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Non-Local Extensions of the Conformal Algebra : Matrix W-Algebras, Matrix KdV-Hierarchies and Non-Abelian Toda Theories

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NON-LOCAL EXTENSIONS OF THE CONFORMAL ALGEBRA : MATRIX W-ALGEBRAS, MATRIX KdV-HIERARCHIES AND NON-ABELIAN TODA THEORIES

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ABSTRACT

In the present contribution, I report on certain *non-linear* and *non-local* extensions of the conformal (Virasoro) algebra. These so-called V-algebras are matrix generalizations of W-algebras. First, in the context of two-dimensional field theory, I discuss the non-abelian Toda model which possesses three conserved (chiral) "currents". The Poisson brackets of these "currents" give the simplest example of a V-algebra. The classical solutions of this model provide a free-field realization of the V-algebra. Then I show that this V-algebra is identical to the second Gelfand-Dikii symplectic structure on the manifold of 2×2-matrix Schrödinger operators $L = -\partial^2 + U$ (with $\mathrm{tr}\,\sigma_3 U = 0$). This provides a relation with matrix KdV-hierarchies and allows me to obtain an infinite family of conserved charges (Hamiltonians in involution). Finally, I work out the general $V_{n,m}$-algebras as symplectic structures based on $n \times n$-matrix m^{th}-order differential operators $L = -\partial^m + U_2\partial^{m-2} + U_3\partial^{m-3} + \ldots + U_m$. It is the absence of U_1, together with the non-commutativity of matrices that leads to the non-local terms in the $V_{n,m}$-algebras. I show that the conformal properties are similar to those of W_m-algebras, while the complete $V_{n,m}$-algebras are much more complicated, as is shown on the explicit example of $V_{n,3}$.

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1. Introduction

In this contribution I will report on certain *non-linear* and *non-local* extensions of the conformal or Virasoro algebra (with central extension). These algebras will be called V-algebras.

The procedure will be to go from physics to mathematics. In particular, I will start with a certain $1+1$ dimensional field theory (non-abelian Toda field theory) defined by its action functional. The corresponding equations of motion admit three conserved "left-moving" and three conserved "right-moving" quantities (spin-two currents), expressed in terms of the interacting fields. I will define canonical Poisson brackets from the action functional, which provide the phase space with the canonical symplectic structure. Then I compute the Poisson brackets of the conserved quantities. This algebra closes in a non-linear and non-local way, reminiscent of W-algebras (which are however local), so it seemed appropriate to call this algebra a V-algebra. The explicit classical solutions of the equations of motion induce a "chiral" realization of the conserved quantities (i.e. of the generators of the V-algebra) in terms of free fields. This will be the contents of section 2.

Then, in the third section, I relate the previously found results to standard mathematical structures. I show that exactly the same algebra is obtained as the second Gelfand-Dikii symplectic structure based on a second-order differential operator, that is a 2×2-matrix Schrödinger operator $L = -\partial^2 + U$. Applying a straightforward matrix generalization of the classical work on the resolvent by Gelfand and Dikii, this provides us with an infinity of Hamiltonians in involution, hence an infinity of conserved charges for the non-abelian Toda field theory I started with. It also immediately implies the connection with matrix KdV-hierarchies.

In the fourth section, I generalize all these developments to $n \times n$-matrix m^{th}-order differential operators $L = -\partial^m + U_2\partial^{m-2} + U_3\partial^{m-3} + \ldots + U_m$ and so-called $V_{n,m}$-algebras. The latter are $n \times n$-matrix generalizations of the W_m-algebras. They are non-local due to the reduction to $U_1 = 0$ and the non-commutativity of matrices. This fourth section is mathematically self-contained, (except that the reader is referred to my original paper for the proofs). Of course, taking this section just by itself, misses the whole point of the present contribution (at least in my physicist's point of view), which is to relate abstract mathematical structures to certain physical theories.

2. The non-abelian Toda field theory

2.1. The action and equations of motion

The one plus one dimensional field theory I will consider is defined by its action functional

$$S \equiv S[r,t,\phi] = \frac{2}{\gamma^2}\int \mathrm{d}\sigma \mathrm{d}\tau \left(\partial_u r \partial_v r + \mathrm{th}^2 r\, \partial_u t \partial_v t + \partial_u \phi \partial_v \phi + \mathrm{ch}2r\, e^{2\phi}\right) , \tag{2.1}$$

where τ is the time and σ the space-coordinate and $u = \tau + \sigma$ and $v = \tau - \sigma$. The three fields are $r(\tau,\sigma), t(\tau,\sigma)$ and $\phi(\tau,\sigma)$. This action has the following physical interpretation. The first two, ϕ-independent terms, constitute a sigma-model describing a string on a two-dimensional black hole background, while the last two, ϕ-dependent terms correspond to an "internal" field ϕ (or a flat third dimension) and a tachyon potential $\mathrm{ch}2r\, e^{2\phi}$. However, this interpretation need not concern us here. The constant γ^2 plays the role of the Planck constant and will be seen later to control the central charge of the conformal algebra. As discussed below, this model is obtained by gauging a *nilpotent* subalgebra of the Lie algebra B_2. The theory defined by the above action is known as the non-abelian Toda theory [1] associated with the Lie algebra B_2 [2]. The general solution of the equations of motion is in principle contained in ref. 2 where it is shown how the solutions for an equivalent system of equations can be obtained from the general scheme of ref. 1. However, it is non-trivial to actually spell out the solution and put it in a compact and useful form. This was done in ref. 3.

The equations of motion obtained from the action (2.1) read

$$\begin{aligned}
\partial_u \partial_v r &= \frac{\mathrm{sh} r}{\mathrm{ch}^3 r}\partial_u t \partial_v t + \mathrm{sh}2r\, e^{2\phi} \\
\partial_u \partial_v t &= -\frac{1}{\mathrm{sh} r\, \mathrm{ch} r}\left(\partial_u r \partial_v t + \partial_u t \partial_v r\right) \\
\partial_u \partial_v \phi &= \mathrm{ch}2r\, e^{2\phi} .
\end{aligned} \tag{2.2}$$

Using these equations of motion it is completely straightforward to show that the following three quantities are conserved [2]:

$$\begin{aligned}
T \equiv T_{++} &= (\partial_u r)^2 + \mathrm{th}^2 r\, (\partial_u t)^2 + (\partial_u \phi)^2 - \partial_u^2 \phi \\
V^{\pm} \equiv V^{\pm}_{++} &= \frac{1}{\sqrt{2}}\left(2\partial_u \phi - \partial_u\right)\left[e^{\pm i\nu}\left(\partial_u r \pm i \mathrm{th} r\, \partial_u t\right)\right]
\end{aligned} \tag{2.3}$$

i.e.

$$\partial_v T = \partial_v V^{\pm} = 0 \ . \tag{2.4}$$

Here ν is defined by

$$\partial_v \nu = \mathrm{ch}^{-2} r\, \partial_v t \quad , \qquad \partial_u \nu = (1 + \mathrm{th}^2 r) \partial_u t \tag{2.5}$$

where the integrability condition is fulfilled due to the equations of motion (2.2).

2.2. The symplectic structure and the constraint algebra

Now I will define the symplectic structure using canonical Poisson brackets. Recall that $u = \tau + \sigma$, $v = \tau - \sigma$. The action (2.1) can then equivalently be written as (as usual, a dot denotes ∂_τ while a prime denotes ∂_σ)

$$S = \frac{1}{\gamma^2} \int \mathrm{d}\tau \mathrm{d}\sigma \left[\frac{1}{2}(\dot{r}^2 - r'^2) + \frac{1}{2} \mathrm{th}^2 r\, (\dot{t}^2 - t'^2) + \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + 2\, \mathrm{ch} 2r\, e^{2\phi} \right] \ . \tag{2.6}$$

The constant γ^2 can be viewed as the Planck constant $2\pi\hbar$, already included ino the classical action, or merely as a coupling constant. The canonical momenta then are

$$\Pi_r = \gamma^{-2}\, \dot{r} \quad , \quad \Pi_t = \gamma^{-2} \mathrm{th}^2 r\, \dot{t} \quad , \quad \Pi_\phi = \gamma^{-2}\, \dot{\phi} \tag{2.7}$$

and the canonical (equal τ) Poisson brackets are

$$\begin{aligned} \{r(\tau,\sigma)\,,\, \Pi_r(\tau,\sigma')\} = \{t(\tau,\sigma)\,,\, \Pi_t(\tau,\sigma')\} = \{\phi(\tau,\sigma)\,,\, \Pi_\phi(\tau,\sigma')\} = \delta(\sigma - \sigma') \\ \{r(\tau,\sigma)\,,\, r(\tau,\sigma')\} = \{r(\tau,\sigma)\,,\, t(\tau,\sigma')\} = \ldots = 0 \\ \{\Pi_i(\tau,\sigma)\,,\, \Pi_j(\tau,\sigma')\} = 0 \ . \end{aligned} \tag{2.8}$$

It follows that the only non-zero equal τ Poisson brackets are

$$\begin{aligned} \{r(\tau,\sigma)\,,\, \dot{r}(\tau,\sigma')\} = \gamma^2 \delta(\sigma - \sigma') \quad &, \quad \{t(\tau,\sigma)\,,\, \dot{t}(\tau,\sigma')\} = \frac{\gamma^2}{\mathrm{th}^2 r} \delta(\sigma - \sigma') \ , \\ \{\phi(\tau,\sigma)\,,\, \dot{\phi}(\tau,\sigma')\} = \gamma^2 \delta(\sigma - \sigma') \quad &, \quad \{\dot{r}(\tau,\sigma)\,,\, \dot{t}(\tau,\sigma')\} = \frac{2\gamma^2}{\mathrm{sh} r\, \mathrm{ch} r}\, \dot{t}\, \delta(\sigma - \sigma') \ , \end{aligned} \tag{2.9}$$

and those derived from them by applying $\partial_\sigma^n \partial_{\sigma'}^m$.

Before one can compute the Poisson bracket algebra of the $T, V^{\pm}$ one has to rewrite them in terms of the fields and their momenta. This means in particular that second (and higher) τ-derivatives have to be eliminated first, using the equations of motion. One might object that one is not allowed to use the equations of motion in a canonical formulation. However, the conserved quantities given above are only defined up to terms that vanish on solutions of the equations of motion. So the correct starting point for a canonical formulation are the expression where all higher τ-derivatives are eliminated, while the expressions given in eq. (2.3) are merely derived from the canonical ones by use of the equations of motion. One has, for example, using the ϕ-equation of motion (2.2)

$$\partial_u^2\phi = \frac{1}{4}(\ddot{\phi} + 2\dot{\phi}' + \phi'') = \frac{1}{2}(\phi'' + \dot{\phi}') + \mathrm{ch}2r\, e^{2\phi} = \partial_\sigma\partial_u\phi + \mathrm{ch}2r\, e^{2\phi} \tag{2.10}$$

where in the canonical formalism

$$\partial_u r = \frac{1}{2}(\gamma^2\Pi_r + r') \ , \ \partial_u t = \frac{1}{2}(\gamma^2 \mathrm{th}^{-2}r\, \Pi_t + t') \ , \ \partial_u\phi = \frac{1}{2}(\gamma^2\Pi_\phi + \phi') \ . \tag{2.11}$$

The canonical expressions for the ++ components of the constraints are

$$\begin{aligned} T &= (\partial_u r)^2 + \mathrm{th}^2 r\,(\partial_u t)^2 + (\partial_u\phi)^2 - (\partial_u\phi)' - \mathrm{ch}2r e^{2\phi} \\ V^{\pm} &= \frac{1}{\sqrt{2}} e^{\pm i\nu}\Big[2\partial_u\phi\partial_u r \pm 2i\mathrm{th}r\partial_u\phi\partial_u t + 2\mathrm{th}^3 r(\partial_u t)^2 \mp 2i\mathrm{th}^2 r\partial_u r\partial_u t \\ &\qquad + \frac{\mathrm{sh}r}{\mathrm{ch}^3 r}\partial_u t\, t' \mp i\frac{\partial_u r\, t' + \partial_u t\, r'}{\mathrm{ch}^2 r} - (\partial_u r)' \mp i\mathrm{th}r(\partial_u t)' - \mathrm{sh}2r\, e^{2\phi}\Big] \end{aligned} \tag{2.12}$$

where the substitutions (2.11) are understood. For the Poisson bracket of T with itself one then obtains

$$\gamma^{-2}\{T(\sigma)\, ,\, T(\sigma')\} = (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma - \sigma')\right] - \frac{1}{2}\delta'''(\sigma - \sigma') \ . \tag{2.13}$$

$\bar{T}$ satisfies the same algebra (with $\sigma \to -\sigma, \sigma' \to -\sigma'$) while $\{T(\sigma)\, ,\, \bar{T}(\sigma')\} = 0$. These are just two copies of the conformal algebra. If σ takes values on the unit circle one can define the modes

$$L_n = \gamma^{-2} \int_{-\pi}^{\pi} \mathrm{d}\sigma \left[T(\tau,\sigma) + \frac{1}{4} \right] e^{in(\tau+\sigma)} \tag{2.14}$$

and similarly for $\bar{L}_n$. Then the bracket (2.13) becomes a Virasoro algebra

$$i\{L_n \, , \, L_m\} = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2.15}$$

and idem for $\{\bar{L}_n \, , \, \bar{L}_m\}$, while $\{L_n \, , \, \bar{L}_m\} = 0$. Here c is the central charge given by

$$c = \frac{12\pi}{\gamma^2} \; . \tag{2.16}$$

The occurrence of a central charge already at the classical, Poisson bracket level is due to the $\partial^2\phi$-term in T. This is reminiscent of the well-known Liouville theory. The factor i on the left hand side of equations (2.15) may seem strange at first sight, but one should remember that quantization replaces i times the canonical Poisson bracket by the commutator. Hence (2.15) is indeed the Poisson bracket version of the (commutator) Virasoro algebra.

In order to compute Poisson brackets involving $V^{\pm}$ one needs the Poisson brackets involving the field ν. Now, ν is only defined through its partial derivatives, and thus only up to a constant. This constant, however, may have a non-trivial Poisson bracket with certain modes of ν and/or of the other fields. From (2.5) one has using (2.7)

$$\nu' = \gamma^2 \Pi_t + t' \tag{2.17}$$

whereas one does not need $\dot{\nu}$ explicitly. Equation (2.17) implies

$$\partial_\sigma \partial_{\sigma'} \{\nu(\sigma) \, , \, \nu(\sigma')\} = \{\nu'(\sigma) \, , \, \nu'(\sigma')\} = 2\gamma^2 \delta'(\sigma - \sigma') \; . \tag{2.18}$$

This is integrated to yield $\{\nu(\sigma) \, , \, \nu(\sigma')\} = -\gamma^2 \epsilon(\sigma - \sigma') + h(\sigma) - h(\sigma')$ where I already used the antisymmetry of the Poisson bracket. $\epsilon(\sigma - \sigma')$ is defined to be $+1$ if $\sigma > \sigma'$, -1 if $\sigma < \sigma'$ and 0 if $\sigma = \sigma'$. The freedom to choose the function h corresponds to the above-mentioned freedom to add a constant ν_0 to ν with $\{\nu(\sigma)\,,\,\nu_0\} = h(\sigma)$. However, if one imposes invariance under translations $\sigma \to \sigma + a$, $\sigma' \to \sigma' + a$ then h can be only linear and one arrives at

$$\{\nu(\sigma)\,,\,\nu(\sigma')\} = -\gamma^2 \epsilon_\alpha(\sigma-\sigma') \equiv -\gamma^2 \left[\epsilon(\sigma-\sigma') - \frac{\alpha}{\pi}(\sigma-\sigma')\right] . \tag{2.19}$$

There is only one parameter α left, related to the zero-mode of ν.

Before doing the actual calculation it is helpful to show how the result is constrained by dimensional and symmetry considerations. First consider $\{V^+(\sigma)\,,\,V^+(\sigma')\}$. Each V^+ contains a factor $e^{i\nu}$. The Poisson bracket of $e^{i\nu(\sigma)}$ with $e^{i\nu(\sigma')}$ leads to a term $\gamma^2\epsilon(\sigma-\sigma')V^+(\sigma)V^+(\sigma')$. All other terms are local, i.e. involve $\delta(\sigma-\sigma')$ or derivatives of $\delta(\sigma-\sigma')$. On dimensional grounds* $\delta(\sigma-\sigma')$ must be multiplied by a dimension 3 object (3 derivatives) and $\delta'(\sigma-\sigma')$ by a dimension 2 object. Furthermore, these objects must contain an overall factor $e^{2i\nu}$. If the T, V^+ and V^- form a closed algebra, there are no such objects. The same reasoning applies to $\{V^-(\sigma)\,,\,V^-(\sigma')\}$. Thus one expects

$$\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\pm(\sigma')\} = \epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') . \tag{2.20}$$

It is a bit tedious, but otherwise straightforward to verify that this is indeed correct. Note that it is enough to do the computation for V^+ since V^- is the complex conjugate of V^+ (treating the fields r, t, ϕ and ν and their derivatives as real):

$$V^- = (V^+)^* . \tag{2.21}$$

What can one say about $\{V^+(\sigma)\,,\,V^-(\sigma')\}$? Using (2.21) one sees that $\{V^+(\sigma)\,,\,V^-(\sigma')\}^* = -\{V^+(\sigma)\,,\,V^-(\sigma')\}|_{\sigma\leftrightarrow\sigma'}$. This fact, together with the same type of arguments as used above, implies

$$\begin{aligned}\gamma^{-2}\{V^+(\sigma)\,,\,V^-(\sigma')\} = &-\epsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma') + (\partial_\sigma - \partial_{\sigma'})[a\delta(\sigma-\sigma')] \\ &+ ib\delta(\sigma-\sigma') + i(\partial_\sigma^2 + \partial_{\sigma'}^2)[d\delta(\sigma-\sigma')] + \tilde{c}\delta'''(\sigma-\sigma')\end{aligned} \tag{2.22}$$

where $a, b, d, \tilde{c}$ are real and have (naive) dimensions 2,3,1 and 0. Hence $\tilde{c}$ is a c-number. Also, b, a, d cannot contain a factor $e^{\pm i\nu}$. If one assumes closure of the algebra one must have

* The naive dimensional counting assigns dimension 1 to each derivative, dimension 0 to all fields r, t, ϕ, ν and functions thereof, except for functions of ϕ. $e^{2\phi}$ has dimension 2 as seen from the action, while $\delta(\sigma-\sigma')$ has dimension 1.

$b = d = 0$ and $a \sim T_{++}$. After a really lengthy computation one indeed finds equation (2.22) with $b = d = 0$, $a = T_{++}$ and $\tilde{c} = -\frac{1}{2}$.

Finally the Poisson bracket of T with $V^\pm$ simply shows that $V^\pm$ are conformally primary fields of weight (conformal dimension) 2. The complete algebra thus is

$$\begin{aligned}
\gamma^{-2}\{T(\sigma)\,,\,T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma') \\
\gamma^{-2}\{T(\sigma)\,,\,V^\pm\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[V^\pm(\sigma')\delta(\sigma-\sigma')\right] \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\pm(\sigma')\} &= \epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\mp(\sigma')\} &= -\epsilon(\sigma-\sigma')V^\pm(\sigma)V^\mp(\sigma') \\
&\quad + (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma')\ .
\end{aligned} \tag{2.23}$$

The algebra of the $--$ components $\bar{T}, \bar{V}^\pm$ looks exactly the same except for the replacements $\sigma \to -\sigma$, $\sigma' \to -\sigma'$ and hence $\partial \to -\partial$, $\epsilon(\sigma-\sigma') \to -\epsilon(\sigma-\sigma')$.

The algebra (2.23) is the correct algebra for $\sigma \in \mathbf{R}$. If $\sigma \in S^1$, one must replace $\epsilon(\sigma-\sigma') \to \epsilon_1(\sigma-\sigma')$ which is a periodic function (cf. eq. (2.19)). Also $\delta(\sigma-\sigma') \to \frac{1}{2}\partial_\sigma\epsilon_1(\sigma-\sigma') = \delta(\sigma-\sigma') - \frac{1}{2\pi}$ while $\delta'(\sigma-\sigma')$ remains unchanged. But since the right hand sides of (2.23) can be written using only $\epsilon(\sigma-\sigma')$, $\delta'(\sigma-\sigma')$ and $\delta'''(\sigma-\sigma')$, only the replacement $\epsilon(\sigma-\sigma') \to \epsilon_1(\sigma-\sigma')$ is relevant. One then defines the modes

$$V_n^\pm = \gamma^{-2}\int_{-\pi}^{\pi} \mathrm{d}\sigma V^\pm(\tau,\sigma)\, e^{in(\tau+\sigma)} \tag{2.24}$$

and similarly for $\bar{V}_n^\pm$. The mode algebra is*

$$\begin{aligned}
i\{L_n\,,\,L_m\} &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\
i\{L_n\,,\,V_m^\pm\} &= (n-m)V_{n+m}^\pm \\
i\{V_n^\pm\,,\,V_m^\pm\} &= \frac{12}{c}\sum_{k\neq 0}\frac{1}{k}V_{n+k}^\pm V_{m-k}^\pm \\
i\{V_n^\pm\,,\,V_m^\mp\} &= -\frac{12}{c}\sum_{k\neq 0}\frac{1}{k}V_{n+k}^\pm V_{m-k}^\mp + (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\ .
\end{aligned} \tag{2.25}$$

This is a non-linear algebra, reminiscent of the W-algebras. What is new here are the non-local terms involving $\epsilon(\sigma-\sigma')$, or the $\frac{1}{k}$ in the mode algebra.

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* As a further consistency check one can verify that the Jacobi identities are satisfied.

2.3. The classical solutions of the equations of motion

The essential point for solving the equations of motion (2.2) is to realize the underlying Lie algebraic structure. Following Gervais and Saveliev [2], one first introduces fields a_1, a_2, a_+ and a_- subject to the following equations of motion

$$\begin{aligned}
&\partial_u \partial_v a_1 = -2(1+2a_+a_-)e^{-a_1} \\
&\partial_u \partial_v (2a_2 - a_1) + 2\partial_u(\partial_v a_+ a_-) = 0 \\
&\partial_u \left(e^{a_1 - 2a_2}\partial_v a_+\right) = 2a_+ e^{-2a_2} \\
&\partial_u \left[e^{-a_1+2a_2}\left(\partial_v a_- - a_-^2 \partial_v a_+\right)\right] = 2a_-(1+a_+a_-)e^{-2a_1+2a_2} \ .
\end{aligned} \tag{2.26}$$

It is then straightforward, although a bit lengthy, to show that we can identify

$$\begin{aligned}
\phi &= -\frac{1}{2}a_1 \\
\mathrm{sh}^2 r &= a_+a_- \\
\partial_u t &= \frac{1}{2i}\left[(1+2a_+a_-)\frac{\partial_u a_-}{a_-} - \frac{\partial_u a_+}{a_+} - 2(1+a_+a_-)\partial_u(a_1 - 2a_2)\right] \\
\partial_v t &= \frac{i}{2}\left[(1+2a_+a_-)\frac{\partial_v a_+}{a_+} - \frac{\partial_v a_-}{a_-}\right]
\end{aligned} \tag{2.27}$$

i.e. with these definitions the equations (2.26) and (2.2) are equivalent. The integrability condition for solving for t is given by the equations (2.26).

The advantage of equations (2.26) over equations (2.2) is that they follow from a Lie algebraic formulation. The relevant algebra is B_2 with generators h_1, h_2 (Cartan subalgebra) and $E_{e_1}, E_{e_2}, E_{e_1-e_2}, E_{e_1+e_2}$ and their conjugates $E_\alpha^+ = E_{-\alpha}$. Then $H = 2h_1 + h_2, J_+ = E_{e_1}$ and $J_- = E_{-e_1}$ span an A_1 subalgebra. H induces a gradation on B_2. The gradation 0 part $\mathcal{G}_0$ is spanned by h_1, h_2, E_{e_2} and $E_{-e_2} = E_{e_2}^+$. The corresponding group elements $g_0 \in G_0$ can be parametrized as

$$g_0 = \exp(a_+ E_{e_2}) \exp(a_- E_{e_2}^+) \exp(a_1 h_1 + a_2 h_2) \ . \tag{2.28}$$

One can then show that equations (2.26) are equivalent to

$$\partial_u(g_0^{-1}\partial_v g_0) = [J_- \, , \, g_0^{-1} J_+ g_0] \ . \tag{2.29}$$

If one now defines g_+ and g_- as solutions of the ordinary differential equations

$$\begin{aligned} \partial_u g_+^{-1} &= g_+^{-1}(g_0^{-1}J_+g_0) \\ \partial_v g_- &= g_-(g_0J_-g_0^{-1}) \ . \end{aligned} \tag{2.30}$$

and $g = g_-g_0g_+$ then one can show [2,3] that eq. (2.29) is in turn equivalent to

$$\partial_u(g^{-1}\partial_v g) = 0 \quad \text{and} \quad \partial_v(\partial_u g g^{-1}) = 0 \ . \tag{2.31}$$

The general solution to these equations is well-known: $g = g_L(u)g_R(v)$. But each group element g_L and g_R has again a Gauss decomposition $g_L(u) = g_{L-}(u)g_{L0}(u)g_{L+}(u)$ and $g_R(v) = g_{R-}(v)g_{R0}(v)g_{R+}(v)$ so that $g = g_{L-}(u)g_{L0}(u)g_{L+}(u)g_{R-}(v)g_{R0}(v)g_{R+}(v)$. On the other hand we also have the decomposition $g = g_-g_0g_+$ where g_+ and g_- must obey the differential equations (2.30). The latter translate into

$$\begin{aligned} \partial_u g_{L+}(u) &= -\mathcal{F}_L(u)g_{L+}(u) \quad , \quad \mathcal{F}_L(u) = g_{L0}^{-1}(u)J_+g_{L0}(u) \ , \\ \partial_v g_{R-}(v) &= g_{R-}(v)\mathcal{F}_R(v) \quad , \quad \mathcal{F}_R(v) = g_{R0}(v)J_-g_{R0}^{-1}(v) \ . \end{aligned} \tag{2.32}$$

The strategy then is

1. Pick some arbitrary $g_{L0}(u),\ g_{R0}(v) \in G_0$.
2. Compute the solutions $g_{L+}(u)$ and $g_{R-}(v)$ from the first order ordinary differential equations (2.32).
3. Let

$$\Gamma = g_{L0}(u)g_{L+}(u)g_{R-}(v)g_{R0}(v) \tag{2.33}$$

and choose a basis $|\lambda_\alpha\rangle$ of states annihilated by $\mathcal{G}_+$ (i.e. by $E_{e_1}, E_{e_1-e_2}$ and $E_{e_1+e_2}$. Then using the different decompositions of g we have

$$G_{\alpha\beta} \equiv \langle\lambda_\beta|g_0|\lambda_\alpha\rangle = \langle\lambda_\beta|\Gamma|\lambda_\alpha\rangle \ . \tag{2.34}$$

This yields all matrix elements of g_0, solution of equation (2.29), which in turn, as shown above, yields the solution for the a_1, a_2, a_+ and a_-, parametrized in terms of the arbitrary $g_{L0}(u)$ and $g_{R0}(v)$.

I will now display the resulting solutions. For details of the derivation, see ref. 3. One introduces three arbitrary ("left-moving") functions $f_1(u), f_+(u)$ and $f_-(u)$ of *one* variable (parametrizing $g_{L0}(u)$), and three arbitrary ("right-moving") functions $g_1(v), g_+(v)$ and $g_-(v)$ of *one* variable (parametrizing $g_{R0}(v)$). To write the results in a more compact way, introduce the functions of one variable

$$\begin{aligned} F_1(u) &= -\int^u e^{f_1}(1+2f_+f_-) \\ F_2(u) &= 2\int^u e^{f_1} f_- \\ F_3(u) &= -2\int^u e^{f_1} f_+(1+f_+f_-) \end{aligned} \tag{2.35}$$

as well as

$$F_+ = F_1 + f_+F_2 \quad , \quad F_- = F_3 - f_+F_1 \tag{2.36}$$

and similarly for $G_1(v)$ etc, with $f_i(u) \to g_i(v)$. Then one introduces the quantities X, Y, Z and V, W that depend on both variables u and v:

$$\begin{aligned} X &= 1 + f_+g_+ + F_+G_+ + F_-G_- \\ Y &= (1+f_+f_-)(1+g_+g_-) + f_-g_- + (F_1 - f_-F_-)(G_1 - g_-G_-) \\ &\quad + (F_2 + f_-F_+)(G_2 + g_-G_-) \\ Z &= 1 + 2F_1G_1 + F_2G_2 + F_3G_3 + (F_1^2 + F_2F_3)(G_1^2 + G_2G_3) \\ V &= -g_- - f_+ - g_+g_-f_+ - g_-F_+G_+ - g_-F_-G_- + F_-G_1 - F_+G_2 \\ W &= -f_- - g_+ - f_+f_-g_+ - f_-F_+G_+ - f_-F_-G_- + F_1G_- - F_2G_+ \ . \end{aligned} \tag{2.37}$$

They are sums of products of left-moving times right-moving quantities. The complete solution then is

$$\begin{aligned} e^{a_1} &= e^{-f_1-g_1} Z \\ e^{-a_2} &= \pm e^{f_2+g_2}\frac{Y}{Z} \\ a_+ &= e^{f_1-2f_2}\frac{V}{Y} \\ a_- &= e^{2f_2-f_1}\frac{YW}{Z} \end{aligned} \tag{2.38}$$

while the fifth equation is the relation $XY - Z = VW$. From equations (2.27) one immediately finds

$$
\begin{aligned}
\phi &= \frac{1}{2}\left(f_1 + g_1 - \log Z\right) \\
\mathrm{sh}^2 r &= \frac{VW}{Z} = \frac{XY}{Z} - 1 \\
t &= t_0 + i\int^u f_- f_+' - i\int^v g_- g_+' + \frac{i}{2}\log\frac{V}{W} \,.
\end{aligned}
\tag{2.39}
$$

which is the general solution of the equations of motion .

Using the equations of motion for ϕ, r and t it was shown above that the quantities $T \equiv T_{++}$ and $V^{\pm} \equiv V^{\pm}_{++}$ are conserved, i.e. can only depend on u. This means that they must be expressible entirely in terms of the $f_i(u)$'s. Given the complexity of the solutions (2.39) and (2.37) this is highly non-trivial and constitutes a severe consistency check. The same considerations apply to $\bar{T} \equiv T_{--}$ and $\bar{V}^{\pm} \equiv V^{\pm}_{--}$. One can indeed show [3] that

$$
\begin{aligned}
T_{++} &= \frac{1}{4}(f_1')^2 - \frac{1}{2}f_1'' + f_+'(f_-' - f_-^2 f_+') \\
V^+ &= \frac{1}{\sqrt{2}}(f_1' - \partial_u)\left[\exp\left(-2\int^u f_- f_+'\right)(f_-' - f_-^2 f_+')\right] \\
V^- &= \frac{1}{\sqrt{2}}(f_1' - \partial_u)\left[\exp\left(+2\int^u f_- f_+'\right) f_+'\right] \,.
\end{aligned}
\tag{2.40}
$$

It is then natural to set

$$
\begin{aligned}
f_- &= e^{\sqrt{2}\varphi_1} \\
f_+' &= \frac{1}{\sqrt{2}} e^{-\sqrt{2}\varphi_1}(\partial_\sigma\varphi_1 + i\partial_\sigma\varphi_2) \\
f_1 &= \sqrt{2}\varphi_3
\end{aligned}
\tag{2.41}
$$

(and with a similar relation between the g_i and $\bar{\varphi}_i$). The conserved quantities $T, V^{\pm}$ are easily expressed in terms of the φ_j as

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{j=1}^{3}(\partial_\sigma\varphi_j)^2 - \frac{1}{\sqrt{2}}\partial_\sigma^2\varphi_3 \\
V^{\pm} &= \frac{1}{2}(\sqrt{2}\partial_\sigma\varphi_3 - \partial_\sigma)\left[e^{\mp i\sqrt{2}\varphi_2}(\partial_\sigma\varphi_1 \mp i\partial_\sigma\varphi_2)\right] \,.
\end{aligned}
\tag{2.42}
$$

Thus in terms of the φ_j, T has the standard form of a stress-energy tensor in a conformal field theory with a background charge. The $V^{\pm}$ are *local* expressions of the fields φ_j, analogous to standard vertex operators. (Of course, their Poisson brackets exhibit the non-local $\epsilon(\sigma - \sigma')$-function.)

2.4. Canonical transformation to free fields

In principle one could now deduce the Poisson brackets of the f_i and g_i through the transformation induced by the classical solution (2.39) and (2.41). More precisely, one would have to allow formally that the f_i and g_i, respectively the φ_i and $\bar{\varphi}_i$, depend both on u and v since one has to consider the full phase space and not only the manifold of solutions to the equations of motion. Nevertheless, equations (2.39) and (2.41), as well as their time derivatives, constitute a phase space transformation from $r(\tau, \sigma)$, $t(\tau, \sigma)$, $\phi(\tau, \sigma)$ and their momenta $\Pi_r(\tau, \sigma)$, $\Pi_t(\tau, \sigma)$, $\Pi_\phi(\tau, \sigma)$ to new phase space variables $f_i(\tau, \sigma)$, $g_i(\tau, \sigma)$ and to $\varphi_i(\tau, \sigma), \bar{\varphi}_i(\tau, \sigma)$. (Of course, the equations of motion still imply $\partial_v f_i = \partial_u g_i = \partial_v \varphi_i = \partial_u \bar{\varphi}_i = 0$.) In practice, this would be very complicated to implement. It is much simpler to use the Poisson brackets of the T and $V^\pm$ derived before, and then consider the T and $V^\pm$ (or $\bar{T}$ and $\bar{V}^\pm$) as given in terms of the φ_i only (or $\bar{\varphi}_i$ only). Thus one does the phase space transformation in two steps: $r(\tau, \sigma)$, $t(\tau, \sigma)$, $\phi(\tau, \sigma)$, $\Pi_r(\tau, \sigma)$, $\Pi_t(\tau, \sigma)$, $\Pi_\phi(\tau, \sigma)$ $\rightarrow$ $T_{\pm\pm}(\tau, \sigma)$, $V^+_{\pm\pm}(\tau, \sigma)$, $V^-_{\pm\pm}(\tau, \sigma)$ $\rightarrow$ $\varphi_i(\tau, \sigma)$, $\bar{\varphi}_i(\tau, \sigma)$. This yields the Poisson brackets of the φ_i and of the $\bar{\varphi}_i$ in a relatively easy way. These Poisson brackets are simple, standard harmonic oscillator Poisson brackets:

$$\{\partial_\sigma \varphi_i(\sigma)\,,\, \partial_{\sigma'} \varphi_j(\sigma')\} = \gamma^2 \delta_{ij} \delta'(\sigma - \sigma') \tag{2.43}$$

or

$$\{\varphi_i(\sigma)\,,\, \varphi_j(\sigma')\} = -\frac{\gamma^2}{2} \delta_{ij} \epsilon(\sigma - \sigma')\ . \tag{2.44}$$

If one considers $\sigma \in S^1$, the mode expansion*

$$\partial_\sigma \varphi_j(\tau, \sigma) = \frac{\gamma}{\sqrt{2\pi}} \sum_n \varphi_n^j e^{-in(\tau+\sigma)} \tag{2.45}$$

---

* Note that this is a Fourier expansion in σ. The factor $e^{-in\tau}$ is only extracted from the φ_n^j for convenience. The φ_n^j are still functions of τ. It is only if one imposes the equations of motion that the φ_n^j are constant.

leads to

$$i\{\varphi_n^j\,,\,\varphi_m^k\} = n\delta^{ij}\delta_{n+m,0} \tag{2.46}$$

as appropriate for three sets of harmonic oscillators. The Poisson brackets of the $\bar{\varphi}_i$ are analogous, while $\{\varphi_i(\sigma)\,,\,\bar{\varphi}_j(\sigma')\} = 0$.

2.5. Associated linear differential equation

From experience with integrable models, in particular the (non-affine, conformally invariant) Toda models [4] one expects that the conserved quantities appear as coefficients of an ordinary linear differential equation, e.g. for the A_{m-1} Toda model

$$\left[\partial_u^m - \sum_{k=2}^{m} W^{(k)}(u)\partial_u^{m-k}\right]\psi(u) = 0 \ . \tag{2.47}$$

The $W^{(k)}(u), k = 2, \ldots m$ are the conserved quantities which form the W_m-algebra, while the solutions $\psi_j(u)$ of this equation, together with solutions $\chi_j(v)$ of a similar equation in v, are the building blocks of the general solution to the Toda equations of motion. The $W^{(k)}(u)$ have (naive) dimension k.

In the present theory all conserved quantities have dimension 2, so one might expect a second-order differential equation of the form $(-\partial^2 + U)\psi = 0$. To fit three conserved quantities into U, it needs to be at least a 2×2-matrix. In ref. 3, I guessed the following linear differential equation

$$\left[\partial_u^2 - \begin{pmatrix} \alpha T(u) & \beta_+ V^+(u) \\ \beta_- V^-(u) & \delta T(u) \end{pmatrix}\right]\Psi(u) = 0 \ . \tag{2.48}$$

Actually, if one inserts the free-field representation (2.42) of T and $V^\pm$ one can see that this equation admits the very simple solution

$$\begin{aligned} \psi_1 &= \exp(a\varphi_1 + ib\varphi_2 + d\varphi_3) \\ \psi_2 &= \exp(a\varphi_1 - ib\varphi_2 + d\varphi_3) \ . \end{aligned} \tag{2.49}$$

if and only if

$$
\begin{aligned}
a = \frac{1}{\sqrt{2}} \quad &, \quad b = d = -\frac{1}{\sqrt{2}} \\
\alpha = \delta = 1 \quad &, \beta_+ = \beta_- = -\sqrt{2} \, .
\end{aligned}
\tag{2.50}
$$

The existence of this very simple solution already is an indication that the differential equation (2.48) is probably the correct generalization of (2.47) for W-algebras to the present V-algebra. That this is indeed so was shown in ref. 5, and will be reviewed in the next section.

3. The V-algebra as second Gelfand-Dikii bracket, the resolvent and matrix KdV-hierarchy

In this section, following ref. 5, I will relate the V-algebra obtained above to well-known mathematical structures. In the case of the standard W_m-algebras it was shown [6] that their classical version coincides with second Gelfand-Dikii symplectic structure [7-12] on the space of ordinary differential operators of order m (as in eq. (2.47)). What I will show here in the remainder of this contribution, is that a straightforward matrix generalization leads to a whole family of V-algebras. In this section I will obtain the algebra of the previous section from the 2×2-matrix second-order differential operator

$$
L = \partial^2 - U \; , \quad U = \begin{pmatrix} T & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T \end{pmatrix} \; . \tag{3.1}
$$

The general case of $n \times n$-matrix m^{th}-order differential operators leading to $V_{n,m}$-algebras will be treated in the next section.

If not indicated otherwise, $\partial \equiv \partial_\sigma$, and U depends on σ. U may also depend on other parameters $t_1, t_2, \ldots$. In the previous section U depended on τ and σ, but, upon imposition of the equations of motion , only through the combination $\sigma + \tau$. Actually, as usual (see below), this is the first flow of the matrix KdV hierarchy: $\frac{\partial}{\partial t_1} U = \partial_\sigma U$, so that τ is identified with t_1.

Let f and g be differential polynomial functionals on the space of second-order differential operators L, i.e. polynomial functionals of U (and its derivatives). One defines the pseudo-differential operator

$$X_f = \partial^{-1} X_1 + \partial^{-2} X_2 \quad , \quad X_1 = \frac{\delta f}{\delta U} \tag{3.2}$$

where* $\partial^{-1}\partial = \partial\partial^{-1} = 1$ and $\frac{\delta}{\delta U}$ is defined as

$$\frac{\delta}{\delta U} = \begin{pmatrix} \frac{1}{2}\frac{\delta}{\delta T} & -\frac{1}{\sqrt{2}}\frac{\delta}{\delta V^-} \\ -\frac{1}{\sqrt{2}}\frac{\delta}{\delta V^+} & \frac{1}{2}\frac{\delta}{\delta T} \end{pmatrix} \tag{3.3}$$

so that $\frac{\delta}{\delta U}\int \mathrm{tr}\, U^n = nU^{n-1}$, and X_2 is determined (cf. e.g. [8,9]) by requiring† $\mathrm{res}[L, X_f] = 0$. As usual, the residue of a pseudo-differential operator, denoted res, is the coefficient of ∂^{-1}. One then has

$$X_2' = \frac{1}{2}\left(\frac{\delta f}{\delta U}\right)'' + \frac{1}{2}\left[U, \frac{\delta f}{\delta U}\right] \ . \tag{3.4}$$

Integrating this equation yields X_2. Here, one observes a new feature as compared to the scalar case: since in general $\left[U, \frac{\delta f}{\delta U}\right] \neq 0$, X_2 will be given by a non-local expression involving an integral. This is the origin of the non-local terms ($\sim \epsilon(\sigma - \sigma')$) in the V-algebra (2.23).

In analogy with the standard procedure [7-12], I define the second Gelfand-Dikii bracket in the matrix case as follows

$$\{f, g\}_{\mathrm{GD2}} = a \int \mathrm{d}\sigma\, \mathrm{tr}\ \ \mathrm{res}\left(L(X_f L)_+ X_g - (LX_f)_+ L X_g\right) \ . \tag{3.5}$$

Using the definitions of L and X_f, X_g it is straightforward to obtain

$$\begin{aligned}\{f, g\}_{\mathrm{GD2}} = a \int \mathrm{d}\sigma\, \mathrm{tr} \Bigg(& \frac{1}{2}\frac{\delta f}{\delta U}\partial^3 \frac{\delta g}{\delta U} + \frac{1}{2}\left[U, \frac{\delta g}{\delta U}\right]\left(\partial^{-1}\left[U, \frac{\delta f}{\delta U}\right]\right) \\ & - \frac{\delta f}{\delta U}(U\partial + \frac{1}{2}U')\frac{\delta g}{\delta U} + \frac{\delta g}{\delta U}(U\partial + \frac{1}{2}U')\frac{\delta f}{\delta U}\Bigg)\end{aligned} \tag{3.6}$$

where the ∂^{-1} is meant to act only on $\left[U, \frac{\delta f}{\delta U}\right]$.

⋆ On the circle S^1 e.g., ∂^{-1} is well-defined on functions without constant Fourier mode, i.e. $f(\sigma) = \sum_{m\neq 0} f_m e^{-im\sigma}$. One easily sees that $(\partial^{-1} f)(\sigma) = \int \mathrm{d}\sigma'\ \frac{1}{2}\epsilon(\sigma - \sigma') f(\sigma')$ with $\epsilon(\sigma - \sigma') = \frac{1}{\pi i}\sum_{m\neq 0} \frac{1}{m} e^{im(\sigma - \sigma')}$.

† This condition is necessary since the coefficient of ∂ in L vanishes. I will discuss this condition in more detail in the next section.

Inserting the definitions of the 2×2-matrices U and $\frac{\delta}{\delta U}$ one obtains

$$
\begin{aligned}
\{f,g\}_{\mathrm{GD2}} = -\frac{a}{2}\int \mathrm{d}\sigma \Bigg[& -\frac{1}{2}\frac{\delta f}{\delta T}\partial^3\frac{\delta g}{\delta T} - \frac{1}{2}\frac{\delta f}{\delta V^+}\partial^3\frac{\delta g}{\delta V^-} - \frac{1}{2}\frac{\delta f}{\delta V^-}\partial^3\frac{\delta g}{\delta V^+} \\
& + T\left(\frac{\delta f}{\delta T}\partial\frac{\delta g}{\delta T} + \frac{\delta f}{\delta V^+}\partial\frac{\delta g}{\delta V^-} + \frac{\delta f}{\delta V^-}\partial\frac{\delta g}{\delta V^+} - (f \leftrightarrow g)\right) \\
& + V^+\left(\frac{\delta f}{\delta T}\partial\frac{\delta g}{\delta V^+} + \frac{\delta f}{\delta V^+}\partial\frac{\delta g}{\delta T} - (f \leftrightarrow g)\right) \\
& + V^-\left(\frac{\delta f}{\delta T}\partial\frac{\delta g}{\delta V^-} + \frac{\delta f}{\delta V^-}\partial\frac{\delta g}{\delta T} - (f \leftrightarrow g)\right)\Bigg] \\
-\frac{a}{2}\int\int \mathrm{d}\sigma\mathrm{d}\sigma'\epsilon(\sigma-\sigma') & \left(V^+\frac{\delta f}{\delta V^+} - V^-\frac{\delta f}{\delta V^-}\right)(\sigma)\left(V^+\frac{\delta g}{\delta V^+} - V^-\frac{\delta g}{\delta V^-}\right)(\sigma') \,.
\end{aligned}
\tag{3.7}
$$

For completeness, let me note that the first Gelfand-Dikii bracket defined by $\{f,g\}_{\mathrm{GD1}} = a\int \mathrm{d}\sigma \,\mathrm{tr}\;\; \mathrm{res}([L, X_f]_+ X_g)$ is simply

$$
\{f,g\}_{\mathrm{GD1}} = -a\int \mathrm{d}\sigma \left(\frac{\delta f}{\delta T}\partial\frac{\delta g}{\delta T} + \frac{\delta f}{\delta V^+}\partial\frac{\delta g}{\delta V^-} + \frac{\delta f}{\delta V^-}\partial\frac{\delta g}{\delta V^+}\right) \,. \tag{3.8}
$$

Taking f, g to be T, V^+ or V^- one concludes that the second Gelfand-Dikii bracket (3.7) coincides with the V-algebra (2.23) provided one chooses

$$
a = -2\gamma^{-2} \,. \tag{3.9}
$$

Henceforth I will adopt this choice and the only Poisson bracket used is the second Gelfand-Dikii bracket, unless otherwise stated. Let me note that the free field representation of $T, V^\pm$ in terms of the φ_i obtained in the previous section constitutes a Miura transformation since it maps the second Gelfand-Dikii symplectic structure to the much simpler free-field Poisson brackets. I will discuss the Miura transformation in more detail in the next section.

Let me now turn to some results that are a bit more specific to second-order differential operators, generalizing the classical work of ref. 13. For a hermitian $n \times n$-matrix U define the $n \times n$-matrix resolvent R as

$$
R(x,y;\xi) = \langle x|(-\partial^2 + U + \xi)^{-1}|y\rangle \tag{3.10}
$$

which is a solution of

$$(-\partial_x^2 + U(x) + \xi)R(x, y; \xi) = \delta(x - y) . \tag{3.11}$$

Just as in the scalar case, $n = 1$, the restriction of the resolvent to the diagonal, $R(x; \xi) \equiv R(x, x; \xi)$ has an asymptotic expansion for $\xi \to \infty$ of the form

$$R(x; \xi) = \sum_{n=0}^{\infty} \frac{R_n[u]}{\xi^{n+1/2}} . \tag{3.12}$$

This equation is to be understood as an equality of the asymptotic expansions in half-integer powers of $1/\xi$, disregarding any terms that vanish exponentially fast as $\xi \to \infty$. From the defining differential equation for $R(x, y; \xi)$ one easily establishes that $R \equiv R(x; \xi) = R(x, x; \xi)$ satisfies

$$R''' - 2(UR' + R'U) - (U'R + RU') + [U, \partial^{-1}[U, R]] = 4\xi R' \tag{3.13}$$

(where $R' \equiv \partial_x R(x; \xi)$ etc.) and hence that the coefficients R_n of the asymptotic expansion (3.12) satisfy

$$4R'_{n+1} = R'''_n - 2(UR'_n + R'_nU) - (U'R_n + R_nU') + [U, \partial^{-1}[U, R_n]] . \tag{3.14}$$

This allows us to determine the R_n recursively:

$$\begin{aligned}
R_0 =& \frac{1}{2} \\
R_1 =& -\frac{1}{4}U \\
R_2 =& \frac{1}{16}(3U^2 - U'') \\
R_3 =& -\frac{1}{64}(10U^3 - 5UU'' - 5U''U - 5U'^2 + U^{(4)}) \\
R_4 =& \frac{1}{256}(35U^4 - 21U^2U'' - 21U''U - 28UU''U - 28U'^2U - 28UU'^2 - 14U'UU' \\
& + 7UU^{(4)} + 7U^{(4)}U + 14U'U''' + 14U'''U' + 21U''^2 + U^{(6)}) .
\end{aligned} \tag{3.15}$$

One of the reasons why one is interested in the coefficients R_n is the following: if one defines an infinite family of Hamiltonians as

$$H_n = \frac{(-4)^n}{2(2n-1)} \int dx \ \mathrm{tr}\ R_n(x) \tag{3.16}$$

one can show [5] that all H_n are in involution*:

$$\{H_n, H_m\} = 0 \ . \tag{3.17}$$

The proof uses the fact that the recursion relation between the H_n (inherited from the one between the R_n) relates the first and second Gelfand-Dikii brackets: $\{H_n, H_m\}_2 \sim \{H_n, H_{m+1}\}_1 = -\{H_{m+1}, H_n\}_1 \sim -\{H_{m+1}, H_{n-1}\}_2 = \{H_{n-1}, H_{m+1}\}_2$, so that by iteration one arrives at $\{H_n, H_m\}_2 = \{H_0, H_{n+m}\}_2 = 0$. The first few H_n are

$$\begin{aligned}
H_1 &= \frac{1}{2}\int \mathrm{tr}\ U = \int T \\
H_2 &= \frac{1}{2}\int \mathrm{tr}\ U^2 = \int (T^2 + 2V^+V^-) \\
H_3 &= \frac{1}{2}\int \mathrm{tr}\ (2U^3 + U'^2) = \int (2T^3 + 12TV^+V^- + T'^2 + 2V^{+\prime}V^{-\prime}) \\
H_4 &= \frac{1}{2}\int \mathrm{tr}\ (5U^4 + 10UU'^2 + U''^2) \\
&= \int (5T^4 + 20V^{+2}V^{-2} + 60T^2V^+V^- + 10TT'^2 \\
&\qquad + 20TV^{+\prime}V^{-\prime} + 20T'V^+V^{-\prime} + 20T'V^{+\prime}V^- + T''^2 + 2V^{+\prime\prime}V^{-\prime\prime}) \ .
\end{aligned} \tag{3.18}$$

Note that within our non-abelian Toda field theory, H_1 is the integral of the (++)-component of the energy-momentum tensor $T \equiv T_{++}$. Thus the Hamiltonian of the theory is $H = L_0 + \bar{L}_0 = H_1 + \bar{H}_1$. The other H_n are all in involution with H_1 (and obviously also with $\bar{H}_1$) and thus are conserved under the conformal evolution of the non-abelian Toda field theory.

As usual one can define an infinite hierarchy of flows by†

$$\frac{\partial U}{\partial t_r} = \gamma^{-2}\{U, H_r\} \ . \tag{3.19}$$

Since $\{H_r, H_s\} = 0$ it follows from the Jacobi identity that all flows commute.

$\star$ Such a family of H_n in involution for the matrix Schrödinger operator L was already obtained a long time ago in ref. 14.

$\dagger$ As before, $\{\cdot,\cdot\}$ is meant to be the second Gelfand-Dikii bracket. But since $4\frac{\partial U}{\partial t_r} = 4\gamma^{-2}\{U, H_r\}_{\mathrm{GD2}} = \gamma^{-2}\{U, H_{r+1}\}_{\mathrm{GD1}}$, one can also use the first Gelfand-Dikii bracket and the next higher Hamiltonian instead.

The flow in t_2 gives the matrix generalisation of the KdV equation:‡

$$\frac{\partial U}{\partial t_2} = (3U^2 - U'')' \tag{3.20}$$

or in components

$$\frac{\partial T}{\partial t_2} = (3T^2 - T'' + 6V^+V^-)' \quad , \quad \frac{\partial V^\pm}{\partial t_2} = (6TV^\pm - V^{\pm\prime\prime})' \tag{3.21}$$

Just as the Virasoro algebra is a subalgebra of the V-algebra (2.23), the KdV equation is simply obtained by setting $V^\pm = 0$. Note that since all H_n are symmetric in V^+ and V^- any non-local term ($\sim \epsilon(\sigma - \sigma')$) that might appear cancels in all flow equations, and the latter are always partial *differential* equations.§

4. The Gelfand-Dikii symplectic structure for general $n \times n$-matrix m^{th}-order differential operators and the $V_{n,m}$-algebras

In this section, following ref. 16, I will compute the second Gelfand-Dikii bracket of two functionals f and g of the $n \times n$ matrix coefficient functions $U_k(\sigma)$ of the linear m^{th}-order differential operator¶

$$L = -\partial^m + \sum_{k=1}^{m} U_k \partial^{m-k} \equiv \sum_{k=0}^{m} U_k \partial^{m-k} \ . \tag{4.1}$$

where $\partial = \frac{d}{d\sigma}$. To make subsequent formula more compact, I formally introduced $U_0 = -\mathbf{1}$. The fuctionals f and g one considers are of the form $f = \int \mathrm{tr}\ P(U_k)$, where P is some polynomial in the U_k, $k = 1, \dots m$, and their derivatives (i.e. a differential polynomial in the U_k). P may also contain other constant or non-constant numerical matrices so that these functionals are fairly general. (Under suitable boundary conditions, *any* functional of the U_k and their derivatives can be approximated to arbitrary "accuracy" by an f of the type considered.) The integral can either be defined in a formal sense as assigning to any function an equivalence class by considering functions only up to total derivatives (see e.g. section 1 of

‡ Matrix KdV flows were already discussed a long time ago in ref. 15.

§ This also follows from the equivalence with the first Gelfand-Dikii bracket which is local, see above.

¶ Throughout the rest of this paper, m will denote the order of L which is a positive integer.

the second ref. 7), or in the standard way if one restricts the integrand, i.e. the U_k, to the class of e.g. periodic functions or sufficiently fast decreasing functions on $\mathbf{R}$, etc. All that matters is that the integral of a total derivative vanishes and that one can freely integrate by parts.

To define the Gelfand-Dikii brackets, it is standard to use pseudo-differential operators [8,9] involving integer powers of ∂^{-1}, as already encountered in the previous section. Again, ∂^{-1} can be defined in a formal sense by $\partial\partial^{-1} = \partial^{-1}\partial = 1$, but one can also give a concrete definition on appropriate classes of functions. For example for C^∞-functions h on $\mathbf{R}$ decreasing exponentially fast as $\sigma \to \pm\infty$ one can simply define $(\partial^{-1}h)(\sigma) = \int_{-\infty}^{\infty} \mathrm{d}\sigma' \frac{1}{2}\epsilon(\sigma - \sigma')h(\sigma')$.

Throughout this section, I will only state the results. The reader is referred to ref. 16 for all proofs.

4.1. The Gelfand-Dikii brackets for general $U_1, \dots U_m$

In analogy with the scalar case (i.e. $n = 1$) [8,9,10,12], I define the second Gelfand-Dikii bracket associated with the $n \times n$-matrix m^{th}-order differential operator L as

$$\{f, g\}_{(2)} = a \int \text{tr res } \big(L(X_f L)_+ X_g - (L X_f)_+ L X_g\big) \tag{4.2}$$

where a is an arbitrary scale factor and X_f, X_g are the pseudo-differential operators

$$\begin{aligned} X_f &= \sum_{l=1}^{m} \partial^{-l} X_l \quad , \quad X_g = \sum_{l=1}^{m} \partial^{-l} Y_l \\ X_l &= \frac{\delta f}{\delta U_{m+1-l}} \quad , \quad Y_l = \frac{\delta g}{\delta U_{m+1-l}} \ . \end{aligned} \tag{4.3}$$

The functional derivative of $f = \int \mathrm{d}\sigma \, \text{tr } P(U)$ is defined as usual by

$$\left(\frac{\delta f}{\delta U_k}(\sigma)\right)_{ij} = \sum_{r=0}^{\infty} \left(-\frac{d}{d\sigma}\right)^r \left(\frac{\partial \text{tr } P(u)(\sigma)}{\partial (U_k^{(r)})_{ji}}\right) \tag{4.4}$$

where $(U_k^{(r)})_{ji}$ denotes the (j,i) matrix element of the r^{th} derivative of U_k. It is easily seen, that for $n = 1$, equations (4.2)-(4.4) reduce to the standard definitions of the Gelfand-Dikii brackets [8,9,10,12]. For $m = 2$, $n = 2$ and with the extra restrictions $U_1 = 0$, $\text{tr } \sigma_3 U_2 = 0$, equation (4.2) was shown in the previous section to reproduce the original V-algebra (2.23) (with $a = -2\gamma^2$). Working through the algebra (see [16] for details) and defining for $l \geq 1$

$$S^{q,j}_{r,l} = \sum_{s=\max(0,r)}^{\min(q,j)} (-)^{s-r} \binom{s-r+l-1}{l-1} \binom{q}{s} , \tag{4.5}$$

with $S^{q,j}_{r,l} = 0$ if $\max(0,r) > \min(q,j)$, one obtains the

Proposition 1 : The second Gelfand-Dikii bracket associated with the $n \times n$-matrix m^{th}-order differential operator L as defined by eqs. (4.1), (4.2) and (4.3) equals

$$\begin{aligned} \{f,g\}_{(2)} &= a \int \mathrm{tr} \sum_{j=0}^{m-1} \mathcal{V}(f)_j Y_{j+1} \\ \mathcal{V}(f)_j &= \sum_{l=1}^{m} \sum_{p=0}^{2m-j-l} \sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} \left(S^{q,j}_{q-p,l} - \binom{q-l}{p} \right) U_{m-q} (X_l U_{m-p-j-l+q})^{(p)} . \end{aligned} \tag{4.6}$$

It is not obvious that (4.6) satisfies antisymmetry or the Jacobi identity, but this will follow from the Miura transformation discussed below.

4.2. The Gelfand-Dikii brackets reduced to $U_1 = 0$

The problem of consistently restricting a given symplectic manifold (phase space) to a symplectic submanifold by imposing certain constraints $\phi_i = 0$ has been much studied in the literature. The basic point is that for a given phase space one cannot set a coordinate to a given value (or function) without also eliminating the corresponding momentum. More generally, to impose a constraint $\phi = 0$ consistently, one has to make sure that for any functional f the bracket $\{\phi, f\}$ vanishes if the constraint $\phi = 0$ is imposed *after* computing the bracket. In general this results in a modification of the original Poisson bracket.

Here, I want to impose $\{f, U_1\}|_{U_1=0} = 0$ for all f. Since $Y_m = \frac{\delta g}{\delta U_1}$, one sees from (4.6) that this requires $\mathcal{V}(f)_{m-1}|_{U_1=0} = 0$. In practice this determines X_m which otherwise would be undefined if one starts with $U_1 = 0$. In the scalar case it is known [8,9] that X_m should be determined by res $[L, X_f] = 0$. The following Lemma shows that this is still true in the matrix case.

Lemma 2 : One has $\mathcal{V}(f)_{m-1} = -\text{res}\,[L, X_f]$, and for $U_1 = 0$, res $[L, X_f] = 0$ is equivalent to

$$X_m = \frac{1}{m}\sum_{l=1}^{m-1}\left(\partial^{-1}[U_{m+1-l}, X_l] + \sum_{k=l}^{m}(-)^{k-l}\binom{k}{l-1}(X_l U_{m-k})^{(k-l)}\right) . \tag{4.7}$$

Note the commutator term which is a new feature of the present matrix case as opposed to the scalar case.

One of the main results then is the following

Theorem 3 : The second Gelfand-Dikii bracket for $n \times n$-matrix m^{th}-order differential operators L with vanishing U_1 is given by

$$\begin{aligned}
\{f, g\}_{(2)} =& a \int \text{tr}\, \sum_{j=0}^{m-2} \tilde{\mathcal{V}}(f)_j Y_{j+1} \,, \\
\tilde{\mathcal{V}}(f)_j =& \frac{1}{m}\sum_{l=1}^{m-1}[U_{m-j}, \partial^{-1}[X_l, U_{m-l+1}]] \\
&+ \frac{1}{m}\sum_{l=1}^{m-1}\Bigg\{\sum_{k=0}^{m-l}(-)^k\binom{k+l}{l-1}(X_l U_{m-k-l})^{(k)} U_{m-j} \\
&\qquad - \sum_{k=0}^{m-j-1}\binom{k+j+1}{j} U_{m-k-j-1}(U_{m-l+1}X_l)^{(k)}\Bigg\} \\
&+ \sum_{l=1}^{m-1}\sum_{p=0}^{2m-j-l}\sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} C^{q,j}_{q-p,l} U_{m-q}(X_l U_{m-p-j-l+q})^{(p)} \,, \\
C^{q,j}_{q-p,l} =& S^{q,j}_{q-p,l} - \binom{q-l}{p} - \frac{1}{m}(-)^{q-p+j}\binom{q}{j}\binom{p-q+j+l}{l-1}
\end{aligned} \tag{4.8}$$

where the $S^{q,j}_{q-p,l}$ are defined by eq. (4.5), and it is understood that $U_0 = -1$ and $U_1 = 0$.

Remark 4 : If one takes $m = 2$, $L = -\partial^2 + U$, so that $U_2 \equiv U$ and $X_1 \equiv X$, only $\tilde{\mathcal{V}}(f)_0$ is non-vanishing:

$$\tilde{\mathcal{V}}(f)_0 = -\frac{1}{2}[U, \partial^{-1}[U, X]] + \frac{1}{2}(XU + UX)' + \frac{1}{2}(X'U + UX') - \frac{1}{2}X''' \tag{4.9}$$

and with $X = \frac{\delta f}{\delta U}$ and $Y = \frac{\delta g}{\delta U}$ one obtains (using $\int x\partial^{-1}y = -\int(\partial^{-1}x)y$)

$$\{f, g\}_{(2)} = a \int \text{tr}\, \left(-\frac{1}{2}[U, X]\partial^{-1}[U, Y] + \frac{1}{2}(X'Y + YX' - XY' - Y'X)U - \frac{1}{2}YX'''\right) \tag{4.10}$$

which obviously is a generalization of the original V-algebra (2.23) to arbitrary $n \times n$-matrices

$U \equiv U_2$. To appreciate the structure of the non-local terms, I explicitly write this algebra in the simplest case for $n = 2$ (but *without* the restriction tr $\sigma_3 U = 0$ which is present for (2.23)). Let

$$U = \begin{pmatrix} T + V_3 & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T - V_3 \end{pmatrix} . \tag{4.11}$$

Then one obtains from (4.10) (with $a = -2\gamma^2$) the algebra

$$\begin{aligned}
\gamma^{-2}\{T(\sigma)\,,\,T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma') \\
\gamma^{-2}\{T(\sigma)\,,\,V^\pm(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[V^\pm(\sigma')\delta(\sigma-\sigma')\right] \\
\gamma^{-2}\{T(\sigma)\,,\,V_3(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[V_3(\sigma')\delta(\sigma-\sigma')\right] \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\pm(\sigma')\} &= \epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\mp(\sigma')\} &= -\epsilon(\sigma-\sigma')(V^\pm(\sigma)V^\mp(\sigma') + V_3(\sigma)V_3(\sigma')) \\
&\quad + (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma') \\
\gamma^{-2}\{V_3(\sigma)\,,\,V^\pm(\sigma')\} &= \epsilon(\sigma-\sigma')V^\pm(\sigma)V_3(\sigma') \\
\gamma^{-2}\{V_3(\sigma)\,,\,V_3(\sigma')\} &= \epsilon(\sigma-\sigma')(V^+(\sigma)V^-(\sigma') + V^-(\sigma)V^+(\sigma')) \\
&\quad + (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma')\ .
\end{aligned} \tag{4.12}$$

Once again, one sees that T generates the conformal algebra with a central charge, while V^+, V^- and V_3 are conformally primary fields of weight (spin) two. It is easy to check on the example (4.10), that antisymmetry and the Jacobi identity are satisfied. For general m this follows from the Miura transformation to which I now turn.

4.3. The Miura transformation : The case of general $U_1, \ldots U_m$

Definition and Lemma 5 : Introduce the $n \times n$-matrix-valued functions $P_j(\sigma)$, $j = 1, \ldots m$. Then for functionals f, g (integrals of traces of differential polynomials) of the P_j the following Poisson bracket is well-defined

$$\{f, g\} = a \sum_{i=1}^{m} \int \text{tr}\left(\left(\frac{\delta f}{\delta P_i}\right)' \frac{\delta g}{\delta P_i} - \left[\frac{\delta f}{\delta P_i}, \frac{\delta g}{\delta P_i}\right] P_i\right) \tag{4.13}$$

or equivalently for $n \times n$-matrix-valued (numerical) test-functions F and G

$$\{ \int \mathrm{tr}\ F P_i, \int \mathrm{tr}\ G P_j \} = a\, \delta_{ij} \int \mathrm{tr}\ \left(F' G - [F, G] P_i \right) \ . \tag{4.14}$$

Note that due to the δ_{ij} in (4.14) one has m decoupled Poisson brackets. In the scalar case ($n = 1$), (4.14) simply gives $\{P_i(\sigma), P_j(\sigma')\} = (-a)\delta_{ij}\delta'(\sigma - \sigma')$. These are m free fields or m $U(1)$ current algebras. In the matrix case, one easily sees that each P_j actually gives a $gl(n)$ current algebra. So one has no longer free fields but m completely decoupled current algebras. This is still much simpler than the bracket (4.6). To connect both brackets one starts with the following obvious

Lemma 6 : Let $P_j, j = 1, \ldots m$ be as in Lemma 5. Then

$$L = -(\partial - P_1)(\partial - P_2) \ldots (\partial - P_m) \tag{4.15}$$

is a m^{th}-order $n \times n$-matrix linear differential operator and can be written $L = \sum_{k=0}^{m} U_{m-k} \partial^k$ with $U_0 = -1$ as before. This identification gives all $U_k, k = 1, \ldots m$ as k^{th}-order differential polynomials in the P_j, i.e. it provides an embedding of the algebra of differential polynomials in the U_k into the algebra of differential polynomials in the P_j. One has in particular

$$U_1 = \sum_{j=1}^{m} P_j \quad , \quad U_2 = -\sum_{i<j}^{m} P_i P_j + \sum_{j=2}^{m} (j-1) P_j' \ . \tag{4.16}$$

I will call the embedding given by (4.15) a (matrix) Miura transformation. The most important property of this Miura transformation is given by the following matrix-generalization of a well-known theorem [11, 12].

Theorem 7 : Let $f(U)$ and $g(U)$ be functionals of the U_k, $k = 1, \ldots m$. By Lemma 6 they are also functionals of the P_j, $j = 1, \ldots m$: $f(U) = \tilde{f}(P)$, $g(U) = \tilde{g}(P)$ where $\tilde{f}(P) = f(U(P))$ etc. One then has

$$\{\tilde{f}(P), \tilde{g}(P)\} = \{f(U), g(U)\}_{(2)} \tag{4.17}$$

where the bracket on the l.h.s. is the Poisson bracket (4.13) and the bracket on the r.h.s. is the second Gelfand-Dikii bracket (4.6).

The previous Theorem states that one can either compute $\{U_k, U_l\}$ using the complicated formula (4.6) or using the simple Poisson bracket (4.13) for more or less complicated functionals $U_k(P)$ and $U_l(P)$. In particular Lemma 5 implies the

Corollary 8 : The second Gelfand-Dikii bracket (4.6) obeys antisymmetry and the Jacobi identity. Bilinearity in f and g being evident, it is a well-defined Poisson bracket.

4.4. The Miura transformation : The case $U_1 = 0$

As seen from (4.16), $U_1 = 0$ corresponds to $\sum_{i=1}^m P_i = 0$. In order to describe the reduction to $\sum_i P_i = 0$ it is convenient to go from the $P_i, i = 1, \ldots m$ to a new "basis": $Q = \sum_{i=1}^m P_i$ and $\mathcal{P}_a, a = 1, \ldots m-1$ where all $\mathcal{P}_a$ lie in the hyperplane $Q = 0$. Of course, Q and each $\mathcal{P}_a$ are still $n \times n$-matrices. More precisely:

Definition and Lemma 9 : Consider a $(m-1)$-dimensional vector space, and choose an overcomplete basis of m vectors h_j, $j = 1, \ldots m$. Denote the components of each h_j by h_j^a, $a = 1, \ldots m-1$. Choose the h_j such that

$$\begin{aligned} \sum_{j=1}^m h_j &= 0 \ , \\ h_i \cdot h_j &= \delta_{ij} - \frac{1}{m} \ , \\ \sum_{i=1}^m h_i^a h_i^b &= \delta_{ab} \end{aligned} \tag{4.18}$$

and define the completely symmetric rank-3 tensor D_{abc} by

$$D_{abc} = \sum_{i=1}^m h_i^a h_i^b h_i^c \ . \tag{4.19}$$

Define Q and $\mathcal{P}_a$, $a = 1, \ldots m-1$ to be the following linear combinations of the P_j's

$$\mathcal{P}_a = \sum_{j=1}^m h_j^a P_j \quad , \quad Q = \sum_{j=1}^m P_j \ . \tag{4.20}$$

If one considers the P_j as an orthogonal basis in a m-dimensional vector-space, then the $\mathcal{P}_a$ are an orthogonal basis in a $(m-1)$-dimensional hyperplane orthogonal to the line spanned by Q.

Proposition 10 : The Poisson bracket (4.13) can be reduced to the symplectic submanifold with $Q \equiv \sum_{j=1}^{m} P_j = 0$. The reduced Poisson bracket is

$$\{f, g\} = a \int \mathrm{tr}\, \left(W_b V_b' - [V_b, W_c] D_{bcd} \mathcal{P}_d - \frac{1}{m} [V_b, \mathcal{P}_b] \partial^{-1} [W_c, \mathcal{P}_c] \right) \tag{4.21}$$

where V_b and W_b denote $V_a = \frac{\delta f}{\delta \mathcal{P}_a}$, $V_0 = \frac{\delta f}{\delta Q}$, $W_a = \frac{\delta g}{\delta \mathcal{P}_a}$ and $W_0 = \frac{\delta g}{\delta Q}$. Equivalently one has

$$\{ \int \mathrm{tr}\, F\mathcal{P}_b, \int \mathrm{tr}\, G\mathcal{P}_c \} = a \int \mathrm{tr}\, \left(GF' \delta_{bc} - [F, G] D_{bcd} \mathcal{P}_d - \frac{1}{m} [F, \mathcal{P}_b] \partial^{-1} [G, \mathcal{P}_c] \right) . \tag{4.22}$$

Corollary 11 : The Poisson bracket (4.14) when reduced to $Q = \sum_{j=1}^{m} P_j = 0$ can be equivalently written as

$$\begin{aligned} \{f, g\} = a \int \mathrm{tr}\, \Big(& GF'(\delta_{ij} - \frac{1}{m}) - [F, G](\delta_{ij} - \frac{2}{m}) \frac{1}{2} (P_i + P_j) \\ & - \frac{1}{m} [F, P_i] \partial^{-1} [G, P_j] \Big) . \end{aligned} \tag{4.23}$$

Theorem 12 : Let $U_1 = 0$ and hence $Q = \sum_{j=1}^{m} P_j = 0$. By the Miura transformation of Lemma 6 any functionals $f(U)$, $g(U)$ of the U_k only ($k = 2, \ldots m$) are also functionals $\tilde{f}(\mathcal{P}) = f(U(\mathcal{P}))$, $\tilde{g}(\mathcal{P}) = g(U(\mathcal{P}))$ of the $\mathcal{P}_a, a = 1, \ldots m$ only. The reduced second Gelfand-Dikii bracket (4.8) of f and g equals the reduced Poisson bracket (4.21) of $\tilde{f}$ and $\tilde{g}$.

Corollary 13 : The second Gelfand-Dikii bracket (4.8) obeys antisymmetry and the Jacobi identity. Bilinearity in f and g being evident, it is a well-defined Poisson bracket.

4.5. The conformal properties

In the scalar case, i.e. for $n = 1$, the second Gelfand-Dikii bracket (with $U_1 = 0$) gives the W_m-algebras [6,17,18]. The interest in the W-algebras stems from the fact that they are extensions of the conformal Virasoro algebra, i.e. they contain the Virasoro algebra as a subalgebra. Furthermore, in the scalar case, it is known that certain combinations of the U_k and their derivatives yield primary fields of integer spins $3, 4, \ldots m$. It is the purpose of this section to establish the same results for the matrix case, $n > 1$. From now on, I only consider the second Gelfand-Dikii bracket (4.8) for the case $U_1 = 0$. I will simply write $\{f, g\}$ instead of $\{f,g\}_{(2)}$. Also, it is often more convenient to replace the scale factor a by γ^2 related to a by

$$a = -2\gamma^2 \; . \tag{4.24}$$

(Note that γ^2 need not be positive.)

The Virasoro subalgebra

For the original V-algebra (2.23) (corresponding to $m = 2$, $n = 2$ and an additional constraint tr $\sigma_3 U_2 = 0$) one sees that $T = \frac{1}{2}\text{tr } U_2$ generates the conformal algebra. I will now show that for general m, n, the generator of the conformal algebra is still given by this formula.

Lemma 14 : For arbitrary $m \geq 2$ one has

$$\begin{aligned}\{\int \text{tr } FU_2, \int \text{tr } GU_2\} = a \int \text{tr } \Big(&-\frac{1}{m}[F,U_2]\partial^{-1}[G,U_2] - [F,G](U_3 - \frac{m-2}{2}U_2') \\ &+ \frac{1}{2}(F'G - G'F + GF' - FG')U_2 - \frac{1}{2}\binom{m+1}{3}GF'''\Big) \; .\end{aligned} \tag{4.25}$$

Note that for $m = 2$ one has to set $U_3 = 0$.

Proposition 15 : Let $T(\sigma) = \frac{1}{2}\text{tr } U_2(\sigma)$. Then

$$\gamma^{-2}\{T(\sigma_1), T(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})\left(T(\sigma_2)\delta(\sigma_1 - \sigma_2)\right) - \frac{n}{4}\binom{m+1}{3}\delta'''(\sigma_1 - \sigma_2) \; . \tag{4.26}$$

Equivalently, if, for $\sigma \in S^1$, one defines for integer r

$$L_r = \gamma^{-2}\int_{-\pi}^{\pi} \mathrm{d}\sigma\, T(\sigma)e^{ir\sigma} + \frac{c}{24}\delta_{r,0} \tag{4.27}$$

where

$$c = \frac{6\pi}{\gamma^2}n\binom{m+1}{3} = \frac{12\pi}{(-a)}n\binom{m+1}{3} \tag{4.28}$$

then the L_r form a Poisson bracket version of the Virasoro algebra with (classical) central charge c :

$$i\{L_r, L_s\} = (r-s)L_{r+s} + \frac{c}{12}(r^3 - r)\delta_{r+s,0} \; . \tag{4.29}$$

Also, if $\{A_\mu\}_{\mu=1,\dots n^2-1}$ is a basis for the traceless $n \times n$-matrices, then each $S_\mu(\sigma) =$ tr $A_\mu U_2(\sigma), \mu = 1, \dots n^2 - 1$ is a conformally primary field of conformal dimension (spin)

2:

$$\gamma^{-2}\{T(\sigma_1), S_\mu(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})\left(S_\mu(\sigma_2)\delta(\sigma_1-\sigma_2)\right) \tag{4.30}$$

or for the modes $(S_\mu)_r = \gamma^{-2}\int_{-\pi}^{\pi} \mathrm{d}\sigma\, S_\mu(\sigma)e^{ir\sigma}$ one has $i\{L_r, (S_\mu)_s\} = (r-s)(S_\mu)_{r+s}$. Equations (4.26) and (4.30) can be written in matrix notation as ($\mathbf{1}$ denotes the $n\times n$ unit matrix)

$$\gamma^{-2}\{T(\sigma_1), U_2(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})\left(U_2(\sigma_2)\delta(\sigma_1-\sigma_2)\right) - \frac{1}{2}\binom{m+1}{3}\mathbf{1}\,\delta'''(\sigma_1-\sigma_2)\ . \tag{4.31}$$

The conformal properties of the U_k for $k \geq 3$

Having computed the conformal properties of the matrix elements of U_2, I will now give those of all the other U_k, i.e. compute $\{T(\sigma_1), U_k(\sigma_2)\}$ or equivalently, for any (test-) function $\epsilon(\sigma)$, compute $\{\int \epsilon T, U_k(\sigma_2)\}$ for all $k \geq 3$. I will find that this Poisson bracket is linear in the U_l and their derivatives and is formally identical to the result of the scalar case. It then follows that appropriately symmetrized combinations W_k can be formed that are $n\times n$-matrices, each matrix element of W_k being a conformal primary field of dimension (spin) k.

Proposition 16 : The conformal properties of all matrix elements of all $U_k, k = 2,\ldots m$ are given by

$$\begin{aligned}\gamma^{-2}\{\int \epsilon T, U_k\} &= -\epsilon U_k' - k\epsilon' U_k + \frac{k-1}{2}\binom{m+1}{k+1}\epsilon^{(k+1)} \\ &+\sum_{l=2}^{k-1}\left[\binom{m-l}{k+1-l} - \frac{m-1}{2}\binom{m-l}{k-l}\right]\epsilon^{(k-l+1)}U_l\end{aligned} \tag{4.32}$$

which is formally the same equation as in the scalar case $n = 1$ [18].

Since the conformal properties (4.32) are formally the same as in the scalar case, and in the latter case it was possible to form combinations W_k that are spin-k conformally primary fields [18], one expects a similar result to hold in the matrix case. Indeed, one has the

Theorem 17 : For matrices $A_1, A_2, \ldots A_r$ denote by $S[A_1, A_2, \ldots, A_r]$ the completely symmetrized product normalized to equal A^r if $A_s = A$ for all $s = 1, \ldots r$. Let

$$
\begin{aligned}
W_k = &\sum_{l=2}^{k} B_{kl} U_l^{(k-l)} + \sum_{\substack{0 \le p_1 \le \cdots \le p_r \\ \sum p_i + 2r = k}} (-)^{r-1} C_{p_1 \ldots p_r} \, S[U_2^{(p_1)}, \ldots, U_2^{(p_r)}] \\
&+ \sum_{\substack{0 \le p_1 \le \cdots \le p_r \\ s \le l \le k - \sum p_i - 2r}} (-)^r D_{p_1 \ldots p_r, l} \, S[U_2^{(p_1)}, \ldots, U_2^{(p_r)}, U_l^{(k-l-\sum p_i - 2r)}]
\end{aligned}
\tag{4.33}
$$

where the coefficiets $B_{kl}, C_{p_1 \ldots p_r}$ and $D_{p_1 \ldots p_r, l}$ are the same as those given in ref. 18 for the scalar case, in particular

$$
B_{kl} = (-)^{k-l} \, \frac{\binom{k-1}{k-l}\binom{m-l}{k-l}}{\binom{2k-2}{k-l}} \, . \tag{4.34}
$$

Then the W_k are spin-k conformally primary $n \times n$-matrix-valued fields, i.e.

$$
\gamma^{-2} \{ \int \epsilon T, W_k \} = -\epsilon W_k' - k \epsilon' W_k \, . \tag{4.35}
$$

For $\sigma \in S^1$ one can define the modes $(W_k)_s = \gamma^{-2} \int_{-\pi}^{\pi} \mathrm{d}\sigma \, W_k(\sigma) e^{is\sigma}$ and the Virasoro generators L_r as in (4.27). Then one has equivalently

$$
i\{L_r, (W_k)_s\} = \big((k-1)r - s\big)(W_k)_{r+s} \tag{4.36}
$$

where each $(W_k)_s$ is a $n \times n$-matrix.

Examples : From the previous Theorem and the results of ref. 18 (their Table I) one has explicitly:

$$
\begin{aligned}
W_3 &= U_3 - \frac{m-2}{2} U_2' \\
W_4 &= U_4 - \frac{m-3}{2} U_3' + \frac{(m-2)(m-3)}{10} U_2'' + \frac{(5m+7)(m-2)(m-3)}{10m(m^2-1)} U_2^2 \\
W_5 &= U_5 - \frac{m-4}{2} U_4' + \frac{3(m-3)(m-4)}{28} U_3'' - \frac{(m-2)(m-3)(m-4)}{84} U_2''' \\
&+ \frac{(7m+13)(m-3)(m-4)}{14m(m^2-1)} (U_2 W_3 + W_3 U_2) \, .
\end{aligned}
\tag{4.37}
$$

4.6. Example of the $V_{n,3}$-algebra

From the previous subsection one might have gotten the impression that the matrix case is not very different from the scalar case. This is however not true. In the previous subsection only the conformal properties, i.e. the Poisson brackets with $T = \frac{1}{2}\mathrm{tr}\ \mathbf{1}U_2$ were studied, and since the unit-matrix $\mathbf{1}$ always commutes, most of the new features due to the non-commutativity of matrices were not seen. Technically speaking, only tr $\tilde{\mathcal{V}}(f)$ was needed, not $\tilde{\mathcal{V}}(f)$ itself (cf. eq. (4.8)). In this subsection, I will give the Poisson brackets, for the (more interesting) reduction to $U_1 = 0$, of any two matrix elements of U_2 or U_3, or equivalently of U_2 or W_3, for $m = 3$. This is the complete algebra, giving a matrix generalization of Zamolodchikov's W_3-algebra [19]. (Recall that F and G are $n \times n$-matrices of test-functions.)

$$\begin{aligned}\{\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GU_2\} = a\int \mathrm{tr}\ \Big(&-\frac{1}{3}[F,U_2]\partial^{-1}[G,U_2] - [F,G]W_3 \\ &+\frac{1}{2}(F'G + GF' - FG' - G'F)U_2 - 2GF'''\Big)\ ,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\{\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GW_3\} = a\int \mathrm{tr}\ \Big(&-\frac{1}{3}[F,U_2]\partial^{-1}[G,W_3] - \frac{1}{6}[F,G]U_2^2 \\ &+ (-\frac{1}{4}[F',G'] + \frac{1}{2}[F'',G] + \frac{1}{12}[F,G''])U_2 \\ &+ (F'G + GF' - \frac{1}{2}FG' - \frac{1}{2}G'F)W_3\Big)\ ,\end{aligned} \tag{4.39}$$

$$\begin{aligned}\{\int \mathrm{tr}\ FW_3, \int \mathrm{tr}\ GW_3\} = a\int \mathrm{tr}\ \Big(&-\frac{1}{3}[F,W_3]\partial^{-1}[G,W_3] \\ -\frac{1}{6}[F,G](W_3U_2 + U_2W_3) &+ \frac{2}{3}(FU_2GW_3 - GU_2FW_3) \\ +\frac{5}{12}(F'U_2GU_2 - G'U_2FU_2) &+ \frac{1}{12}(FG' - GF')U_2^2 + \frac{1}{12}[F,G]U_2'U_2 \\ +\frac{7}{12}[F',G']W_3 - \frac{1}{6}[F,G]''W_3 &+ \frac{1}{12}(FG''' + G'''F - F'''G - GF''')U_2 \\ +\frac{1}{8}(F''G' + G'F'' - F'G'' - G''F')U_2 &+ \frac{1}{6}GF^{(5)}\Big)\ .\end{aligned} \tag{4.40}$$

One remarks that in the scalar case ($n = 1$) this reduces to the Poisson bracket version of Zamolodchikov's W_3-algebra [19], as it should. In the matrix case however, even if $F = f\mathbf{1}$, $G = g\mathbf{1}$ (with scalar f, g) this is a different algebra, i.e. $\{\int \mathrm{tr}\ W_3(\sigma), \int \mathrm{tr}\ W_3(\sigma')\}$ does not reduce to the W_3-algebra, since the r.h.s. contains the non-linear terms and tr $U_2^2 \neq (\mathrm{tr}\ U_2)^2$.

In other words, the scalar ($n = 1$) W_m-algebras are not subalgebras of the matrix $V_{n,m}$-algebras. The only exception is $m = 2$, since one always has a Virasoro subalgebra.

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