_{X,Y,\phi}}{\rightrightarrows}} \coprod_{X''} F(X'')^* \otimes F(X'') \to A_{X,Y,\phi}.$$

Here $f_{X,Y,\phi} = F(\phi)^* \otimes \mathrm{id}$ and $g_{X,Y,\phi} = \mathrm{id} \otimes F(\phi)$ if $X' = X, Y' = Y$ and are otherwise zero. This is very similar to the well-known case when $\mathcal{V} = Vec$ [27, Sec. I.1.3.2.1] or [31]. Another name for this abstract construction for A is the "tensor product" of functors $F^* = * \circ F$ and F, see [10, Sec. IX.6] where it is identified as a coend. In the notation of [10, Sec. IX.6] we could write $A = F^* \otimes F = \int^X F^*(X) \otimes F(X)$. Our assumption of cocompleteness of $\mathcal{V}$ ensures that this tensor product exists. On the other hand, such fancy notations do not substitute for the actual construction. We have indicated it in detail because we shall later need to re-examine it more carefully in order to construct suitable colimits in the weaker setting of Section 3.

Once we have obtained an object A obeying (11) the rest follows in a standard way by repeated use of (11) without recourse to details of A. This is an important advantage of the present approach. Thus, let $\beta_X = \theta_A(\mathrm{id}_A)_X : F(X) \to F(X) \otimes A$ correspond in (11) to the identity $A \to A$. From this, θ_V is recovered as $\theta_V(f)_X = (\mathrm{id}\otimes f)\circ\beta_X$ for all $f \in \mathrm{Hom}(A,V)$. Let $\beta^2 \in Nat(F, F_{A\otimes A})$ be defined by $\beta^2_X = (\beta_X\otimes\mathrm{id})\circ\beta_X$. Then $\Delta = \theta^{-1}_{A\otimes A}(\beta^2)$ defines a comultiplication on A in the usual way. Now consider $F^n : \mathcal{C}^n \to \mathcal{V}$ sending $(X_1,\cdots,X_n)$ to $F(X_1)\otimes\cdots\otimes F(X_n)$, and the corresponding $\widetilde{F}^n : V \to \mathrm{Nat}(F^n, F^n_V)$.

Lemma 2.3 *The $\widetilde{F}^n$ are representable by the n-fold tensor products A^n. Explicitly the isomorphisms $\theta^n_V : \mathrm{Hom}(A^n,V)\cong\mathrm{Nat}(F^n,F^n_V)$ are given by,*

$$\theta^n_V(f)_{X_1,\cdots,X_n} = (\mathrm{id}\otimes f)\circ\Psi^{\mathcal{V}}_{A,F(X_2)\otimes\cdots\otimes F(X_n)}\circ\cdots\circ\Psi^{\mathcal{V}}_{A,F(X_n)}\circ(\beta_{X_1}\otimes\cdots\otimes\beta_{X_n}).$$

Proof This follows by induction from the case $n=1$ proven so far. If $\Theta \in \mathrm{Nat}(F^n, F^n_V)$ for $V \in \mathcal{V}$, we consider $\Theta'_{X_1,\cdots,X_{n-1}} = \Theta_{X_1,\cdots,X_{n-1},X_n}\circ(\mathrm{id}\otimes\pi_{X_n})$ as $\Theta' \in \mathrm{Nat}(F^{n-1}, F^{n-1}_{V'})$ where $V' = F(X_n)\otimes V\otimes F(X_n)^*$. Functoriality of Θ' follows from that of Θ. Hence by induction hypothesis we have $\Theta' = \theta^{n-1}_{V'}(f')$ for some $f' : A^{n-1} \to F(X_n)\otimes V\otimes F(X_n)^*$. Along the lines of Lemma 2.1, this corresponds also to $f'' : A^{n-1}\otimes F(X_n) \to F(X_n)\otimes V$ defined by $f'' = \mathrm{ev}_{F(X_n)}\circ(f'\otimes\mathrm{id})$. Let $W = A^{n-1}$ and regard $f''\circ{\Psi^{\mathcal{V}}_{W,F(X_n)}}^{-1}\circ\pi_W : F(X_n) \to F(X_n)\otimes V\otimes W^*$ as defining an element of $\mathrm{Nat}(F, F_{V\otimes W^*})$. Hence this expression is $\theta_{V\otimes W^*}(f''')$ for some $f''' : A \to V\otimes W^*$. Finally, again along the lines of Lemma 2.1 this corresponds to $f : W\otimes A \to V$ defined by $f = \mathrm{ev}_W\circ f'''\circ\Psi^{\mathcal{V}}_{W,A}$. Tracing through the inverses of these correspondences we have $f''' = f\circ{\Psi^{\mathcal{V}}_{W,A}}^{-1}\circ\pi_W$ and hence that $f'' = f\circ\Psi^{\mathcal{V}}_{W,F(X_n)}\circ\beta_{X_n}$. Now $\Theta_{X_1,\cdots,X_n} = \mathrm{ev}_{F(X_n)}\circ(\Theta'_{X_1,\cdots,X_{n-1}}\otimes\mathrm{id})$. We put $\Theta' = \theta^{n-1}_{V'}(f')$ into the right hand side and recognize $f'' = \mathrm{ev}_{F(X_n)}\circ(f'\otimes\mathrm{id})$. Using our expression for this and the braid relations in $\mathcal{V}$, we obtain finally $\Theta = \theta^n_V(f)$.

Lemma 2.4 *Let $\cdot : A\otimes A \to A$ be defined as the inverse image under θ^2_A of the element of $\mathrm{Nat}(F^2, F^2_A)$ defined by*

$$\theta^2_A(\cdot)_{X,Y} = c^{-1}_{X,Y}\circ\beta_{X\otimes Y}\circ c_{X,Y}.$$

Then the multiplication $\cdot$ is associative.

Proof From the definitions the element of $\mathrm{Nat}(F^3, F^3_A)$ corresponding to the two-fold multiplication $\cdot(\ \cdot\ ) : a \otimes b \otimes c \mapsto a\cdot(b\cdot c)$ is $\theta^3_A(\cdot(\ \cdot\ )) = c^{-1}_{Y,Z} \circ c^{-1}_{X,Y\otimes Z} \circ \beta_{X\otimes(Y\otimes Z)} \circ c_{X,Y\otimes Z} \circ c_{Y,Z}$. Likewise, $\theta^3_A((\ \cdot\ )\cdot) = c^{-1}_{X,Y} \circ c^{-1}_{X\otimes Y,Z} \circ \beta_{(X\otimes Y)\otimes Z} \circ c_{X\otimes Y,Z} \circ c_{X,Y}$. Comparing these the functoriality $\beta_{(X\otimes Y)\otimes Z}$ under the morphism $\Phi_{X,Y,Z}$ implies that

$$\begin{array}{ccc} F(X)\otimes F(Y)\otimes F(Z) & \xrightarrow{c^{-2}F(\Phi_{X,Y,Z})c^2} & F(X)\otimes F(Y)\otimes F(Z) \\ \theta^3_A(\cdot(\ \cdot\ ))\downarrow & & \downarrow\theta^3_A((\ \cdot\ )\cdot) \\ F(X)\otimes F(Y)\otimes F(Z)\otimes A & \xrightarrow{c^{-2}F(\Phi_{X,Y,Z})c^2} & F(X)\otimes F(Y)\otimes F(Z)\otimes A. \end{array}$$

That F is monoidal just says $c^{-2}F(\Phi)c^2 = \Phi^{\mathcal{V}}$. So the two natural transformations in $\mathrm{Nat}(F^3, F^3_A)$ corresponding to $\cdot(\ \cdot\ )$ and $(\ \cdot\ )\cdot$ are the same up to associativity in $\mathcal{V}$. Hence these maps are the same up to associativity in $\mathcal{V}$.

Lemma 2.5 $\Delta(ab) = \Delta(a)\Delta(b)$ *for all* $a, b \in A$. *Likewise for the counit.*

Proof The corresponding members of $\mathrm{Nat}(F^2, F^2_{A\otimes A})$ are as follows. The morphism $\Delta\circ\cdot$ corresponds on $F(X)\otimes F(Y)$ to $\Delta \circ c^{-1}_{X,Y} \circ \beta_{X\otimes Y} \circ c_{X,Y} = c^{-1}_{X,Y} \circ (\beta_{X\otimes Y}\otimes \mathrm{id}) \circ \beta_{X\otimes Y} \circ c_{X,Y} = (\mathrm{id}\otimes\cdot)\circ(\cdot\otimes\mathrm{id})\circ\Psi^{\mathcal{V}}_{A,F(Y)}\circ(\beta_X\otimes\beta_Y)\circ\Psi^{\mathcal{V}}_{A,F(Y)}\circ(\beta_X\otimes\beta_Y)$. The last expression is just the natural transformation corresponding to $(\cdot\otimes\cdot)\circ\Delta_{A\otimes A}$ as required. The counit is equally straightforward.

Lemma 2.6 ${}^A\mathcal{V}$ *is a monoidal category and there is a monoidal functor* $\beta : \mathcal{C} \to {}^A\mathcal{V}$ *precomposing with the forgetful functor to give* F. *It assigns underlying object* $F(X)$ *in* $\mathcal{V}$ *and comodule structure* β_X *to object* X *in* $\mathcal{C}$, *and assigns intertwiner* $F(\psi)$ *to morphism* ψ *in* $\mathcal{C}$. *Here* $c_{X,Y} : F(X)\otimes F(Y) \to F(X\otimes Y)$ *is an intertwiner inducing* $\beta_X\otimes\beta_Y \cong \beta_{X\otimes Y}$ *in* ${}^A\mathcal{V}$.

Proof The tensor product of comodules is of course determined from the multiplication in A in the usual way, cf (6). That c is indeed an intertwiner then reduces to the definition of $\cdot$ in Lemma 2.4. That β is monoidal (with intertwiner c) reduces to F monoidal.

Lemma 2.7 *(Universal Property of A). Let* A' *be a Hopf algebra in* $\mathcal{V}$ *such that the functor* F *factors through* ${}^{A'}\mathcal{V}$ *via the forgetful functor, and the functor* $\mathcal{C} \to {}^{A'}\mathcal{V}$ *is monoidal* *by intertwiner c. Then there is a map of $\mathcal{V}$-Hopf algebras $f : A \to A'$, uniquely determined, such that the diagram in Theorem 2.2 commutes.*

Proof The assumption is that of a monoidal functor $\beta' : \mathcal{C} \to {}^{A'}\mathcal{V}$, i.e. a certain collection of maps $\beta'_X : F(X) \to F(X) \otimes A'$. These are each comodule structures and form together an element of $\mathrm{Nat}(F, F_{A'})$. Let f be the inverse image under $\theta_{A'}$ of this element. Because each β'_X is a comodule structure, it is easy to see that f is a coalgebra map, $(f \otimes f) \circ \Delta = \Delta' \circ f$, $\epsilon = \epsilon' \circ f$. This is standard. The proof that f respects the multiplication is obtained by computing the elements of $\mathrm{Nat}(F^2, F^2_{A'})$ corresponding to $\cdot_{A'} \circ (f \otimes f)$ and $f \circ \cdot_A$ using the definition of the multiplication in A and that β' is monoidal by c. They coincide. Hence these two maps coincide.

Theorem 2.8 *In Theorem 2.2 suppose that $\mathcal{C}$ is a braided monoidal or "quasitensor" category with quasisymmetry Ψ. Let $\mathcal{R}$ be the inverse image under $\theta^2_{\underline{1}}$ of the element of* $\mathrm{Nat}(F^2, F^2)$ *given by*

$$\theta^2_{\underline{1}}(\mathcal{R})_{X,Y} = \Psi^{\mathcal{V}}_{F(Y),F(X)} \circ c^{-1}_{Y,X} \circ F(\Psi^{-1}_{Y,X}) \circ c_{X,Y}.$$

Then this makes $(A, \mathcal{R})$ a dual quasitriangular Hopf algebra in $\mathcal{V}$. That is,

$$\mathcal{R} \circ (\cdot \otimes \mathrm{id}) = (\mathcal{R} \otimes \mathcal{R}) \circ \Psi^{\mathcal{V}}_{A,A} \circ (\mathrm{id} \otimes \Delta)$$

$$\mathcal{R} \circ (\mathrm{id} \otimes \cdot^{op}) = (\mathcal{R} \otimes \mathcal{R}) \circ \Psi^{\mathcal{V}}_{A,A} \circ (\Delta \otimes \mathrm{id})$$

$$(\cdot^{op} \otimes \mathcal{R}) \circ \Delta_{A \otimes A} = (\mathcal{R} \otimes \cdot) \circ \Delta_{A \otimes A}.$$

Here $\Psi^{\mathcal{V}}_{A,A}$ act as in $\Delta_{A \otimes A}$ on the middle $A \otimes A$ of $A \otimes A \otimes A \otimes A$. $\cdot^{op}$ is uniquely determined by the condition

$$\tilde{\theta}^2_A(\cdot \circ {\Psi^{\mathcal{V}}_{A,A}}^{-1}) = \theta^2_A(\cdot^{op})$$

where $\tilde{\theta}^n$ is defined as in Lemma 2.3 but with the opposite braiding, $\tilde{\Psi}^{\mathcal{V}}_{V,W} = {\Psi^{\mathcal{V}}_{W,V}}^{-1}$ for all objects V, W.

$$
\begin{tikzcd}
& & F(X)\otimes F(Y)\otimes F(Z) \arrow[d, "\beta_Z"] & & \\
& & F(X)\otimes F(Y)\otimes F(Z)\otimes A \arrow[dl, "\beta_X\otimes\beta_Y"'] \arrow[dddr, "c^{-1}\circ\beta_{X\otimes Y}\circ c"] & & \\
& F(X)\otimes A\otimes F(Y)\otimes A\otimes F(Z)\otimes A \arrow[dl, "\Psi^{\mathcal{V}}_{A,F(Z)}"'] \arrow[dr, "\Psi^{\mathcal{V}}_{A,F(Y)}"] & & & \\
F(X)\otimes A\otimes F(Y)\otimes F(Z)\otimes A\otimes A \arrow[dr, "\Psi^{\mathcal{V}}_{A,F(Y)\otimes F(Z)}"'] & & F(X)\otimes F(Y)\otimes A\otimes A\otimes F(Z)\otimes A \arrow[dl, "\Psi^{\mathcal{V}}_{A\otimes A,F(Z)}"] \arrow[dr, "\cdot"] & & \\
& F(X)\otimes F(Y)\otimes F(Z)\otimes A\otimes A\otimes A \arrow[dr, "\cdot\otimes\mathrm{id}"'] & & F(X)\otimes F(Y)\otimes A\otimes F(Z)\otimes A \arrow[dl, "\Psi^{\mathcal{V}}_{A,F(Z)}"] & \\
& & F(X)\otimes F(Y)\otimes F(Z)\otimes A\otimes A \arrow[d, "\mathcal{R}"] & & \\
& & F(X)\otimes F(Y)\otimes F(Z) & &
\end{tikzcd}
$$

Figure 2: Diagram computing $\theta^3_{\underline{1}}(\mathcal{R}\circ(\cdot\otimes\mathrm{id}))$

Proof The proof that this $\mathcal{R}$ obeys the analog in $\mathcal{V}$ of the axioms (3)(4) as shown follows exactly the same strategy as above. Since it is one of the novel aspects of the work, we shall give the proof in some detail. We compute first the element of $\mathrm{Nat}(F^3, F^3)$ corresponding under $\theta^3_{\underline{1}}$ to the morphism $\mathcal{R}\circ(\cdot\otimes\mathrm{id})$. From the definition of $\theta^3_{\underline{1}}$, the braid relations in $\mathcal{V}$ and functoriality under $\cdot$ we obtain it on $F(X)\otimes F(Y)\otimes F(Z)$ in the intermediate form $\mathcal{R}\circ\Psi^{\mathcal{V}}{}_{A,F(Z)}\circ\cdot\circ\Psi^{\mathcal{V}}{}_{A,F(Y)}\circ(\beta_X\otimes\beta_Y\otimes\beta_Z)$. The $\beta_X\otimes\beta_Y$ then combines with $\cdot$ to give $c^{-1}\circ\beta_{X\otimes Y}\circ c$ which combines with β_Z and $\mathcal{R}$ to give $c^{-1}\circ\theta^2_{\underline{1}}(\mathcal{R})_{X\otimes Y,Z}\circ c$. The proof of these steps is spelled out explicitly in Figure 2. The result appears as the clockwise path. The left hand cell uses the braid relations in $\mathcal{V}$ while the lower cell is functoriality.

The remaining computations will continue to use the same techniques. For the natural transformation corresponding to $(\mathcal{R}\otimes\mathcal{R})\circ\Psi^{\mathcal{V}}_{A,A}\circ(\mathrm{id}\otimes\Delta)$ we use the definition of Δ to turn β_Z to β^2_Z. After some braid relations in $\mathcal{V}$ we bring the outer β_Z past the right most $\mathcal{R}$ to obtain $\mathcal{R}\circ\Psi^{\mathcal{V}}{}_{A,F(Y)\otimes F(Z)}\circ\beta_Z\circ\mathcal{R}\circ\Psi_{A,F(Z)}\circ(\beta_X\otimes\beta_Y\otimes\beta_Z)$. The $\beta_Y\otimes\beta_Z$ with $\mathcal{R}$ contributes $\theta^2_{\underline{1}}(\mathcal{R})_{Y,Z}$. Commuting β_X past this, using $\Psi^{\mathcal{V}}{}_{F(X),F(Y)}$ (and later its inverse) to put β_X next to the remaining β_Z and combining with $\mathcal{R}$ contributes $\theta^2_{\underline{1}}(\mathcal{R})_{X,Z}$. In this

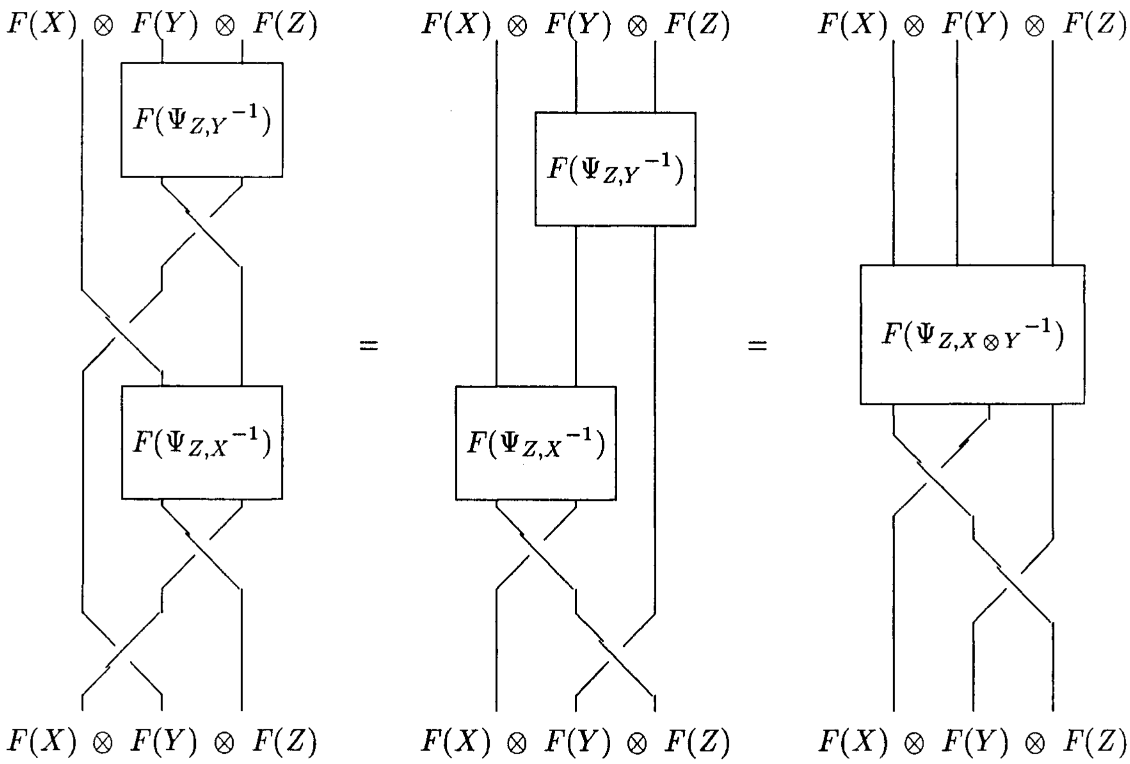


Figure 3: Diagram for verifying (12)

way, for the elements of $\mathrm{Nat}(F^3, F^3)$ to coincide we require

$$\Psi^{\mathcal{V}}{}_{F(X),F(Y)}{}^{-1} \circ \theta_{\underline{1}}^2(\mathcal{R})_{X,Z} \circ \Psi^{\mathcal{V}}{}_{F(X),F(Y)} \circ \theta_{\underline{1}}^2(\mathcal{R})_{Y,Z} = c^{-1} \circ \theta_{\underline{1}}^2(\mathcal{R})_{X\otimes Y,Z} \circ c. \tag{12}$$

This is solved by $\theta_{\underline{1}}^2(\mathcal{R})$ as stated because the $F(\Psi^{-1})$ can be brought together using functoriality of the $\Psi^{\mathcal{V}}$: when this is done (using of course the ubiquitous braid relations in $\mathcal{V}$) it reduces to F applied to the hexagon for Ψ. This is shown in Figure 3.

Likewise, applying $\theta_{\underline{1}}^3$ to the morphism $\mathcal{R} \circ (\mathrm{id} \otimes \cdot^{op})$ gives $\Psi^{\mathcal{V}}{}_{F(Z),F(Y)} \circ \theta_{\underline{1}}^2(\mathcal{R})_{X,Z\otimes Y} \circ \Psi^{\mathcal{V}}{}_{F(Z),F(Y)}{}^{-1}$. Here the braid relations in $\mathcal{V}$ work out provided we use $\cdot^{op}$ defined by $\theta_A^2(\cdot^{op})_{Y,Z} = \cdot \circ \Psi^{\mathcal{V}}{}_{A,A}{}^{-1} \circ \Psi^{\mathcal{V}}{}_{F(Z),A}{}^{-1} \circ (\beta_Y \otimes \beta_Z) = \Psi^{\mathcal{V}}{}_{F(Z),F(Y)} \circ c^{-1} \circ \beta_{Z\otimes Y} \circ c \circ \Psi^{\mathcal{V}}{}_{F(Z),F(Y)}{}^{-1}$. The computation of the element of $\mathrm{Nat}(F^3, F^3)$ corresponding to $(\mathcal{R} \otimes \mathcal{R}) \circ \Psi^{\mathcal{V}}_{A,A} \circ (\Delta \otimes \mathrm{id})$ is similar to that above. We thus obtain the requirement corresponding to (4) in $\mathcal{V}$ as

$$\theta_{\underline{1}}^2(\mathcal{R})_{X,Y} \circ \Psi^{\mathcal{V}}{}_{F(X),F(Y)}{}^{-1} \circ \theta_{\underline{1}}^2(\mathcal{R})_{X,Z} \circ \Psi^{\mathcal{V}}{}_{F(X),F(Y)}$$

$$= \Psi^{\mathcal{V}}{}_{F(Z),F(Y)} \circ c^{-1} \circ \theta_{\underline{1}}^2(\mathcal{R})_{X,Z\otimes Y} \circ c \circ \Psi^{\mathcal{V}}{}_{F(Z),F(Y)}{}^{-1}.$$

This is similarly solved by $\theta^2_{\underline{1}}(\mathcal{R})$ as stated, reducing to F applied to the other hexagon for Ψ^{-1}.

For the proof of the analog of (5) we follow the same steps as in the proof of Lemma 2.5. Thus the map on the right of (5) (now a morphism in $\mathcal{V}$) is $\cdot\circ(\mathcal{R}\otimes\mathrm{id})\circ\Delta_{A\otimes A}$. We compute the corresponding element under θ^2_A of $\mathrm{Nat}(F^2,F^2_A)$. Using the definition of Δ, the $\beta_X\otimes\beta_Y$ becomes $\beta^2_X\otimes\beta^2_Y$ as in Lemma 2.5. Using the outer $\beta_X\otimes\beta_Y$ with $\mathcal{R}$ gives $\theta^2_{\underline{1}}(\mathcal{R})_{X,Y}$. Commuting $\cdot$ past this and combining with the remaining $\beta_X\otimes\beta_Y$ gives $\theta^2_{\underline{1}}(\mathcal{R})_{X,Y}\circ c^{-1}\circ\beta_{X\otimes Y}\circ c$. Putting in its definition gives as result $\Psi^{\mathcal{V}}_{F(Y),F(X)}\circ c^{-1}_{Y,X}\circ F(\Psi^{-1}_{Y,X})\circ\beta_{X\otimes Y}\circ c_{X,Y}$. Likewise the map on the left in (5), namely $(\mathrm{id}\otimes\mathcal{R})\circ(\cdot^{op}\otimes\mathrm{id})\circ\Delta_{A\otimes A}$ corresponds under θ^2_A to $\Psi^{\mathcal{V}}_{F(Y),F(X)}\circ c^{-1}_{Y,X}\circ\beta_{Y\otimes X}\circ F(\Psi^{-1}_{Y,X})\circ c_{X,Y}$. These two natural transformations are the same by functoriality of $\beta\in\mathrm{Nat}(F,F_A)$ for the morphism $\Psi^{-1}_{Y,X}$. This completes the proof of the theorem.

Note that in the case $\mathcal{V}=Vec$ and $\mathcal{C}={}^A\mathcal{M}$ for an ordinary dual quasitriangular Hopf algebra $(A,\mathcal{R})$, this theorem returns not the original dual quasitriangular structure on A but the opposite one, $\mathcal{R}^{-1}\circ\Psi^{Vec}_{A,A}$. This is a feature of our conventions: in Theorem 2.8 there is also an opposite dual quasitriangular structure on A obeying similar axioms.

Proposition 2.9 *In Theorem 2.2 suppose that $\mathcal{C}$ is rigid. Then A has an antipode. Explicitly, $S:A\to A$ is defined as the inverse image under θ_A of the natural transformation*

$$\theta_A(S)_X=\mathrm{ev}_{F(X)}\circ{\Psi^{\mathcal{V}}_{A,F(X)^*}}^{-1}\circ d_X^{-1}\circ\beta_{X^*}\circ d_X\circ(\pi_{F(X)}\otimes\mathrm{id}).$$

Here $d_X:F(X)^*\to F(X^*)$ are the isomorphisms induced in a natural way by the property that F is monoidal, cf [30].

Proof We evaluate the elements of $\mathrm{Nat}(F,F_A)$ corresponding to the morphisms $\cdot(S\otimes\mathrm{id})\Delta$ and $\cdot(\mathrm{id}\otimes S)\Delta$ from $A\to A$. For the first of these the result is $\mathrm{ev}_{F(X)}\circ\beta_{X^*\otimes X}\circ\pi_{F(X)}$ (omitting c,d for brevity). The computation is spelled out in Figure 4. The hexagonal cell is the definition of $\theta_A(S)_X$. The central cell uses the braid relations as usual. But now functoriality of β under the morphism $\mathrm{ev}_X:X^*\otimes X\to\underline{1}$ implies that $\mathrm{ev}_{F(X)}\circ\beta_{X^*\otimes X}\circ\pi_{F(X)}$

$$
\begin{tikzcd}
& & F(X) \arrow[ll, "\beta_X"'] \arrow[d, "\beta_X"] \arrow[rr, "\pi_{F(X)}"] & & \\
F(X)\otimes A \arrow[d, "\Delta"'] & & F(X)\otimes A \arrow[dll, "\beta_X"'] \arrow[dr, "\pi_{F(X)}"] & & F(X)\otimes F(X)^*\otimes F(X) \arrow[dl, "\beta_X"] \arrow[ddddd, "d^{-1}c^{-1}\beta_{X^*\otimes X}cd"] \\
F(X)\otimes A\otimes A \arrow[dddd, "S\otimes \mathrm{id}"'] & & & F(X)\otimes F(X)^*\otimes F(X)\otimes A \arrow[dl, "d^{-1}\beta_{X^*}d"] & \\
& & F(X)\otimes F(X)^*\otimes A\otimes F(X)\otimes A \arrow[dl, "{\Psi^{\mathcal{V}}_{A,F(X)^*}}^{-1}"'] \arrow[dd, "\Psi^{\mathcal{V}}_{A,F(X)}"] & & \\
& F(X)\otimes A\otimes F(X)^*\otimes F(X)\otimes A \arrow[ddl, "\mathrm{ev}_{F(X)}"'] \arrow[dr, "\Psi^{\mathcal{V}}_{A,F(X)^*\otimes F(X)}"] & & & \\
& & F(X)\otimes F(X)^*\otimes F(X)\otimes A\otimes A \arrow[dll, "\mathrm{ev}_{F(X)}"] \arrow[drr, "\cdot"] & & \\
F(X)\otimes A\otimes A \arrow[drr, "\cdot"'] & & & & F(X)\otimes F(X)^*\otimes F(X)\otimes A \arrow[dll, "\mathrm{ev}_{F(X)}"] \\
& & F(X)\otimes A & &
\end{tikzcd}
$$

Figure 4: Diagram in proof of Proposition 2.9

is the same as $\mathrm{id}\otimes\eta \in \mathrm{Nat}(F, F_A)$. But this is just $\theta_A(\eta\circ\epsilon)$ since β_X is a comodule structure for A as seen above. Likewise, $\cdot(\mathrm{id}\otimes S)\Delta$ corresponds under θ_A to the natural transformation $\mathrm{ev}_{F(X)} \circ {\Psi^{\mathcal{V}}{}_{A,F(X)^*}}^{-1} \circ \beta_{X\otimes X^*} \circ \pi_{F(X)}$ (omitting c, d). Functoriality of β under the morphism $\pi_X : \underline{1} \to X\otimes X^*$ implies that this is also the same as the natural transformation corresponding to $\eta\circ\epsilon$.

3 MAIN COROLLARY: BRAIDED GROUPS

In this section we obtain as a corollary of the theorems above that every rigid braided monoidal category has a braided group of automorphisms $\mathrm{Aut}(\mathcal{C})$.

Definition 3.1 *A* braided group *is a pair $(A, \mathcal{O})$ where A is Hopf algebra in a braided monoidal category, $\mathcal{O}$ is a subcategory of A-comodules with respect to which A is braided-commutative in the sense cf.[12]*

$$(\mathrm{id}\otimes\cdot)\circ\beta_V = (\mathrm{id}\otimes\cdot)\circ Q_{V,A}\circ\Psi_{A,A}\circ\beta_V$$

on $V\otimes A$ for all right comodules β_V in $\mathcal{O}$. Here $Q_{V,A}=\Psi_{A,V}\circ\Psi_{V,A}$.

The notion of braided-commutativity introduced here is a weak one defined with respect to a class of comodules in the category. This is useful even for ordinary Hopf algebras: So long as the Hopf algebra coacts on comodules in the class, it behaves like a commutative one, i.e. like a group. Since our Hopf algebras live in a braided monoidal category, it is natural to denote the (weakly) braided-commutative ones as "braided groups". Some applications of this concept of commutativity with respect to a class are in [20] (in a dual setting of co-commutativity). In fact every Hopf algebra A in a braided category can be regarded as a braided group by taking for $\mathcal{O}$ the subcategory $\mathcal{O}(A)$ of A-comodules obeying the condition in Definition 3.1. The class $\mathcal{O}(A)$ thus measures the degree to which any A is braided-commutative. It is closed under $\otimes$, see [21] (in the dual setting). In practice we can specify any convenient subcategory $\mathcal{O}$. The idea behind weak commutativity then is that instead of limiting ourselves to declaring that a given Hopf algebra is commutative or not, we can systematically treat any one as being commutative in a limited context. This broader view of commutativity seems to be the one that generalizes most easily to braided monoidal categories.

Of course, more conventional intrinsic (not weak) conditions such as $\cdot=\Psi^{-1}_{A,A}\circ\cdot$ or $\cdot=\Psi_{A,A}\circ\cdot$ can also be imposed but neither of these is particularly natural in a truly braided monoidal category where $\Psi^2\neq\mathrm{id}$, i.e. there do not appear to be many non-trivial examples. On the other hand, in the weak form above there are many examples of A equipped with a non-trivial class $\mathcal{O}$. Indeed, the specific form of the definition of braided groups is motivated by considering dual quasitriangular Hopf algebras in braided monoidal categories as arising in Theorem 2.8 above.

Proposition 3.2 *Let $F:\mathcal{C}\to\mathcal{V}$ be a monoidal functor between rigid braided monoidal category in the setting of Section 2. Then $A=\mathrm{Aut}(\mathcal{C},F,\mathcal{V})$ is a braided group with respect to the image of $\mathcal{C}\to{}^A\mathcal{V}$ iff $\cdot=\cdot^{\mathrm{op}}$. This happens if $F:\mathcal{C}\to\mathcal{V}$ respects the quasisymmetries in the form $c^{-1}\circ F(\Psi)\circ c=\Psi^{\mathcal{V}}$.*

Proof (i) $\cdot^{\rm op}$ was defined in Theorem 2.8 (it is a second Hopf algebra structure on $A = \mathrm{Aut}\,(\mathcal{C},F,\mathcal{V})$). We see that $\cdot = \cdot^{\rm op}$ corresponds to equality of the natural transformations $\theta^2_A(\cdot)_{X,Y}, \widetilde{\theta}^2_A(\cdot\circ{\Psi^{\mathcal{V}}_{A,A}}^{-1})_{X,Y}$. These are morphisms $F(X)\otimes F(Y)\to F(X)\otimes F(Y)\otimes A$. As in Lemma 2.3 we can write these equivalently as two morphisms $f', g' : F(X)\to F(X)\otimes V$ where $V = F(Y)\otimes A\otimes F(Y)^*$. Explicitly $f' = \cdot\circ\Psi^{\mathcal{V}}_{A,F(Y)}\circ\beta_Y\circ(\mathrm{id}\otimes\pi_{F(Y)})\circ\beta_X$ and $g' = \cdot\circ{\Psi^{\mathcal{V}}_{A,A}}^{-1}\circ{\Psi^{\mathcal{V}}_{F(Y),A}}^{-1}\circ\beta_Y\circ(\mathrm{id}\otimes\pi_{F(Y)})\circ\beta_X$. Our original natural transformations are recovered by applying $\mathrm{ev}_{F(Y)}$ to these, hence $\cdot = \cdot^{\rm op}$ iff $f' = g'$ for all X, Y. On the other hand we see that $f' = \theta_V(f)_X$ and $g' = \theta_V(g)_X$ for certain $f, g : A \to F(Y)\otimes A\otimes F(Y)^*$ and we have equality iff $f = g$. This is $\cdot\circ\Psi^{\mathcal{V}}_{A,F(Y)}\circ\beta_Y\circ(\mathrm{id}\otimes\pi_{F(Y)}) = \cdot\circ{\Psi^{\mathcal{V}}_{A,A}}^{-1}\circ{\Psi^{\mathcal{V}}_{F(Y),A}}^{-1}\circ\beta_Y\circ(\mathrm{id}\otimes\pi_{F(Y)})$. Applying $\mathrm{ev}_{F(Y)}$ to both sides we have equivalently

$$\cdot\circ\Psi^{\mathcal{V}}_{A,F(Y)}\circ\beta_Y = \cdot\circ{\Psi^{\mathcal{V}}_{A,A}}^{-1}\circ{\Psi^{\mathcal{V}}_{F(Y),A}}^{-1}\circ\beta_Y$$

as morphisms $A\otimes F(Y)\to F(Y)\otimes A$. This is for all Y, i.e. for all comodules $(F(Y),\beta_Y)$ in the image of $\mathcal{C}\to{}^A\mathcal{V}$ in Lemma 2.6. After precomposing with the isomorphisms $\Psi^{\mathcal{V}}_{F(Y),A}$ one both sides and using the braid relations and functoriality of $\Psi^{\mathcal{V}}$ under β_Y, we obtain the relations in Definition 3.1. (ii) If F respects the quasisymmetry as stated then we see in Theorem 2.8 that $\theta^2_{\underline{1}}(\mathcal{R}) = \mathrm{id}$. In general it was given as the ratio of $\Psi^{\mathcal{V}}$ and $F(\Psi)$. $\mathcal{R}$ controls the degree of non-commutativity so when $\mathcal{R}$ is trivial in this way we can expect that $\cdot = \cdot^{\rm op}$. To see this we simply write out the natural transformations corresponding to $(\cdot^{\rm op}\otimes\mathcal{R})\circ\Delta_{A\otimes A} = (\mathcal{R}\otimes\cdot)\circ\Delta_{A\otimes A}$. These were computed in the proof of Theorem 2.8. When $\theta^2_{\underline{1}}(\mathcal{R})$ is trivial we obtain the condition $c^{-1}\circ\beta_{X\otimes Y}\circ c = \Psi^{\mathcal{V}}_{F(Y),F(X)}\circ c^{-1}\circ\beta_{Y\otimes X}\circ c\circ{\Psi^{\mathcal{V}}_{F(Y),F(X)}}^{-1}$. The left hand side is $\theta^2_A(\cdot)_{X,Y}$. The right hand side is $\Psi^{\mathcal{V}}_{F(Y),F(X)}\circ\cdot\circ\Psi^{\mathcal{V}}_{A,F(X)}\circ(\beta_Y\otimes\beta_X)\circ{\Psi^{\mathcal{V}}_{F(Y),F(X)}}^{-1}$. Using the braid relations and functoriality, this is just the same as $\widetilde{\theta}^2_A(\cdot\circ{\Psi^{\mathcal{V}}_{A,A}}^{-1})_{X,Y}$. This completes the proof of the proposition.

Finally, we note two variants of the work of Section 2 and Proposition 3.2. In the first variant we drop the assumption that $\mathcal{V}$ is cocomplete. In this case we do not automatically have that $A = \mathrm{Aut}(\mathcal{C},F,\mathcal{V})$ exists. Instead we suppose that A exists anyway in $\mathcal{V}$. If $\mathcal{V}$ is rigid we will still be able to apply Lemma 2.3 and all the remaining results of Section 2 and Proposition 3.2. If $\mathcal{V}$ is not rigid we can still proceed provided we establish the conclusion of Lemma 2.3 directly by some other means. In either case, if $\mathcal{C}$ is also rigid we obtain a Hopf algebra, if not we obtain only a bialgebra. This was the point of view taken in [17]. It is useful in actually computing examples and will be used in Section 4. For example, taking $F : \mathcal{C} \to \mathcal{C}$ as the identity functor we will compute some examples of the form $\mathrm{Aut}(\mathcal{C}, \mathrm{id}, \mathcal{C})$. From Proposition 3.2 these will necessarily be braided groups.

In the present section we drop the assumption that $\mathcal{V}$ is rigid but keep cocompleteness. This is a second variant but enables us to associate a braided group to any rigid braided monoidal category $\mathcal{C}$. This is because in general $\mathrm{Aut}(\mathcal{C}, \mathrm{id}, \mathcal{C})$ need not exist in $\mathcal{C}$. Instead, by working with a cocompletion $i : \mathcal{C} \to \bar{\mathcal{C}}$, we will be assured of existence in $\bar{\mathcal{C}}$. We concentrate on the cocompletion provided by the Yoneda embedding of any $\mathcal{C}$ in its presheaf category. Some related work is in [2]. For our limited purposes, however, it is easy enough to proceed directly by elementary arguments. $\mathcal{C}$ itself should be equivalent to a small category.

Corollary 3.3 *Let $\mathcal{C}$ be a rigid braided monoidal category and $\bar{\mathcal{C}}$ its cocompletion given by the Yoneda embedding $i : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C}, i, \bar{\mathcal{C}})$ exists in $\bar{\mathcal{C}}$ and is a braided group, i.e. braided-commutative in the sense of Definition 3.1. We call it the automorphism braided group of $\mathcal{C}$. $A = \mathrm{Aut}(\mathcal{C})$ is the universal object with the property that the forgetful functor ${}^{A}\bar{\mathcal{C}} \to \bar{\mathcal{C}}$ has a section (i.e. a functor $\mathcal{C} \to {}^{A}\bar{\mathcal{C}}$ composing with the forgetful functor to the identity functor).*

Proof The Yoneda embedding $i : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is given by

$$i(X) = \mathrm{Hom}(\ ,X), \quad i(\phi) = \phi\circ$$

for objects and morphisms of $\mathcal{C}$. Here $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the presheaf category of contravariant functors from $\mathcal{C}$ to Set and $\mathrm{Hom}(\ ,X)$ is the contravariant functor $Z \mapsto \mathrm{Hom}(Z, X)$ and $\mathrm{Hom}(\psi, X) = \circ\psi$ for $\psi : Z \to Z'$. It is known that we can take a cocompletion $\bar{\mathcal{C}}$ of the image of $\mathcal{C}$. Hence there is an object A (a certain contravariant functor) in $\bar{\mathcal{C}}$ such that $\mathrm{Hom}_{\bar{\mathcal{C}}}(A,V) = \mathrm{Nat}(i, i_V)$. Explicitly, the tensor product $i(X)\otimes V$ is defined by $(i(X)\otimes V)(Z) = V(X^*\otimes Z)$ for all $Z \in \mathcal{C}$. Here $V \in \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ can be any contravariant functor from $\mathcal{C}$ to Set. From Lemma 2.1 we note also that $(i(X)\otimes i(Y))(Z) = \mathrm{Hom}(X^*\otimes Z, Y) = \mathrm{Hom}(Z, X\otimes Y) = i(X\otimes Y)(Z)$. Now $\Theta \in \mathrm{Nat}(i, i_V)$ denotes a coherent family $\{\Theta_X : i(X) \to i(X)\otimes V\}$ i.e. defines a certain family $\{\Theta_{X,Z} : \mathrm{Hom}(Z,X) \to V(X^*\otimes Z)\}$ for all $X, Z \in \mathcal{C}$. Meanwhile, A and V are functors and an element of $\mathrm{Hom}(A,V)$ is a natural transformation between them. If we take A defined by the coequalizer $\coprod_{X,Y} i(Y^*)\otimes i(X) \rightrightarrows \coprod_X i(X^*)\otimes i(X) \to A$ then clearly any $f : A \to V$ corresponds to a coherent family of morphisms $f_X : i(X^*)\otimes i(X) \to V$. This means a coherent family $f_{X,Z} : \mathrm{Hom}(Z, X^*\otimes X) \to V(Z)$. The coequalizers are chosen so that coherent families of this sort are in one-to-one correspondence with coherent families of the sort $\Theta_{X,Z}$ by means of the rigidity of $\mathcal{C}$. We can similarly describe $\mathrm{Hom}(A\otimes A, V)$ etc as coherent families and by arguments of this type we can verify the conclusion of Lemma 2.3 explicitly. Hence the remaining results of Section 2 apply. That $\mathcal{C}$ is rigid braided monoidal also means that the resulting $\bar{\mathcal{C}}$-Hopf algebra A has an antipode (corresponding to rigidity of $\mathcal{C}$) and a dual quasitriangular structure (corresponding to $\mathcal{C}$ braided monoidal). As explained in Proposition 3.2, the dual quasitriangular structure is given by the ratio of the braidings in $\mathcal{C}$ and $\bar{\mathcal{C}}$. This is trivial and hence A is a braided group with respect to the class stated.

Clearly, these conclusions also apply to any other cocompletion for which the canonical inclusion is well-behaved. Among these our $\mathrm{Aut}(\mathcal{C})$ is universal. This is because $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the free cocomplete extension of $\mathcal{C}$ as a category, see for example [22, Sec. 1.2]. In particular, for any $F : \mathcal{C} \to \mathcal{V}$ to a cocomplete category $\mathcal{V}$ we have an induced functor $p : \bar{\mathcal{C}} \to \mathcal{V}$. Here $p \circ i = F$ and p preserves all small colimits. Hence if $\mathcal{V}$ is another cocompletion of $\mathcal{C}$ we will have $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V}) = p(\mathrm{Aut}(\mathcal{C}))$ according to our explicit coequalizer descriptions of both sides.

Another observation, useful for computations is as follows. If i factors through a braided monoidal category $\mathcal{V}$ as $\mathcal{C} \overset{F}{\hookrightarrow} \mathcal{V} \overset{j}{\hookrightarrow} \bar{\mathcal{C}}$ (j a full embedding) and $\mathrm{Aut}(\mathcal{C})$ happens to lie in the image of $\mathcal{V}$, then $\mathrm{Aut}\,(\mathcal{C},F,\mathcal{V})$ must exist, and its image coincides with $\mathrm{Aut}\,(\mathcal{C})$ up to unique isomorphism. This is because an element of $\mathrm{Nat}(F,F_V)$ is mapped by j to an element of $\mathrm{Nat}(i,i_{j(V)})$. This corresponds to a unique morphism $\mathrm{Aut}\,(\mathcal{C})\to j(V)$ and hence to a unique morphism in $\mathcal{V}$. Here $\mathrm{Aut}\,(\mathcal{C},F,\mathcal{V})$ is characterized up to isomorphism by this representing property. For example, if $\mathcal{C}$ has finite coproducts, we may consider for $\mathcal{V}$ the category $\mathrm{Ind}\,(\mathcal{C})$ of [6].

Finally, let us note that the constructions $\mathrm{Aut}\,(\mathcal{C},F,\mathcal{V})$ and $\mathrm{Aut}\,(\mathcal{C})$ are quite natural and have the obvious functorial properties. Assuming the data for the relevant automorphism Hopf algebras, we have

(i)
$$\begin{tikzcd} \mathcal{C}_1 \arrow[d, "f"'] \arrow[dr, "F_1"] & \\ \mathcal{C}_2 \arrow[r, "F_2"'] & \mathcal{V} \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \mathrm{Aut}\,(\mathcal{C}_1,F_1,\mathcal{V}) \arrow[d, "f_*"] \\ \mathrm{Aut}\,(\mathcal{C}_2,F_2,\mathcal{V}) \end{tikzcd}$$

(ii)
$$\begin{tikzcd} & \mathcal{V}_1 \arrow[d, "f"] \\ \mathcal{C} \arrow[ur, "F_1"] \arrow[r, "F_2"'] & \mathcal{V}_2 \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} f(\mathrm{Aut}\,(\mathcal{C},F_1,\mathcal{V}_1)) \\ \mathrm{Aut}\,(\mathcal{C},F_2,\mathcal{V}_2) \arrow[u, "f^*"] \end{tikzcd}$$

for all monoidal functors f which are compatible as shown (in the second case f should respect the two quasisymmetries also). The f_* and f^* are Hopf algebra morphisms constructed in the obvious way from the universal property in Theorem 2.2. Thus for (i), we compose f with the functor $\mathcal{C}_2\to{}^{\mathrm{Aut}\,(\mathcal{C}_2,F_2,\mathcal{V})}\mathcal{V}$ to obtain a functor that composes with the forgetful functor to give F_1. By the universal property for $\mathrm{Aut}\,(\mathcal{C}_1,F_1,\mathcal{V})$ we obtain a morphism f_*, uniqely determined. For (ii), apply f to the comodules in the image of $\mathcal{C}\to{}^{\mathrm{Aut}(\mathcal{C},F_1,\mathcal{V}_1)}\mathcal{V}_1$ with f to obtain a functor $\mathcal{C}\to{}^{f(\mathrm{Aut}(\mathcal{C},F_1,\mathcal{V}_1))}\mathcal{V}_2$. This composes with the forgetful functor to give F_2. By the universal property for $\mathrm{Aut}\,(\mathcal{C},F_2,\mathcal{V}_2)$ we obtain a morphism f^*, uniquely determined. Note that if either of the functors f is an equivalence then the corresponding f_* or f^* is an isomorphism. From this observation it also follows that if $f:\mathcal{C}_1\to\mathcal{C}_2$ is an equivalence of braided monoidal categories then $f(\mathrm{Aut}\,(\mathcal{C}_1))\cong\mathrm{Aut}\,(\mathcal{C}_2)$ after extending f to the cocompletions.

4 MAIN APPLICATION: TRANSMUTATION

We are now in a position to use the results of the preceding section to associate to any quantum group a braided group analog. This arises as the automorphism braided group in

Corollary 3.3 computed in the case $\mathcal{C} = {}^A\mathcal{M}^{f.d.}$. In practice, to make such computations it is not necessary to work with cocompletions. Instead, what we actually compute is $\mathrm{Aut}\,(\mathcal{C},\mathrm{id},\mathcal{C})$ or $\mathrm{Aut}\,(\mathcal{C},\mathrm{id},{}^A\mathcal{M})$ according to the first variant mentioned in the preceding section. We introduced this process of *transmutation* in [12, Sec. 3]. We obtain its dual formulation.

Theorem 4.1 *Let A be a dual quasitriangular Hopf algebra in the usual sense. Then there is a braided group $\underline{A}$ in the category ${}^A\mathcal{M}$ described as follows in terms of A. As a linear space and coalgebra, $\underline{A}$ coincides with A. The algebra structure and antipode are transmuted to*

$$a\underline{\cdot}b = \sum a_{(2)}b_{(3)}\mathcal{R}(a_{(3)}\otimes Sb_{(1)})\mathcal{R}(a_{(1)}\otimes b_{(2)}) \tag{13}$$

$$\underline{S}a = \sum Sa_{(2)}\mathcal{R}((S^2a_{(3)})Sa_{(1)}\otimes a_{(4)}). \tag{14}$$

$\underline{A}$ is an object in ${}^A\mathcal{M}$ by the adjoint right coaction

$$\beta_{\underline{A}}(a) = \sum a_{(2)}\otimes(Sa_{(1)})a_{(3)} = \sum a^{(\bar{1})}\otimes a^{(\bar{2})}. \tag{15}$$

As such, there is an action of the braid group on tensor powers of $\underline{A}$ defined by $\Psi_{\underline{A},\underline{A}}$ in (7). This makes $\underline{A}$ into a braided Hopf algebra. It is braided-commutative in the sense

$$b\underline{\cdot}a = \sum a^{(\bar{1})}\underline{\cdot}b_{(2)}{}^{(\bar{1})}\mathcal{R}(b_{(2)}{}^{(\bar{2})}\otimes a^{(\bar{2})}{}_{(2)})Q(b_{(1)}\otimes a^{(\bar{2})}{}_{(1)}). \tag{16}$$

This makes $\underline{A}$ into a braided group. Here $Q : A\otimes A\to k$ is the convolution product $\mathcal{R}_{21}*\mathcal{R}$, i.e., $Q(a\otimes b) = \sum\mathcal{R}(b_{(1)}\otimes a_{(1)})\mathcal{R}(a_{(2)}\otimes b_{(2)})$.

Proof We set $\mathcal{C} = {}^A\mathcal{M}^{f.d.}$. We concentrate for brevity on the case when A is finite-dimensional, taking $\mathcal{V} = \mathcal{C}$, i.e. we show that $\underline{A} = \mathrm{Aut}\,(\mathcal{C},\mathrm{id},\mathcal{C})$ exists and has the structure shown. For the infinite-dimensional case we use $\mathcal{V} = {}^A\mathcal{M}$ with much the same proof for $\underline{A} = \mathrm{Aut}\,(\mathcal{C},\mathrm{id},{}^A\mathcal{M})$. We show first (for either case) that $\underline{A}$ is indeed a representing object for the functor $\widetilde{F}$ in Lemma 2.3. In our case $F = \mathrm{id}$. In one direction, if $f\in\mathrm{Hom}(\underline{A},V)$ we define $\theta_V(f)_X = (1\otimes f)\circ\beta_X$. This is a morphism $X\to X\otimes V$ for every $X\in{}^A\mathcal{M}^{f.d.}$ when $\underline{A}$ has the adjoint coaction as shown. In the inverse direction, if $\Theta\in\mathrm{Nat}(F,F_V)$ we define $f = (\epsilon\otimes\mathrm{id})\circ\Theta_{A_R}$. Here A_R denotes A as a right comodule by the diagonal coaction $\beta_{A_R} = \Delta$. We must show that f is a morphism when viewed as a map $f : \underline{A} \to V$. To see this note the two useful identities

$$\sum \Theta(a_{(1)})\otimes a_{(2)} = \sum \Theta_{A_R}(a)^{(1)}{}_{(1)}\otimes\Theta_{A_R}(a)^{(2)(\bar{1})}\otimes\Theta_{A_R}(a)^{(1)}{}_{(2)}\Theta_{A_R}(a)^{(2)(\bar{2})}$$

$$\sum a_{(1)}\otimes\Theta_{A_R}(a_{(2)}) = \sum \Theta_{A_R}(a)^{(2)}{}_{(1)}\otimes\Theta_{A_R}(a)^{(1)}{}_{(2)}\otimes\Theta_{A_R}(a)^{(2)}$$

for all $a\in A$. Here $\Theta(a) = \sum\Theta(a)^{(1)}\otimes\Theta(a)^{(2)}$ is our explicit notation. The first identity is the statement that $\Theta_{A_R} : A_R \to A_R\otimes V$ is a morphism. The second is the statement that Θ is functorial under the morphisms $A_R \to A_R$ defined by $a\mapsto\sum f(a_{(1)})a_{(2)}$ for all $f\in A^*$. Then we compute

$$\begin{aligned}
f\circ\beta_{\underline{A}}(a) &= \sum \epsilon(\Theta(a_{(2)(1)})^{(1)})\Theta(a_{(2)(1)})^{(2)}\otimes(Sa_{(1)})a_{(2)(2)}\\
&= \sum \epsilon(\Theta(a_{(2)})^{(1)}{}_{(1)})\Theta(a_{(2)})^{(2)(\bar{1})}\otimes(Sa_{(1)})\Theta(a_{(2)})^{(1)}{}_{(2)}\Theta(a_{(2)})^{(2)(\bar{2})}\\
&= \sum \Theta(a_{(2)})^{(2)(\bar{1})}\otimes(Sa_{(1)})\Theta(a_{(2)})^{(1)}\Theta(a_{(2)})^{(2)(\bar{2})}\\
&= \sum \epsilon(\Theta(a_{(2)})^{(1)}{}_{(2)})\Theta(a_{(2)})^{(2)(\bar{1})}\otimes(Sa_{(1)})\Theta(a_{(2)})^{(1)}{}_{(1)}\Theta(a_{(2)})^{(2)(\bar{2})}\\
&= \sum \epsilon(\Theta(a_{(3)})^{(1)})\Theta(a_{(3)})^{(2)(\bar{1})}\otimes(Sa_{(1)})a_{(2)}\Theta(a_{(3)})^{(2)(\bar{2})}\\
&= \sum \epsilon(\Theta(a)^{(1)})\Theta(a)^{(2)(\bar{1})}\otimes\Theta(a)^{(2)(\bar{2})} = \beta_V\circ f.
\end{aligned}$$

The first equality uses (15), the second uses the first of the above identities for Θ_{A_R}. The third and fourth use the counit axioms. The fifth uses the second of the above Θ_{A_R} identities. The sixth uses the antipode axioms. Thus $f : \underline{A}\to V$ is a morphism. Next, if $\Theta = \theta_V(f')$ then $f(a) = (\epsilon\otimes\mathrm{id})\circ\theta_V(f')_{A_R}(a) = (\epsilon\otimes\mathrm{id})\circ(\mathrm{id}\otimes f')\circ\beta_{A_R}(a) = \sum\epsilon(a_{(1)})f'(a_{(2)}) = f'(a)$ by the counit axioms. In the other direction if Θ is given we now show that $\theta_V(f) = \Theta$ as natural transformations. To see this fix an arbitrary $X\in\mathcal{C}$ and consider for all $g\in X^*$ the family of morphisms $i_g = (g\otimes\mathrm{id})\circ\beta_X : X\to A_R$. These are morphisms because $\sum i_g(x^{(\bar{1})})\otimes x^{(\bar{2})} = \sum g(x^{(\bar{1})(\bar{1})})x^{(\bar{1})(\bar{2})}\otimes x^{(\bar{2})} = \sum g(x^{(\bar{1})})x^{(\bar{2})}{}_{(1)}\otimes x^{(\bar{2})}{}_{(2)} = \beta_{A_R}\circ i_g(x)$, using the comodule property of $\beta_X(x) = \sum x^{(\bar{1})}\otimes x^{(\bar{2})}$. Now functoriality of Θ under this morphism means that $(i_g\otimes\mathrm{id})\circ\Theta_X = \Theta_{A_R}\circ i_g$, i.e. $\sum g(\Theta_X(x)^{(1)(\bar{1})})\Theta_X(x)^{(1)(\bar{2})} = g(x^{(\bar{1})})\Theta_{A_R}(x^{(\bar{2})})$. Applying $\epsilon\otimes\mathrm{id}$ to both sides we obtain $\sum g(\Theta_X(x)^{(1)})\Theta_X(x)^{(2)} =$

$g(x^{(\bar{1})})f(x^{(\bar{2})}) = (g\otimes \mathrm{id})\circ\theta_V(f)_X(x)$. Since this is for all $g\in X^*$ we conclude that $\Theta_X = \theta_V(f)$. This concludes our proof that $\underline{A}$ represents $\widetilde{F}$ as needed in Lemma 2.3. Since $\mathcal{V}=\mathcal{C}$ is rigid we can apply the proof of Lemma 2.3 to conclude that $\underline{A}^n$ represent the higher $\widetilde{F}^n$. When A is infinite dimensional we take $\mathcal{V}$ a little bigger as $\mathcal{V} = {}^A\mathcal{M}$. In this case we have to verify the conclusion of Lemma 2.3 directly. This is done by explicit computations in exactly same way as for $n=1$ already given. Hence the results of Sections 2 and 3 apply and we conclude that the object $\underline{A}$ is a braided group.

We can now compute the explicit formulae (13)-(16) by tracing through the abstract constructions in Sections 2 and 3. For example, we verify here that $\theta^2_{\underline{A}}(\underline{\cdot})_{X,Y}$ is as defined in Lemma 2.4. Thus let X,Y be two objects in ${}^A\mathcal{M}^{f.d.}$. We denote the comodule structures by $\beta_X(x)=\sum x^{(\bar{1})}\otimes x^{(\bar{2})}$ etc. F,c are the identity. Then from Lemma 2.3, and (7) for $\Psi_{\underline{A},F(Y)}$, we have

$$\begin{aligned}
\theta^2_{\underline{A}}(\underline{\cdot})_{X,Y}(x\otimes y) &= \sum x^{(\bar{1})}\otimes y^{(\bar{1})}\otimes x^{(\bar{2})}{}_{(2)}\underline{\cdot}\, y^{(\bar{2})}{}_{(2)}\mathcal{R}((Sx^{(\bar{2})}{}_{(1)})x^{(\bar{2})}{}_{(3)}\otimes y^{(\bar{2})}{}_{(1)})\\
&= \sum x^{(\bar{1})}\otimes y^{(\bar{1})}\otimes x^{(\bar{2})}{}_{(3)}y^{(\bar{2})}{}_{(5)}\\
&\qquad \mathcal{R}(x^{(\bar{2})}{}_{(4)}\otimes Sy^{(\bar{2})}{}_{(3)})\mathcal{R}(x^{(\bar{2})}{}_{(2)}\otimes y^{(\bar{2})}{}_{(4)})\mathcal{R}(Sx^{(\bar{2})}{}_{(1)}\otimes y^{(\bar{2})}{}_{(1)})\mathcal{R}(x^{(\bar{2})}{}_{(5)}\otimes y^{(\bar{2})}{}_{(2)})\\
&= \sum x^{(\bar{1})}\otimes y^{(\bar{1})}\otimes x^{(\bar{2})}{}_{(3)}y^{(\bar{2})}{}_{(3)}\mathcal{R}(x^{(\bar{2})}{}_{(2)}\otimes y^{(\bar{2})}{}_{(2)})\mathcal{R}(Sx^{(\bar{2})}{}_{(1)}\otimes y^{(\bar{2})}{}_{(1)})\\
&= \sum x^{(\bar{1})}\otimes y^{(\bar{1})}\otimes x^{(\bar{2})}y^{(\bar{2})} = \beta_{X\otimes Y}(x\otimes y).
\end{aligned}$$

Here for the second equality we used (13) and the suffix notation of [29] for higher comultiplications. We expanded $\mathcal{R}$ using (3). For the third equality we combine the first and last $\mathcal{R}$ factors using (4). Using (3) on the remaining two $\mathcal{R}$ factors gives the last expression as in Lemma 2.4. Note that $\mathcal{R}(a\otimes 1)=\mathcal{R}(1\otimes a)=\epsilon(a)$ for any dual quasitriangular Hopf algebra or bialgebra (this is easily deduced from the axiom of existence of a convolution inverse of $\mathcal{R}$). Likewise, to obtain the formula for the antipode we compute from Proposition 2.9. Note that in ${}^A\mathcal{M}^{f.d.}$, $\pi_X(1)=\sum_a e_a\otimes f^a$ where $\{e_a\}$ is a basis of $F(X)$ and $\{f^a\}$ a dual basis. d is trivial as well as F,c. Thus $\underline{S}$ is defined by the proposition via

$$\theta_{\underline{A}}(\underline{S})_X(x) = \sum x^{(\bar{1})}\otimes \underline{S}x^{(\bar{2})} = \sum \mathrm{ev}_{F(X)}\circ\Psi^{-1}_{\underline{A},F(X)^*}(e_a\otimes f^{a(\bar{1})}\otimes f^{a(\bar{2})}\otimes x)$$

$$= \sum e_a \otimes f^{a(\bar 2)(\bar 1)} < f^{a(\bar 1)(\bar 1)}, x > \mathcal{R}^{-1}(f^{a(\bar 2)(\bar 2)} \otimes f^{a(\bar 1)(\bar 2)})$$
$$= \sum e_a \otimes {f^{a(\bar 2)}}_{(2)}{}^{(\bar 1)} < f^{a(\bar 1)}, x > \mathcal{R}^{-1}({f^{a(\bar 2)}}_{(2)}{}^{(\bar 2)} \otimes {f^{a(\bar 2)}}_{(1)})$$
$$= \sum x^{(\bar 1)} \otimes (Sx^{(\bar 2)})_{(2)}{}^{(\bar 1)} \mathcal{R}^{-1}((Sx^{(\bar 2)})_{(2)}{}^{(\bar 2)} \otimes (Sx^{(\bar 2)})_{(1)})$$

Here we obtain the third equality from (7) for Ψ in terms of the comodule structures on A and $F(X)^*$ (both denoted by the ${}^{(\bar 1)} \otimes {}^{(\bar 2)}$ notation). We obtain the fourth equality since β_{X^*} is a comodule. For the fifth equality we use the relation between β_X and β_{X^*}, namely $\sum < f^{(\bar 1)}, x > f^{(\bar 2)} = \sum < f, x^{(\bar 1)} > Sx^{(\bar 2)}$ where $< \ , \ >$ denotes evaluation. Comparing these expressions and using that S is an anticoalgebra map we see that $\underline{S}a = \sum (Sa_{(1)})^{(\bar 1)} \mathcal{R}^{-1}((Sa_{(1)})^{(\bar 2)} \otimes Sa_{(2)})$. Since $\mathcal{R}^{-1}(a \otimes b) = \mathcal{R}(Sa \otimes b)$ and $\mathcal{R}(Sa \otimes Sb) = \mathcal{R}(a \otimes b)$ for any dual quasitriangular Hopf algebra (from (3)-(4)), we have $\underline{S}a = \sum (Sa_{(1)})^{(\bar 1)} \mathcal{R}((Sa_{(1)})^{(\bar 2)} \otimes a_{(2)}) = \sum (Sa_{(1)})_{(2)} \mathcal{R}((S(Sa_{(1)})_{(1)})(Sa_{(1)})_{(3)}) \otimes a_{(2)})$. Using again that S is an anticoalgebra map now gives the formula for the antipode stated in the theorem. More precisely, the formula verifies Proposition 2.9. Likewise we can compute the content of Definition 3.1 in this context. The class $\mathcal{O}$ here is the class of right $\underline{A}$-comodules (V, β_V) which coincide as linear maps $V \to V \otimes A$ with the tautological comodule structure by which V is an object in ${}^A\mathcal{M}$. On $v \in V, a \in \underline{A}$ we have $(\mathrm{id} \otimes \underline{\cdot}) \circ \beta'_V(v \otimes a) = \sum v^{(\bar 1)} \otimes v^{(\bar 2)} \underline{\cdot} a$ while

$$(\mathrm{id} \otimes \underline{\cdot}) \circ Q_{V,\underline{A}} \circ \Psi_{\underline{A},\underline{A}} \circ \beta'_V(v \otimes a) = \sum (\mathrm{id} \otimes \underline{\cdot}) \circ Q_{V,\underline{A}}(v^{(\bar 1)} \otimes a^{(\bar 1)} \otimes v^{(\bar 2)(\bar 1)} \mathcal{R}(v^{(\bar 2)(\bar 2)} \otimes a^{(\bar 2)}))$$
$$= \sum v^{(\bar 1)(\bar 1)} \otimes a^{(\bar 1)(\bar 1)} \underline{\cdot} v^{(\bar 2)(\bar 1)} \mathcal{R}(v^{(\bar 2)(\bar 2)} \otimes a^{(\bar 2)}) Q(v^{(\bar 1)(\bar 2)} \otimes a^{(\bar 1)(\bar 2)})$$
$$= \sum v^{(\bar 1)} \otimes a^{(\bar 1)} \underline{\cdot} {v^{(\bar 2)}}_{(2)}{}^{(\bar 1)} \mathcal{R}({v^{(\bar 2)}}_{(2)}{}^{(\bar 2)} \otimes {a^{(\bar 2)}}_{(2)}) Q({v^{(\bar 2)}}_{(1)} \otimes {a^{(\bar 2)}}_{(1)}).$$

We used here the axioms of the comodule structures on V and on A itself (both denoted by the ${}^{(\bar 1)} \otimes {}^{(\bar 2)}$ notation above) and the definitions of $\Psi_{\underline{A},\underline{A}}, Q_{V,\underline{A}}$. We see that Definition 3.1 obtains on all comodules in the tautological class $\mathcal{O}$ iff (16) holds. This completes the proof of the theorem. We note that putting the comodule structure (15) into (16) and using (3)-(4) also gives the explicit form

$$b \underline{\cdot} a = \sum a_{(5)} \underline{\cdot} b_{(7)} \mathcal{R}(b_{(5)} \otimes a_{(2)}) \mathcal{R}(Sb_{(6)} \otimes a_{(8)}) \mathcal{R}(b_{(9)} \otimes Sa_{(1)}) \mathcal{R}(b_{(8)} \otimes a_{(9)})$$

$$\mathcal{R}(b_{(3)}\otimes a_{(7)})\mathcal{R}(b_{(4)}\otimes Sa_{(3)})\mathcal{R}(Sa_{(4)}\otimes b_{(1)})\mathcal{R}(a_{(6)}\otimes b_{(2)}). \qquad (17)$$

This is used in [14].

This theorem is an important application of our category-theoretic constructions because it transmutes a non-commutative object A in the ordinary category of vector spaces into a commutative object, albeit in a braided category. This is a shift in view-point from non-commutative geometry in the sense of [1] to the philosophy of supergeometry and its extensions. That is, the braided group $\underline{A}$ is commutative, i.e. a classical and not quantum object. Thus a braided group is like a supergroup but in an even more non-commutative category (since the quasisymmetry no-longer has square one). Because of the many successes of the super philosophy, it does seem worthwhile to transmute quantum groups in this way. The notions of super-manifolds, super-vector fields, and super-integration all have natural generalization to the symmetric monoidal setting as stressed in [23][7] and further to the present braided monoidal setting. This will be explored elsewhere. The structure of $\underline{A}$ first turned up in [13] in another context.

Analogously, the main theorem itself, as well as Theorem 2.8, implies in this setting,

Theorem 4.2 *Let $f: A_1 \to A_2$ be a Hopf algebra map where A_2 is dual quasitriangular. Then there is a braided Hopf algebra $B(A_1, f, A_2)$ in ${}^{A_2}\mathcal{M}$. If A_1 is also dual quasitriangular then $B(A_1, f, A_2)$ is dual quasitriangular in ${}^{A_2}\mathcal{M}$ with $\mathcal{R}$ induced by f. Here $B(A, \mathrm{id}, A) = \underline{A}$, $B(A, \epsilon, k) = A$ and $B(k, \eta, A) = k$.*

Proof The strategy is the same. We compute a realization of $\mathrm{Aut}\,(\mathcal{C}, F, \mathcal{V})$ where $\mathcal{C} = {}^{A_1}\mathcal{M}^{f.d.}$ and F is induced by pushout along f. As an object for B we take the space A_1 which becomes an A_2 comodule by the inner adjoint coaction induced by f,

$$\beta_B(a) = \sum a_{(2)} \otimes (Sf(a_{(1)}))f(a_{(3)}).$$

The proof that Lemma 2.3 holds for this is similar to the preceding theorem. The computation of the necessary formulae then follows in the same way from the results of Section 2.

We conclude with the simplest concrete example. We transmute the Hopf algebra A dual to the four-dimensional triangular Hopf algebra H described in [26] and in the present context in [14]. As a Hopf algebra H is self-dual and is due to Sweedler.

Example 4.3 *Let k be a field of characteristic not 2. Let $A = \mathrm{span}\{1, f, y, fy\}$ be the 4-dimensional dual quasitriangular Hopf algebra with generators $1, y, f$ and relations, comultiplication and counit*

$$y^2 = 0, \quad f^2 = 1, \quad yf = -fy, \quad \Delta y = y \otimes f + 1 \otimes y, \quad \Delta f = f \otimes f, \quad \epsilon(y) = 0,\ \epsilon(f) = 1$$

(the reduced quantum plane at $q = -1$). There is also an antipode $Sy = fy$, $Sf = f$. The dual quasitriangular structure in the basis $\{1, f, y, fy\}$ is the two form

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & \alpha & \alpha \end{pmatrix}.$$

The category $\mathcal{C} = {}^A\mathcal{M}$ has objects given by a pair consisting of a superspace $V = V_0 \oplus V_1$ and an odd operator D_V on V with $D_V^2 = 0$ (cf [25]). The braiding in this category is $\Psi_{V \otimes W}(v \otimes w) = w \otimes v(-1)^{|v||w|} + \alpha D_W w \otimes D_V v(-1)^{|v|(1-|w|)}$ [14]. In this category the transmutation $\underline{A}$ is the same as a coalgebra but with a new multiplication $\underline{\cdot}$ given in terms of the multiplication in A by

$$f \underline{\cdot} y = fy = y \underline{\cdot} f, \quad y \underline{\cdot} y = \alpha(1 - f), \quad f \underline{\cdot} f = f^2 = 1.$$

As an object in the category, $\underline{A} = \{1, f\} \oplus \{y, fy\}$, i.e. $1, f$ have degree 0 and y, fy have degree 1. The operator $D_{\underline{A}}$ is

$$D_{\underline{A}}(1) = D_{\underline{A}}(f) = 0, \quad D_{\underline{A}}(y) = 1 - f, \quad D_{\underline{A}}(fy) = f - 1.$$

Accordingly, the action of the braid group $\Psi_{\underline{A},\underline{A}}$ is

$$\Psi(y \otimes y) = -y \otimes y + \alpha(1-f) \otimes (1-f),\ \Psi(y \otimes f) = f \otimes y,\ \Psi(f \otimes y) = y \otimes f,\ \Psi(f \otimes f) = f \otimes f.$$

With this braid group action extended to all of $\underline{A}$, the latter becomes a Hopf algebra in the category ${}^A\mathcal{M}$. The antipode is given by

$$\underline{S}f = f, \qquad \underline{S}y = -fy, \qquad (\underline{S}fy = -y).$$

Note that in this example the category ${}^{A}\mathcal{M}$ is actually symmetric monoidal rather than braided monoidal. It is, however, probably the simplest example demonstrating transmutation. y is transmuted from a bosonic element in A (a Hopf algebra in Vec, i.e. with the usual transposition Ψ^{Vec}) into an almost fermionic element!

Note also that $\underline{A}$ can also be defined intrinsically as generated by $1, f, y$ and relations

$$f\underline{\cdot}f = 1, \quad f\underline{\cdot}y = y\underline{\cdot}f, \quad y\underline{\cdot}y = \alpha(1-f).$$

Writing $z = y + f\underline{\cdot}y$ and $w = y - f\underline{\cdot}y$ we have equivalently,

Example 4.4 *The braided group $\underline{A}$ can be intrinsically described as the commutative algebra generated by $1, f, z, w$ and relations*

$$z^2 = 0, \quad w^2 = 4\alpha(1-f), \quad fz = z, \quad fw = -w.$$

The comultiplication counit and antipode are

$$\Delta z = (\frac{1+f}{2}) \otimes z + z \otimes (\frac{1+f}{2}) + (\frac{1-f}{2}) \otimes w + w \otimes (\frac{1-f}{2})$$

$$\Delta w = (\frac{1+f}{2}) \otimes w + w \otimes (\frac{1+f}{2}) + (\frac{1-f}{2}) \otimes z + z \otimes (\frac{1-f}{2})$$

$$\Delta f = f \otimes f, \quad \epsilon f = 1, \quad \epsilon z = \epsilon w = 0, \quad Sf = f \quad Sz = -z, \quad Sw = w.$$

Here $1, f$ are bosonic, z is fermionic and w is almost fermionic in the sense

$$\Psi(f \otimes f) = f \otimes f,\ \Psi(f \otimes z) = z \otimes f,\ \Psi(z \otimes f) = f \otimes z,\ \Psi(f \otimes w) = w \otimes f,\ \Psi(w \otimes f) = f \otimes w$$

$$\Psi(z \otimes z) = -z \otimes z, \quad \Psi(z \otimes w) = -w \otimes z, \quad \Psi(w \otimes z) = -z \otimes w,$$

$$\Psi(w \otimes w) = -w \otimes w + 4\alpha(1-f) \otimes (1-f).$$

A DIRECT PROOF THAT $\underline{A}$ IS A HOPF ALGEBRA IN ${}^{A}\mathcal{M}$

In this section we provide a direct algebraic proof of Theorem 4.1 by directly verifying that $\underline{A}$ as stated really obeys the axioms of a Hopf algebra in the category ${}^{A}\mathcal{M}$ as explained in the Preliminaries. This will be for useful readers with an algebraic background who aim to apply the theorem to the theory of ordinary (dual) quasitriangular Hopf algebras. Nothing in the body of the paper depends on this appendix. Throughout the section $(A, \mathcal{R})$ is an ordinary dual quasitriangular Hopf algebra over a commutative ring k. We use the notation of [29] for the comultiplication structure of A. We use standard Hopf algebra methods.

Note that one of the axioms of a dual quasitriangular Hopf algebra or bialgebra is the existence of an inverse of $\mathcal{R}$ in the convolution algebra $\mathrm{Hom}_k(A\otimes A, k)$. This means a map $A\otimes A\to k$ such that

$$\sum \mathcal{R}^{-1}(a_{(1)}\otimes b_{(1)})\mathcal{R}(a_{(2)}\otimes b_{(2)}) = \epsilon(a)\epsilon(b) = \sum \mathcal{R}(a_{(1)}\otimes b_{(1)})\mathcal{R}^{-1}(a_{(2)}\otimes b_{(2)}).$$

This implies in particular that we have $\mathcal{R}(a\otimes 1) = \sum \mathcal{R}^{-1}(a_{(1)}\otimes 1)(\mathcal{R}(a_{(2)}\otimes 1)\mathcal{R}(a_{(3)}\otimes 1))$ $= \sum \mathcal{R}^{-1}(a_{(1)}\otimes 1)\mathcal{R}(a_{(2)}\otimes 1.1) = \epsilon(a)$ (using (4)). Likewise on the other side. Hence the axioms imply that $\mathcal{R}(a\otimes 1) = \epsilon(a) = \mathcal{R}(1\otimes a)$ for all $a\in A$. These elementary facts corresponds in the Tannaka-Krein Theorem for dual quasitriangular Hopf algebras[18, Sec. 2][15, Thm. 4.1] to $\Psi_{X,\underline{1}} = \mathrm{id}_X = \Psi_{\underline{1},X}$ in ${}^A\mathcal{M}$. Also, if $\mathcal{R}^{-1}$ exists it is unique. Hence for A a Hopf algebra it is given by $\mathcal{R}^{-1}(a\otimes b) = \mathcal{R}(Sa\otimes b)$ (use axioms (3)-(4)). In this case $a\otimes b\mapsto \mathcal{R}(Sa\otimes Sb)$ is convolution inverse to $\mathcal{R}^{-1}$ because $\sum \mathcal{R}(Sa_{(1)}\otimes Sb_{(1)})\mathcal{R}(Sa_{(2)}\otimes b_{(2)}) =$ $\sum \mathcal{R}(Sa\otimes (Sb_{(1)})b_{(2)}) = \mathcal{R}(Sa\otimes 1)\epsilon(b) = \epsilon(a)\epsilon(b)$ etc. Hence

$$\mathcal{R}(Sa\otimes Sb) = \mathcal{R}(a\otimes b).$$

We shall use these various elementary facts about dual quasitriangular Hopf algebras quite freely.

Recall that as a coalgebra $\underline{A}$ coincides with A. It is an object in ${}^A\mathcal{M}$ by the adjoint coaction (15). It is an elementary property of the adjoint coaction that $\Delta : \underline{A}\to \underline{A}\otimes\underline{A}$ and $\epsilon : \underline{A}\to k$ are intertwiners, i.e. morphisms in ${}^A\mathcal{M}$. Likewise for the map $k\to\underline{A}$ defined by the identity of A.

Lemma A.1 *The map $\underline{\cdot}$ in (13) makes $\underline{A}$ into an associative algebra. The identity coincides with that of A.*

Proof We compute for $a, b, c \in \underline{A}$, using the definitions and axioms (3)-(4) to break down multiplications in the argument of $\mathcal{R}$.

$$a\underline{\cdot}(b\underline{\cdot}c) = \sum a_{(2)}(b\underline{\cdot}c)_{(3)}\mathcal{R}(a_{(3)}\otimes S(b\underline{\cdot}c)_{(1)})\mathcal{R}(a_{(1)}\otimes(b\underline{\cdot}c)_{(2)})$$

$$= \sum a_{(2)}b_{(4)}c_{(5)}\mathcal{R}(a_{(3)}\otimes S(b_{(2)}c_{(3)}))\mathcal{R}(a_{(1)}\otimes b_{(3)}c_{(4)})\mathcal{R}(b_{(5)}\otimes Sc_{(1)})\mathcal{R}(b_{(1)}\otimes c_{(2)})$$

$$= \sum a_{(3)}b_{(4)}c_{(5)}\mathcal{R}(b_{(5)}\otimes Sc_{(1)})\mathcal{R}(a_{(4)}\otimes Sb_{(2)})\mathcal{R}(a_{(5)}\otimes Sc_{(3)})$$
$$\mathcal{R}(a_{(2)}\otimes b_{(3)})\mathcal{R}(a_{(1)}\otimes c_{(4)})\mathcal{R}(b_{(1)}\otimes c_{(2)})$$

$$= \sum a_{(3)}b_{(4)}c_{(5)}\mathcal{R}(b_{(5)}\otimes Sc_{(1)})\mathcal{R}(a_{(4)}\otimes Sb_{(2)})\mathcal{R}(a_{(5)}\otimes Sc_{(3)})$$
$$\mathcal{R}(Sa_{(2)}\otimes Sb_{(3)})\mathcal{R}(Sa_{(1)}\otimes Sc_{(4)})\mathcal{R}(Sb_{(1)}\otimes Sc_{(2)})$$

$$= \sum a_{(2)}b_{(2)}c_{(3)}\mathcal{R}(b_{(3)}\otimes Sc_{(1)})\mathcal{R}((Sa_{(1)})_{(2)}\otimes(Sc_{(2)})_{(1)})\mathcal{R}((Sa_{(1)})_{(1)}\otimes(Sb_{(1)})_{(1)})$$
$$\mathcal{R}(a_{(3)(1)}\otimes(Sb_{(1)})_{(2)})\mathcal{R}(a_{(3)(2)}\otimes(Sc_{(2)})_{(2)})\mathcal{R}((Sb_{(1)})_{(3)}\otimes(Sc_{(2)})_{(3)}).$$

To obtain the fourth equality we used $\mathcal{R}(Sa\otimes Sb) = \mathcal{R}(a\otimes b)$ and for the fifth that S is an anticoalgebra map. Likewise we compute

$$(a\underline{\cdot}b)\underline{\cdot}c = \sum(a\underline{\cdot}b)_{(2)}c_{(3)}\mathcal{R}((a\underline{\cdot}b)_{(3)}\otimes Sc_{(1)})\mathcal{R}((a\underline{\cdot}b)_{(1)}\otimes c_{(2)})$$

$$= \sum a_{(3)}b_{(4)}c_{(3)}\mathcal{R}(a_{(4)}b_{(5)}\otimes Sc_{(1)})\mathcal{R}(a_{(2)}b_{(3)}\otimes c_{(2)})\mathcal{R}(a_{(5)}\otimes Sb_{(1)})\mathcal{R}(a_{(1)}\otimes b_{(2)})$$

$$= \sum a_{(3)}b_{(4)}c_{(5)}\mathcal{R}(b_{(5)}\otimes Sc_{(1)})\mathcal{R}(a_{(4)}\otimes Sc_{(2)})\mathcal{R}(a_{(5)}\otimes Sb_{(1)})$$
$$\mathcal{R}(Sa_{(2)}\otimes Sc_{(3)})\mathcal{R}(Sa_{(1)}\otimes Sb_{(2)})\mathcal{R}(Sb_{(3)}\otimes Sc_{(4)})$$

$$= \sum a_{(2)}b_{(2)}c_{(3)}\mathcal{R}(b_{(3)}\otimes Sc_{(1)})\mathcal{R}((Sa_{(1)})_{(2)}\otimes(Sb_{(1)})_{(2)})\mathcal{R}((Sa_{(1)})_{(1)}\otimes(Sc_{(2)})_{(2)})$$
$$\mathcal{R}(a_{(3)(1)}\otimes(Sc_{(2)})_{(3)})\mathcal{R}(a_{(3)(2)}\otimes(Sb_{(1)})_{(3)})\mathcal{R}((Sb_{(1)})_{(1)}\otimes(Sc_{(2)})_{(1)}).$$

Now these two expression are equal because $\mathcal{R}$ obeys the higher order "cocycle" condition proven in the next lemma, applied in our case by putting $a \to Sa_{(1)}$, $b \to a_{(3)}$, $c \to Sb_{(1)}$, $d \to Sc_{(2)}$ in the lemma. That $1\underline{\cdot}a = a = a\underline{\cdot}1$ follows at once from $\mathcal{R}(a\otimes 1) = \epsilon(a) = \mathcal{R}(1\otimes a)$.

Lemma A.2 *Let $(A, \mathcal{R})$ be a dual quasitriangular bialgebra or Hopf algebra. Then for all $a, b, c, d \in A$,*

$$\sum \mathcal{R}(a_{(2)}\otimes d_{(1)})\mathcal{R}(a_{(1)}\otimes c_{(1)})\mathcal{R}(b_{(1)}\otimes c_{(2)})\mathcal{R}(b_{(2)}\otimes d_{(2)})\mathcal{R}(c_{(3)}\otimes d_{(3)})$$

$$= \sum \mathcal{R}(a_{(2)} \otimes c_{(2)})\mathcal{R}(a_{(1)} \otimes d_{(2)})\mathcal{R}(b_{(1)} \otimes d_{(3)})\mathcal{R}(b_{(2)} \otimes c_{(3)})\mathcal{R}(c_{(1)} \otimes d_{(1)})$$

Proof Firstly, (5) implies $\sum \mathcal{R}(a_{(2)}b_{(2)} \otimes c)\mathcal{R}(a_{(1)} \otimes b_{(1)}) = \sum \mathcal{R}(b_{(1)}a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b_{(2)})$. Expanding both sides by (3) gives that $\mathcal{R}$ obeys as usual the Quantum Yang-Baxter Equations (QYBE):

$$\sum \mathcal{R}(a_{(1)} \otimes b_{(1)})\mathcal{R}(b_{(2)} \otimes c_{(2)})\mathcal{R}(a_{(2)} \otimes c_{(1)}) = \sum \mathcal{R}(a_{(2)} \otimes b_{(2)})\mathcal{R}(b_{(1)} \otimes c_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(2)}).$$

To prove the lemma we apply this twice on the left hand side of the lemma. Thus, the left hand side is

$$\begin{aligned}
&\sum \mathcal{R}(a_{(2)} \otimes d_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(1)})\mathcal{R}(b_{(1)} \otimes c_{(2)(1)})\mathcal{R}(b_{(2)} \otimes d_{(2)(1)})\mathcal{R}(c_{(2)(2)} \otimes d_{(2)(2)}) \\
&\quad = \sum \mathcal{R}(a_{(2)} \otimes d_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(1)})\mathcal{R}(b_{(2)} \otimes c_{(2)(2)})\mathcal{R}(b_{(1)} \otimes d_{(2)(2)})\mathcal{R}(c_{(2)(1)} \otimes d_{(2)(1)}) \\
&\quad = \sum \mathcal{R}(a_{(2)} \otimes d_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(1)})\mathcal{R}(c_{(2)} \otimes d_{(2)})\mathcal{R}(b_{(1)} \otimes d_{(3)})\mathcal{R}(b_{(2)} \otimes c_{(3)}).
\end{aligned}$$

For the first equality we apply the QYBE to the last three $\mathcal{R}$s by putting $a \to b$, $b \to c_{(2)}$, $c \to d_{(2)}$ in the QYBE. Now apply the QYBE again to the first three $\mathcal{R}$s of the resulting expression by putting $a \to a$, $b \to c$ and $c \to d$ in the QYBE. This yields precisely the right hand side of the lemma.

Lemma A.3 *The map $\underline{\cdot}$ is an intertwiner (i.e. a morphism in ${}^{A}\mathcal{M}$).*

Proof

$$\begin{aligned}
\beta_{\underline{A}}(a\underline{\cdot}b) &= \sum a_{(3)}b_{(4)} \otimes (S(a_{(2)}b_{(3)}))a_{(4)}b_{(5)}\mathcal{R}(a_{(5)} \otimes Sb_{(1)})\mathcal{R}(a_{(1)} \otimes b_{(2)}) \\
&= \sum a_{(3)}b_{(4)} \otimes (S(b_{(2)}a_{(1)}))a_{(4)}b_{(5)}\mathcal{R}(a_{(5)} \otimes Sb_{(1)})\mathcal{R}(a_{(2)} \otimes b_{(3)}) \\
&= \sum a_{(3)}b_{(4)} \otimes (Sa_{(1)})a_{(5)}(Sb_{(1)})b_{(5)}\mathcal{R}(a_{(4)} \otimes Sb_{(2)})\mathcal{R}(a_{(2)} \otimes b_{(3)}) \\
&= \sum a_{(2)}\underline{\cdot}b_{(2)} \otimes (Sa_{(1)})a_{(3)}(Sb_{(1)})b_{(3)} = (\underline{\cdot} \otimes \mathrm{id}) \circ \beta_{\underline{A} \otimes \underline{A}}(a \otimes b).
\end{aligned}$$

We used axiom (5) to obtain the second equality and again to obtain the third.

Lemma A.4 *For all $a, b \in \underline{A}$,*

$$\Delta(a\underline{\cdot}b) = (\Delta a)\underline{\cdot}(\Delta b).$$

Here on the right hand side the multiplication is in $\underline{A} \otimes \underline{A}$ with the algebra structure (8). Likewise $\epsilon(a \underline{\cdot} b) = \epsilon(a)\epsilon(b)$.

Proof

$$(\Delta a)\underline{\cdot}(\Delta b) = \sum (a_{(1)} \otimes a_{(2)})\underline{\cdot}(b_{(1)} \otimes b_{(2)}) = \sum a_{(1)}\underline{\cdot} b_{(1)}{}^{(\bar{1})} \otimes a_{(2)}{}^{(\bar{1})}\underline{\cdot} b_{(2)} \mathcal{R}(a_{(2)}{}^{(\bar{2})} \otimes b_{(1)}{}^{(\bar{2})})$$

$$= \sum a_{(1)}\underline{\cdot} b_{(2)} \otimes a_{(3)}\underline{\cdot} b_{(4)} \mathcal{R}((Sa_{(2)})a_{(4)} \otimes (Sb_{(1)})b_{(3)})$$

$$= \sum a_{(2)}b_{(4)} \otimes a_{(6)}b_{(8)} \mathcal{R}(a_{(3)} \otimes Sb_{(2)})\mathcal{R}(a_{(1)} \otimes b_{(3)})$$

$$\mathcal{R}(a_{(7)} \otimes Sb_{(6)})\mathcal{R}(a_{(5)} \otimes b_{(7)})\mathcal{R}((Sa_{(4)})a_{(8)} \otimes (Sb_{(1)})b_{(5)})$$

$$= \sum a_{(2)}b_{(5)} \otimes a_{(7)}b_{(10)} \mathcal{R}(a_{(3)} \otimes Sb_{(3)})\mathcal{R}(a_{(1)} \otimes b_{(4)})\mathcal{R}(a_{(6)} \otimes b_{(9)})$$

$$\mathcal{R}(a_{(4)} \otimes b_{(2)})\mathcal{R}(a_{(10)} \otimes Sb_{(1)})\mathcal{R}(Sa_{(5)} \otimes b_{(6)})\mathcal{R}(a_{(9)} \otimes b_{(7)})\mathcal{R}(a_{(8)} \otimes Sb_{(8)})$$

$$= \sum a_{(2)}b_{(5)} \otimes a_{(7)}b_{(8)} \mathcal{R}(a_{(3)} \otimes Sb_{(3)})\mathcal{R}(a_{(1)} \otimes b_{(4)})$$

$$\mathcal{R}(a_{(4)} \otimes b_{(2)})\mathcal{R}(a_{(8)} \otimes Sb_{(1)})\mathcal{R}(Sa_{(5)} \otimes b_{(6)})\mathcal{R}(a_{(6)} \otimes b_{(7)})$$

$$= \sum a_{(2)}b_{(5)} \otimes a_{(5)}b_{(6)} \mathcal{R}(a_{(3)} \otimes Sb_{(3)})\mathcal{R}(a_{(4)} \otimes b_{(2)})\mathcal{R}(a_{(1)} \otimes b_{(4)})\mathcal{R}(a_{(6)} \otimes Sb_{(1)})$$

$$= \sum a_{(2)}b_{(3)} \otimes a_{(3)}b_{(4)} \mathcal{R}(a_{(4)} \otimes Sb_{(1)})\mathcal{R}(a_{(1)} \otimes b_{(2)}) = \Delta a\underline{\cdot} b.$$

For the fourth equality we used the definitions of $\underline{\cdot}$. For the fifth we applied (3)-(4) to break down multiplications in the argument of $\mathcal{R}$. We then successively recombine $\mathcal{R}$ in pairs for the sixth, seventh and eighth equalities, using each time the antipode property and identities of the form $\mathcal{R}(a \otimes 1) = \epsilon(a)$ or $\mathcal{R}(1 \otimes a) = \epsilon(a)$ to eliminate the pair. These identities also imply at once that $\epsilon(a\underline{\cdot} b) = \epsilon(a)\epsilon(b)$.

Proposition A.5 *Let $(A, \mathcal{R})$ be a dual quasitriangular Hopf algebra. Let $v : A \to k$ and $v^{-1} : A \to k$ be defined by*

$$v(a) = \sum \mathcal{R}(a_{(1)} \otimes Sa_{(2)}), \qquad v^{-1}(a) = \sum \mathcal{R}(S^2 a_{(1)} \otimes a_{(2)}).$$

Then (i) $\sum a_{(1)} v(a_{(2)}) = \sum v(a_{(1)}) S^2 a_{(2)}$. (ii) v^{-1} is the inverse of v in the convolution algebra $\mathrm{Hom}_k(A, k)$. *(iii) The antipode of A is bijective.*

Proof This is just the dual of results well known for quasitriangular Hopf algebras. One can likewise define $u(a) = \sum \mathcal{R}(a_{(2)} \otimes Sa_{(1)})$, $u^{-1}(a) = \sum \mathcal{R}(S^2 a_{(2)} \otimes a_{(1)})$ with $\sum u(a_{(1)})a_{(2)} = \sum S^2 a_{(1)} u(a_{(2)})$. To prove (i) we compute

$$\begin{aligned}\sum a_{(1)}v(a_{(2)}) &= \sum a_{(1)}\mathcal{R}(a_{(2)} \otimes Sa_{(3)}) \\ &= \sum (S(Sa_{(3)})_{(1)})(Sa_{(3)})_{(2)}a_{(1)}\mathcal{R}(a_{(2)} \otimes (Sa_{(3)})_{(3)}) \\ &= \sum (S(Sa_{(3)})_{(1)})a_{(2)}(Sa_{(3)})_{(3)}\mathcal{R}(a_{(1)} \otimes (Sa_{(3)})_{(2)}) \\ &= \sum \mathcal{R}(a_{(1)} \otimes Sa_{(4)})(S^2 a_{(5)})a_{(2)}Sa_{(3)} = \sum v(a_{(1)})S^2 a_{(2)}.\end{aligned}$$

Here we inserted $\epsilon(b) = \sum (Sb_{(1)})b_{(2)}$ (where $b = Sa_{(3)}$)) and then used axiom (5) to change the order. To prove (ii) we first note that $\sum \mathcal{R}(a_{(1)} \otimes a_{(3)})v(a_{(2)}) = \sum \mathcal{R}(a_{(1)} \otimes a_{(4)})\mathcal{R}(a_{(2)} \otimes Sa_{(3)})$ $= \sum \mathcal{R}(a_{(1)} \otimes (Sa_{(2)})a_{(3)}) = \epsilon(a)$. We used (4). Hence using (i) we have

$$\epsilon(a) = \sum \mathcal{R}(a_{(1)} \otimes a_{(3)})v(a_{(2)}) = \sum v(a_{(1)})\mathcal{R}(S^2 a_{(2)} \otimes a_{(3)}) = \sum v(a_{(1)})v^{-1}(a_{(2)})$$

$$\epsilon(a) = \sum v(a_{(2)})\mathcal{R}(S^2 a_{(1)} \otimes S^2 a_{(3)}) = \sum \mathcal{R}(S^2 a_{(1)} \otimes a_{(2)})v(a_{(3)}) = \sum v^{-1}(a_{(1)})v(a_{(2)}).$$

We used for the latter that $\mathcal{R}(S^2 a \otimes S^2 b) = \mathcal{R}(a \otimes b)$. To prove (iii) we define $S^{-1}(a) = \sum Sa_{(2)}v(a_{(1)})v^{-1}(a_{(3)})$ and verify from (i) and (ii) that

$$\begin{aligned}\sum (S^{-1}a_{(2)})a_{(1)} &= \sum (Sa_{(3)})a_{(1)}v(a_{(2)})v^{-1}(a_{(4)}) \\ &= \sum (Sa_{(3)})(S^2 a_{(2)})v(a_{(1)})v^{-1}(a_{(4)}) = \sum v(a_{(1)})v^{-1}(a_{(2)}) = \epsilon(a)\end{aligned}$$

$$\begin{aligned}\sum a_{(2)}S^{-1}a_{(1)} &= \sum v^{-1}(a_{(3)})a_{(4)}Sa_{(2)}v(a_{(1)}) \\ &= \sum (S^2 a_{(3)})Sa_{(2)}v(a_{(1)})v^{-1}(a_{(4)}) = \sum v(a_{(1)})v^{-1}(a_{(2)}) = \epsilon(a).\end{aligned}$$

Lemma A.6 *$\underline{S}$ defined in (14) makes $\underline{A}$ into a Hopf algebra, i.e.*

$$\sum a_{(1)} \underline{\cdot} \underline{S} a_{(2)} = 1\epsilon(a) = \sum (\underline{S} a_{(1)}) \underline{\cdot} a_{(2)}.$$

Proof We use the preceding proposition to compute

$$\begin{aligned}\sum a_{(1)}\underline{\cdot}\underline{S}a_{(2)} &= \sum a_{(2)}(\underline{S}a_{(4)})_{(3)}\mathcal{R}(a_{(3)}\otimes S(\underline{S}a_{(4)})_{(1)})\mathcal{R}(a_{(1)}\otimes(\underline{S}a_{(4)})_{(2)})\\
&= \sum a_{(2)}Sa_{(5)}\mathcal{R}(a_{(3)}\otimes S^2a_{(7)})\mathcal{R}(a_{(1)}\otimes Sa_{(6)})\mathcal{R}((S^2a_{(8)})Sa_{(4)}\otimes a_{(9)})\\
&= \sum a_{(2)}Sa_{(5)}\mathcal{R}(a_{(3)}\otimes S^2a_{(7)})\mathcal{R}(a_{(1)}\otimes Sa_{(6)})\mathcal{R}(S^2a_{(8)}\otimes a_{(9)})\mathcal{R}(Sa_{(4)}\otimes a_{(10)})\\
&= \sum a_{(2)}Sa_{(5)}\mathcal{R}(a_{(3)}\otimes S^2a_{(7)})\mathcal{R}(a_{(1)}\otimes Sa_{(6)})v^{-1}(a_{(8)})\mathcal{R}(Sa_{(4)}\otimes a_{(9)})\\
&= \sum a_{(2)}Sa_{(5)}\mathcal{R}(a_{(3)}\otimes S^2a_{(7)})\mathcal{R}(a_{(1)}\otimes Sa_{(6)})\mathcal{R}(a_{(4)}\otimes Sa_{(8)})v^{-1}(a_{(9)})\\
&= \sum a_{(2)}Sa_{(4)}\mathcal{R}(a_{(3)}\otimes(Sa_{(7)})S^2a_{(6)})\mathcal{R}(a_{(1)}\otimes Sa_{(5)})v^{-1}(a_{(8)})\\
&= \sum a_{(2)}Sa_{(3)}\mathcal{R}(a_{(1)}\otimes Sa_{(4)})v^{-1}(a_{(5)}) = \sum \mathcal{R}(a_{(1)}\otimes Sa_{(2)})v^{-1}(a_{(3)})\\
&= \sum v(a_{(1)})v^{-1}(a_{(2)}) = \epsilon(a).\end{aligned}$$

Here the first equality uses the definition of $\underline{\cdot}$, the second that of $\underline{S}$, the third uses (3), the fourth recognizes v^{-1} and the fifth uses the preceding proposition and $\mathcal{R}(Sa\otimes Sb) = \mathcal{R}(a\otimes b)$. We now use (4) to obtain the sixth equality and then proceed to collapse using the antipode property as shown. For $\underline{S}$ on the other side we have more simply

$$\begin{aligned}\sum(\underline{S}a_{(1)})\underline{\cdot}a_{(2)} &= \sum(\underline{S}a_{(1)})_{(2)}a_{(4)}\mathcal{R}((\underline{S}a_{(1)})_{(3)}\otimes Sa_{(2)})\mathcal{R}((\underline{S}a_{(1)})_{(1)}\otimes a_{(3)})\\
&= \sum(Sa_{(3)})a_{(9)}\mathcal{R}(Sa_{(2)}\otimes Sa_{(7)})\mathcal{R}(Sa_{(4)}\otimes a_{(8)})\mathcal{R}((S^2a_{(5)})Sa_{(1)}\otimes a_{(6)})\\
&= \sum(Sa_{(3)})a_{(10)}\mathcal{R}(a_{(2)}\otimes a_{(8)})\mathcal{R}(Sa_{(4)}\otimes a_{(9)})\mathcal{R}((S^2a_{(5)})\otimes a_{(6)})\mathcal{R}(Sa_{(1)}\otimes a_{(7)})\\
&= \sum(Sa_{(1)})a_{(6)}\mathcal{R}(Sa_{(2)}\otimes a_{(5)})\mathcal{R}(S^2a_{(3)}\otimes a_{(4)}) = \sum(Sa_{(1)})a_{(2)} = \epsilon(a).\end{aligned}$$

Here for the first equality we used the definition on $\underline{\cdot}$, for the second that of $\underline{S}$, (3) and $\mathcal{R}(Sa\otimes Sb) = \mathcal{R}(a\otimes b)$ for the third. We then recombined $\mathcal{R}$'s using (3) for the fourth and fifth identities.

Lemma A.7 *$\underline{S}:\underline{A}\to\underline{A}$ is an intertwiner (i.e. a morphism in ${}^A\mathcal{M}$).*

Proof

$$\begin{aligned}\beta_{\underline{A}}(\underline{S}a) &= \sum(Sa_{(2)})_{(2)}\otimes(S(Sa_{(2)})_{(1)})(Sa_{(2)})_{(3)}\mathcal{R}((S^2a_{(3)})Sa_{(1)}\otimes a_{(4)})\\
&= \sum Sa_{(3)}\otimes(S^2a_{(4)})Sa_{(2)}\mathcal{R}(S^2a_{(5)}\otimes a_{(6)})\mathcal{R}(Sa_{(1)}\otimes a_{(7)})\end{aligned}$$

$$= \sum Sa_{(3)} \otimes (S^2 a_{(4)}) Sa_{(2)} v^{-1}(a_{(5)}) \mathcal{R}(Sa_{(1)} \otimes a_{(6)})$$

$$= \sum Sa_{(3)} \otimes a_{(6)} Sa_{(2)} \mathcal{R}(S^2 a_{(4)} \otimes a_{(5)}) \mathcal{R}(Sa_{(1)} \otimes a_{(7)})$$

$$= \sum Sa_{(2)} \otimes (Sa_{(1)}) a_{(7)} \mathcal{R}(S^2 a_{(4)} \otimes a_{(5)}) \mathcal{R}(Sa_{(2)} \otimes a_{(6)})$$

$$= \sum Sa_{(3)} \otimes (Sa_{(1)}) a_{(6)} \mathcal{R}((S^2 a_{(4)}) Sa_{(2)} \otimes a_{(5)})$$

$$= \sum Sa_{(2)(2)} \otimes (Sa_{(1)}) a_{(3)} \mathcal{R}((S^2 a_{(2)(3)}) Sa_{(2)(1)} \otimes a_{(2)(4)})$$

$$= \sum \underline{S} a_{(2)} \otimes (Sa_{(1)}) a_{(3)} = (\underline{S} \otimes \mathrm{id}) \circ \beta_{\underline{A}}(a).$$

Here the first equality is the definitions. The second is by axiom (3), the third recognizes v^{-1}, the fourth is by Proposition A.5, the fifth by axiom (5), the sixth by axiom (3). This expression can then be recognized as the right hand side. This concludes the proof of the lemma and hence that $\underline{A}$ is a Hopf algebra in ${}^{A}\mathcal{M}$.

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