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EQUIVARIANT COHOMOLOGY

AND TOPOLOGICAL THEORIES

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Abstract

The basic concepts and definitions of equivariant cohomology are summarized. Its role in the construction of topological theories is exemplified in the case of the 4-d topological Yang Mills. Some other examples are briefly mentioned.

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1 Introduction

In the preceding lecture devoted to a description of a few aspects of the Slavnov symmetry, we have insisted on its limits of applicability to the perturbative set up which we have interpreted as a clash between locality and geometry. It is, to say the least, embarassing that the Slavnov symmetry emerges algebraically from the Faddeev Popov gauge fixing procedure, thus providing an algebraic substitute for the meaningless integration over the gauge group, and, as stressed in the preceding lecture, a conceptual introduction of this symmetry is still missing.

The construction by E. Witten, in 1988, of topological field theories[6] has started a similar -but different- debate.

These theories [1] are indeed the realm of equivariant cohomology [4] which we shall discuss here rather than "twisted $N = 2$ Supersymmetry" which led to their discovery. This cohomology describes the topology of orbit spaces and, in spite of the formal similarities -which boil down to the use of integral representaions of δ functions, both bosonic and fermionic- should not be confused with the cohomology associated with gauge fixing.

2 Equivariant Cohomology

This section is mostly based on Cartan 1950 and Kalkman 1993 [4].

The situation is as follows:

Let M be a smooth manifold, $\Omega^*(M)$ the differential forms on M, d_M the differential. Let G be a connected Lie group acting smoothly on M. Each element $\lambda \in$ Lie G is represented by a vector field $\underline{\lambda}$ and, to it are associated two operations: on $\Omega^*(M)$: $i_M(\lambda)$, the inner product with $\underline{\lambda}$ and the Lie derivative $\ell_M(\lambda) = [i_M(\lambda), d_M]_+$. One has

$$[\ell_M(\lambda), i_M(\lambda')] = i_M([\lambda, \lambda']) \tag{2.1}$$

where $[\lambda, \lambda']$ is the commutator in Lie G

$$\begin{aligned} [i_M(\lambda), i_M(\lambda')] &= 0 \\ [\ell_M(\lambda), \ell_M(\lambda')] &= \ell_M([\lambda, \lambda']) \\ [\ell_M(\lambda), d_M] &= 0 \end{aligned} \tag{2.2}$$

The question is to define a cohomology which coincides with the de Rham cohomology of M/G when this is a smooth manifold. i.e. when M is a principal G bundle over M/G. Modulo global effects, forms on M/G can be identified with forms on M which are both horizontal i.e. such that

$$i_M(\lambda)\omega = 0 \qquad \forall\lambda \tag{2.3}$$

and invariant

$$\ell_M(\lambda)\omega = 0 \qquad \forall\lambda \tag{2.4}$$

Such forms are called basic.

The basic cohomology of M is the de Rham cohomology of M restricted to the complex of basic forms. When the action of G is good, this is the cohomology of M/G. Then, it contains the characteristic classes obtained by substituting into a symmetric G-invariant (for the adjoint action) polynomial on Lie G the curvature Ω of a G-connection ω. Those classes are independent of ω. A related problem is to extend the theory of characteristic classes to associated bundles: if $P(B,G)$ is a principal G bundle, and M as above (with the action considered as a left action, the associated bundle $E(B,M) = P(B,G) \underset{G}{\times} M$ (i.e; the quotient of $P \times M$ by the simultaneous right action on P and left action on M) is a generalization of $P(B,G) = P(B,G) \underset{G}{\times} G$. Characteristic classes involve a connection ω on P and its curvature Ω. This motivates the following definition:

The equivariant cohomology of $(M, d_M, i_M(\lambda)\ell_M(\lambda))$ is the basic cohomology of $(\Omega^*(M) \otimes W(G), d_M + d_W, , i_M + i_W, \ell_M + \ell_W)$, where $W(G)$ is the Weil algebra of G, a graded commutative differential algebra defined in terms of the generators ω (deg. $\omega = 1$), Ω (deg $\Omega = 2$) with values in Lie G by: the structure equations

$$\begin{aligned}
d_W\omega &= \Omega - \frac{1}{2}[\omega,\omega] \\
d_W\Omega &= -[\omega,\Omega] \\
i_W(\lambda)\omega &= \lambda \qquad \ell_W(\lambda)\Omega = 0 \\
\ell_W(\lambda)\omega &= [\lambda,\omega] \quad \ell_W(\lambda)\Omega = -[\lambda,\Omega].
\end{aligned} \tag{2.5}$$

This is the so-called Weil model for equivariant cohomology.

Equivalently (Kalkman 93) equivariant cohomology is defined as the basis cohomology of $(\Omega^*(M) \otimes W(G), d_M + d_W + \ell_M(\omega) - i_M(\Omega), \ell_W(\lambda), \ell_M(\lambda) + \ell_W(\lambda))$ which we shall call the intermediate model. One goes from the Weil scheme to the intermediate scheme by the algebra automorphism

$$x_W \to x_{Int} = e^{i_M(\omega)}x_W \tag{2.6}$$

which transforms the differential and operation as indicated. The easiest way to do the computation is to establish and solve differential equations for the interpolating family

$$x_W \to x_t = e^{t\ i_M(\omega)}x_W \qquad 0 \le t \le 1 \tag{2.7}$$

The interesting feature of the intermediate scheme is to replace $i_M(\lambda) + i_W(\lambda)$ by $i_W(\lambda)$, and accordingly produce the generalized covariant differential $D = d_M + d_W + \ell_M(\omega) - i_M(\Omega)$.

Since basic cochains are polynomials in ω, Ω, with coefficients in $\Omega^*(M)$, the condition $i_W(\lambda)X = 0$ allows to consider only polynomials in Ω. In view of the invariance property, the differential can then be reduced to $d_C = d_M - i_M(\Omega)$. This is the Cartan differential. It is a differential because <u>on invariant cochains</u> $d_C^2 = \ell_M(\Omega) \equiv \ell_M(\Omega) + \ell_W(\Omega) = 0$.

We shall see in the applications that it is useful to use both the Weil scheme and the intermediate scheme.

The initial requirement that the cohomology thus defined coincides with the basic cohomology of M when the action is good, i.e. M is a principal bundle is fulfilled thanks to Cartan's "theorem 3" according to which, equivariant cohomology maps isomorphically onto the basic cohomology of M, through the replacement $\omega \to \tilde{\omega}, \Omega \to \tilde{\Omega}$ where $\tilde{\omega}$ is a connection on M

and $\tilde{\Omega}$ its curvature. (This is easily proved using the homotopy which insures the triviality of the cohomology of $W(G)$.

Most applications to be found in the next section are concerned with the construction [7] of equivariant cohomology classes associated with a closed invariant form on M, χ_M, which is automatically horizontal in the intermediate scheme. There are in general obstructions to such extensions [7]. One case of general interest has led V. Mathai and D. Quillen (1986) [4] to interesting integral representations of the Thom Class [2] of a vector bundle $E(B,V)$ with base B, fiber V a real vector space of even dimension $|V|$. By the introduction of a metric $\|\ \|$ on the fiber, we can assume that the structure group is reduced to $SO|V|$. One writes

$$E(B,V) = P(B,SO|V|) \underset{SO(|V|)}{\times} V \tag{2.8}$$

where P is the orthonormal frame bundle associated with E.

The Poincaré dual of the zero section of E, χ_0, is a cohomology class of degree $|V|$ with the property

$$\int_{V=0} \omega^{|E|-|V|=|B|} = \int_E \omega \wedge \chi_0 \tag{2.9}$$

for all forms ω of degree $|B|$, where, in the left hand side ω stands for the restriction of ω to the submanifold $V=0$. One candidate is

$$\chi_\delta = \delta(V) \wedge dV = N_0 \int e^{ibV+\bar{\omega}dV} db d\bar{\omega} \tag{2.10}$$

with $b \in V^*, \bar{\omega} \in \wedge V^*$, N_0 a normalisation constant such that $\int_V \chi_\delta = 1$. This can be written, in the intermediate scheme

$$\chi_\delta \equiv \delta(V) \wedge dV = N_0 \int e^{S_{top}(\bar{\omega}V)} db\ d\bar{\omega} \tag{2.11}$$

with

$$\begin{aligned} S_{top}V &= DV = dV + \omega V \\ S_{top}dV &= DdV = \Omega V + \omega dV \\ S_{top}\bar{\omega} &\equiv D_{\bar{\omega}b}\bar{\omega} = ib - \omega\bar{\omega} \\ S_{top}ib &\equiv D_{\bar{\omega}b}ib = \Omega\bar{\omega} - \omega ib \\ \text{i.e.} \qquad S_{top} &= D_{V,dV} + D_{\bar{\omega}b} \end{aligned} \tag{2.12}$$

From the integral formula, we get

$$\begin{aligned} D_{V,dV}\chi_\delta &= N_0 \int D\, e^{S_{top}(\bar{\omega}V)} db\ d\bar{\omega} \\ &= N_0 \int -D_{\bar{\omega}b}\, e^{S_{top}(\bar{\omega}V)} db\ d\bar{\omega} \\ &= -N_0 \int (b - \omega\bar{\omega})\frac{\partial}{\partial\bar{\omega}} + (\Omega\bar{\omega} - \omega b)\frac{\partial}{\partial b}\, e^{S_{top}(\bar{\omega}V)} db\ d\bar{\omega} \\ &= -N_0 \int \left(b\frac{\partial}{\partial\bar{\omega}} + \Omega\bar{\omega}\frac{\partial}{\partial b} - \omega\bar{\omega}\frac{\partial}{\partial\bar{\omega}} + \omega b\frac{\partial}{\partial b} \right) e^{S_{top}(\bar{\omega}V)} db\ d\bar{\omega} \end{aligned} \tag{2.13}$$

The terms from the first parenthesis yield zero by integration by parts, both in b in the sense of distributions, and in ω for algebraic reason. The second term is $\ell_{\bar{\omega}b}(\omega)$ which can be replaced by $\ell_{V,dV}(\omega)$, which is zero because, by invariance of $\bar{\omega}V$, χ_δ does not depend on ω and the result is an invariant combination of V, dV. Since invariance under $i_V(\lambda)$ is obvious in the intermediate scheme, χ_δ defines an element of equivariant cohomology.

Of course χ_δ has a distributional character. By the same method, one can construct a smooth representative

$$\chi_\epsilon = N_0 \int db\, d\bar{\omega}\; e^{S_{top}(\bar{\omega}V - \epsilon\frac{i<\bar{\omega},b>}{2})} \tag{2.14}$$

where $<\ ,\ >$ is an invariant metric on V^*. χ_ϵ is normalized in such a way that $\int_V \chi_\epsilon = 1$. The only change in the previous proofs is that the result now depends on Ω, and, in the last step of the proof $\ell_{\bar{\omega}b}(\omega)$ can be replaced by $\ell_{V,dV}(\omega) + \ell_W(\omega)$. Similarly S_{top} is replaced by $D_{V,dV} + D_{\bar{\omega}b} + D_W$

$$\begin{aligned} S_{top}\omega &= \Omega - \frac{1}{2}[\omega,\omega] = D_W\omega \\ S_{top}\Omega &= -[\omega,\Omega] \end{aligned} \tag{2.15}$$

Going back to the Weil scheme merely replaces dV by $dV + \omega V$.

Differentiating χ_ϵ with respect to ϵ (or other parameters involved in the metric $<\ ,\ >$ yields

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\chi_\epsilon &= -N_0 \int S_{top}\left(\frac{i<\omega b>}{2}\right) e^{S_{top}(\bar{\omega}V - \epsilon\frac{i<\bar{\omega},b>}{2})} db\; d\bar{\omega} \\ &= -N_0\left(S_{top\ V} + S_{top\ W}\right) \int \frac{i<\bar{\omega}\ b>}{2}\, e^{S_{top}(\bar{\omega}V - \epsilon\frac{i<\bar{\omega},b>}{2})} db\; d\bar{\omega} \\ &\quad -N_0 \int S_{top}\frac{i<\bar{\omega},b>}{2}\, e^{S_{top}(\bar{\omega}V - \epsilon\frac{i<\bar{\omega},b>}{2})} db\; d\bar{\omega} \end{aligned} \tag{2.16}$$

The last term vanishes by the same argument according to which χ_ϵ is closed. Thus, the cohomology class of χ is independent of the parameters involved in the metric. Similar properties hold for different choices of $\tilde{\omega}, \tilde{\Omega}$ on $P(\cdot B \cdot G\cdot)$. Similarly, the pull back of χ_ϵ by a section $V = V(p)$, $V(p\gamma) = \gamma^{-1}V(p)$ describes the Poincaré dual of the manifold of zeroes of that section, and the corresponding class is independent of the choice of section, provided it is transverse to the zero section (so that the intersection of the two sections defines a manifold).

In the next section, and in F. Thuillier's talk, we shall meet another class of constructions which yield equivariant cohomology classes.

In conclusion, the Mathaï Quillen formulae are, in the equivariant set up the exact analogues of the integral representations of δ functions or gaussians which are the core of the Faddeev Popov gauge fixing procedure.

3 Application to topological field theories

One of the challenges of topological field theories is whether some strict field theory rules are able to produce "topology". The main difficulty seems to be connected with gauge fixing or, put differently, with finding a good procedure to integrate basic forms over field space in such a way that such general principles as locality can be used. In what follows we shall mainly be concerned with topological Yang Mills theories. The case of 2d topological gravity is both easier ($2 << 4$) and more difficult (diffeomorphism groups are more subtle than gauge groups). See Becchi's talk [3] and F. Thuillier's talk [5].

Whereas YM_4^{top} was found by twisted $N = 2$ supersymmetry arguments (Witten 88), it soon became apparent it had to do with equivariant cohomology in spite of confusions due to the similarities with the Slavnov symmetry covered up by the abuse of symbols such as $Q_{BRST...}$. YM_4^{top} is supposed to be the characterisitc cohomology theory -intersection theory of $\mathcal{A}/\mathcal{G}$ resp. its restriction to the manifold

$$F - *F = F^{-} = 0 \tag{3.17}$$

The operation S_{top} is defined as follows

$$\begin{aligned} S_{top}\ a &= \psi - D_a\omega \\ S_{top}\ \psi &= [\omega, \psi] - D_a\Omega \\ S_{top}\ \omega &= \Omega - \frac{1}{2}[\omega, \omega] \\ S_{top}\ \Omega &= -[\omega, \Omega] \end{aligned} \tag{3.18}$$

In the Weil scheme

$$\psi = \delta a + D_a\omega \tag{3.19}$$

In the intermediate scheme

$$\psi = \delta a; \qquad -D_a\omega = \ell_{\mathcal{A}}(\omega)a, \tag{3.20}$$

ω is an element of the Weil algebra (to be later replaced by a connection on $\mathcal{A}$), Ω, its curvature.

Since the idea is to transform integration over $\mathcal{A}/\mathcal{G}$ into integration over $\mathcal{A}$, we have in particular to transform cohomology classes of $\mathcal{A}/\mathcal{G}$ into equivariant cohomology classes which become basic cohomology classes upon the replacement of ω, Ω by a connection $\tilde{\omega}$ and its curvature $\tilde{\Omega}$.

Those cohomology classes which give rise to the Donaldson polynomials are constructed according to a standard scheme (cf. F. Thuillier's talk [5]), which, in the present case, reduces down to the following: consider the G bundle $P(B, G) \times \mathcal{A}$ over $B \times \mathcal{A}$, and, on it, the $\mathcal{G}$ invariant connection a. In the intermediate scheme its equivariant curvature is

$$\begin{aligned} \mathcal{F}^{interm.}_{equiv.} &= \left[d_P + d_{\mathcal{A}} + \left(\ell_P + \ell_{\mathcal{A}}\right)(\omega) - \left(i_P + i_{\mathcal{A}}\right)(\Omega)\right] a + \frac{1}{2}[a, a] \\ &= F(a) + \delta a + \Omega \\ &= F(a) + \psi^{interm.} + \Omega \end{aligned} \tag{3.21}$$

In the Weil scheme, it is

$$\mathcal{F}^{Weil}_{equiv.} = F(a) + \psi^{Weil} + \Omega \tag{3.22}$$

where

$$v^{Weil} = \delta a + D_a \omega \tag{3.23}$$

$\mathcal{F}_{eq}$ becomes a mixed form of degree 2 upon substituting ω, Ω by $\tilde{\omega}$, a $\mathcal{G}$-connection on $\mathcal{A}$ and its curvature $\tilde{\Omega}$. Any G invariant polynomial of $\mathcal{F}_{eq}$ (G-characteristic class) can be split into a sum of terms of fixed bidegree in $\Omega^*(B)$, because of G basicity, and $\Omega^*_{basic}(\mathcal{A})$. Integration over a homology class in B yields an element of $H^*_{basic}(\mathcal{A})$, independently of the choice of $\tilde{\omega}, \tilde{\Omega}$. Consider now some such element of degree dim. $\mathcal{M}$, where $\mathcal{M} = \{$ solutions of $F = *F$, up to gauge transformations $\}$. Restricting to $\mathcal{M}$ the corresponding class in $H^*(\mathcal{A}/\mathcal{G})$ is represented by exterior multiplication by a representative of the Poincaré dual of $F - *F = 0$, considered as the zero set of a section of an appropriate $\mathcal{G}$ bundle:

$$\chi_{F-*F=0} = \int \mathcal{D}\bar{\omega}\mathcal{D}b \; e^{S_{top}(\bar{\omega}^- F^- + i<\bar{\omega}^-, b^->)} \Big|_{\substack{\omega = \tilde{\omega} \\ \Omega = \tilde{\Omega}}} \tag{3.24}$$

where

$$F^- = F - *F \tag{3.25}$$

$$\begin{aligned} S_{top}\, \bar{\omega}^- &= ib^- - [\omega, \bar{\omega}^-] \\ S_{top}\, ib^- &= [\Omega, \bar{\omega}^-] - [\omega, ib^-] \end{aligned} \tag{3.26}$$

The choice of $\tilde{\omega}, \tilde{\Omega}$ can be expressed, using the Faddeev Popov identity in the Weil algebra

$$\int \mathcal{D}\omega\mathcal{D}\Omega \; \delta(\omega - \tilde{\omega}) \; \delta(\Omega - \tilde{\Omega}) = 1 \tag{3.27}$$

where the δ functions are either fermionic or bosonic. Given a $\mathcal{G}$ covariant Lie $\mathcal{G}$ valued fermionic gauge function

$$H(a, v) = H(a) \cdot v \tag{3.28}$$

whose vanishing defines a connection

$$\tilde{\omega} = -\frac{1}{H(a)D_a} H(a) \cdot \delta a \tag{3.29}$$

one has:

$$\int \delta\left(H(a, v)\delta(S^{top} H(a, v))\right) \mathcal{D}\omega\mathcal{D}\Omega = 1. \tag{3.30}$$

Indeed

$$\begin{aligned} \delta\left(H(a, v)\right) &= \det H(a)D_a \; \delta(\omega - \tilde{\omega}) \\ S^{top} H(a, v) &= \frac{\delta H}{\delta \underline{a}} \underline{\psi} v + H(a)D_a\Omega \\ &= H(a)D_a(\Omega - \tilde{\Omega}) \end{aligned} \tag{3.31}$$

with $\tilde{\Omega}$, the curvature of $\tilde{\omega}$: as one may check, using the first δ function: by differentiating

$$H(a)\tilde{v} = 0 \tag{3.32}$$

and using the gauge covariance of $H(a)\tilde{\psi}$, one does get

$$\tilde{\Omega} = -\frac{1}{H(a)D_a}\frac{\delta H}{\delta \underline{a}}\underline{\tilde{\psi}\tilde{\psi}} \tag{3.33}$$

The Faddeev Popov Weil identity can be rewritten as

$$\int \mathcal{D}\omega \mathcal{D}\Omega \; \mathcal{D}\bar{\psi}\mathcal{D}\bar{\Omega} \;\; e^{S^{top}(\bar{\Omega}H(a)\psi)} = 1 \tag{3.34}$$

with

$$\begin{aligned} S^{top}\bar{\Omega} &= \bar{\psi} + [\omega, \bar{\Omega}] \\ S^{top}\bar{\psi} &= [\Omega, \bar{\Omega}] + [\omega, \bar{\psi}] \end{aligned} \tag{3.35}$$

The usual choice is

$$H(a) = D_a^* \tag{3.36}$$

It excludes reducible connections.

Given the grading and power counting arguments which follow from this construction, it is licit to add to the argument of the exponential in Eq.(3.34) a term of the form

$$S^{top}tr[\Omega, \bar{\Omega}]\bar{\psi} \tag{3.37}$$

(This is also necessary if one foresees a perturbative treatment).

The above arguments which are rather general (compare with Atiyah Jeffrey, 1990), allow to recover both the observables and the action first introduced by Witten. There remains to look at the integration variables, which, so far, are $\bar{\omega}^-, b^-, \bar{\psi}, \bar{\Omega}, \omega, \Omega$. Recall however the meaning of ψ:

$$\psi = \delta a + D_a \omega \tag{3.38}$$

One may consider ψ as a free variable by inserting an extra δ function:

$$\int \mathcal{D}\psi \underbrace{\delta\left(\psi - (\delta a + D_a\omega)\right)}_{\Lambda(\psi - (\delta a + D_a\omega))} = 1 \tag{3.39}$$

One thus gets the following basic form:

$$\begin{aligned} &\int \mathcal{D}\bar{\omega}^- \; \mathcal{D}b^- \; \mathcal{D}\omega \; \mathcal{D}\Omega \; \mathcal{D}\bar{\Omega} \; \mathcal{D}\bar{\psi} \; \mathcal{D}\psi \; \Lambda\left(\psi - (\delta a + D_a\omega)\right) \\ &\mathcal{O} e^{S^{top}(\bar{\omega}^- F^- + \bar{\omega}^- b^- + i\bar{\Omega}D^*\psi + \Omega[\bar{\Omega},\bar{\psi}])} \\ &= \int \mathcal{D}\bar{\omega}^- \; \mathcal{D}b^- \; \mathcal{D}\omega \; \mathcal{D}\Omega \; \mathcal{D}\bar{\Omega} \; \mathcal{D}\bar{\psi} \; \mathcal{D}\psi \Lambda(\psi - (\delta a + D_a\omega)) \\ &\mathcal{O} e^{ib^- F^- - (b^-)^2 + \bar{\omega}^-(D_a\psi)^- + \bar{\omega}^-[\Omega,\bar{\omega}^-] + \bar{\psi}D_a^*\psi} \\ &e^{i\bar{\Omega}D_a^* D_a\Omega + \bar{\Omega}[\psi^*,\psi] + \Omega[\bar{\psi},\bar{\psi}] + [\Omega,\bar{\Omega}]\cdot[\Omega,\bar{\Omega}]} \end{aligned} \tag{3.40}$$

where $\mathcal{O}$ is one of the above mentioned observables, an equivariant form of degree dim $\mathcal{M}$. At this point, one is facing again the problem of integrating a basic top form over field space.

Formally, up to zero modes, the integrand is a top form in ψ so that ψ can be (almost) forgotten in $\Lambda(\psi - (\delta a + D_a\omega))$. One may then multiply through

$$\int_{\mathcal{G}} \delta(\mathrm{g}(a))\Lambda\delta\mathrm{g}(a) = 1 \tag{3.41}$$

where $\int_{\mathcal{G}}$ denotes fiber integration. Using the standard notation, this may be replaced by (Atiyah Jeffrey 90)

$$\int_{\mathcal{G}} \delta(\mathrm{g}(a))\Lambda\mathrm{m}(a)\tilde{\omega} = 1 \tag{3.42}$$

and $\tilde{\omega}$ can be in turn replaced by ω under the integral in Eq.(3.40). This operation is licit locally over $\mathcal{A}/\mathcal{G}$, when $\mathrm{m}(a)$ is invertible i.e., within the Gribov horizon. One thus gets a top form in ω so that $D_a\omega$ can be deleted from the integration form in Eq.(3.40). So, provided one exercises all the necessary care in using the Faddeev Popov gauge fixing procedure one gets

$$\begin{aligned} <\mathcal{O}> \;=\; & \sum_\alpha \int_{U_\alpha} \mathcal{D}\bar{\omega}^- \ \mathcal{D}b^- \ \mathcal{D}\omega \ \mathcal{D}\Omega \ \mathcal{D}\bar{\Omega}\mathcal{D}\bar{\psi}\mathcal{D}\psi(\mathcal{D}a)_0[\psi]_0\mathcal{D}\bar{\omega} \ \mathcal{D}b \\ & \theta(U_\alpha)\mathcal{O} \ e^{ib^-F^- - (b^-)^2 + \bar{\omega}^-(D_a\psi)^- + \bar{\omega}^-[\Omega,\bar{\omega}^-] + \bar{\psi}D_a^*\psi} \\ & e^{i\bar{\Omega}D_a^*D_a\Omega + \bar{\Omega}[\psi^*,\psi] + \Omega[\bar{\psi},\bar{\psi}] + [\Omega,\bar{\Omega}][\Omega,\bar{\Omega}]} \\ & e^{ib\mathrm{g}_\alpha(a) + \bar{\omega}\mathrm{m}_\alpha\omega} \end{aligned} \tag{3.43}$$

where $\{\theta(U_\alpha)\}$ is a gauge invariant partition of unity such that m_α is invertible inside U_α and the subscripts 0 refer to the zero modes. At this point, one has recovered a local field theory -up to zero mode problems-, whose ultraviolet stability is however not very well expressed: recall that before gauge fixing the action is of the form $S^{top}\chi$, with χ basic:

$$\delta(\lambda)\chi = i(\mu)\chi = 0 \qquad \lambda, \mu \in \ \mathrm{Lie}\mathcal{G} \tag{3.44}$$

In order to express this property in terms of a Ward identity, we introduce (Horne 89, Ouvry Stora Van Baal 89):

$$W = \delta(\lambda) + i(\mu) \qquad \underline{\lambda} \in \Lambda \ \mathrm{Lie}\ \mathcal{G}, \qquad \underline{\mu} \in S \ \mathrm{Lie}\ \mathcal{G} \tag{3.45}$$

where λ, μ are the ghosts corresponding to the graded Lie algebra generated by $\delta(\lambda), i(\mu)$ (respectively odd and even).

Extending the operation S^{top} by

$$S^{top}\lambda = \mu \tag{3.46}$$

and W by

$$\begin{aligned} W\lambda &= -\frac{1}{2}[\lambda, \lambda] \\ W\mu &= [\lambda, \mu] \end{aligned} \tag{3.47}$$

we have

$$W^2 = 0 \quad [S^{top}, W] = 0 \tag{3.48}$$

It is desirable to write the gauge fixing action as $S^{top}\chi_{gf}$, with $W\chi_{gf}=0$ upon a suitable extension of S^{top} and W on the corresponding Lagrange multipliers. This can be done as follows:

$$S^{top}W\left(\bar{\mu}\ \mathrm{g}(a)+\frac{i\bar{\mu}\ell}{2}\right)=S^{top}\left(\bar{m}\ \mathrm{g}(a)+\bar{\mu}\ \mathrm{m}(a)\lambda+\frac{i\bar{m}\ell}{2}\right)$$

$$=i\ell\mathrm{g}(a)+\frac{-\ell^2}{2}+\bar{m}\left(\frac{\delta\mathrm{g}(a)}{\delta a}\psi+\mathrm{m}(a)\omega\right)+\bar{\lambda}\mathrm{m}(a)\lambda+\bar{\mu}\mathrm{m}(a)\mu+\bar{\mu}\frac{\delta\mathrm{m}}{\delta a}(\psi+D\omega)\lambda \quad (3.49)$$

with

$$\begin{array}{llll} W\bar{\lambda}=\ell & W\bar{\mu}=\bar{m} & S^{top}\bar{\mu}=\bar{\lambda} & S^{top}\bar{m}=i\ell \\ W\ell=0 & W\bar{m}=0 & S^{top}\bar{\lambda}=0 & S^{top}\ell=0 \end{array} \quad (3.50)$$

$(\ell,\bar{\lambda},\lambda$, odd ; $\bar{m},\mu\bar{\mu}$ even). Here ℓ and $\bar{m}$ replace $b,\bar{\omega}$ in terms of which the naïve gauge fixing is expressed. ℓ has ghost number 0, ω,λ have ghost number 1, μ has ghost number 2, $\bar{m},\bar{\lambda}$ have ghost number -1, $\bar{\mu}$ has ghost number -2. This should of course go with the integration over the corresponding variables. More generally, one could add to the bosonic gauge fixing term $\bar{\mu}\mathrm{g}(a)$ in Eq.(3.49), a fermionic gauge fixing term of the form $\bar{\lambda}H(a,\psi)$. One missing link here is a derivation of this gauge fixing procedure via an identity involving graded fiber integration over the vertical tangent bundle of $\mathcal{A}$. Also, one should verify that the introduction of W solves the ultraviolet stability problem. This also deserves further investigation.

4 Conclusion and Outlook

In this lecture we have attempted to justify the point of view that equivariant cohomology is the appropriate framework to discuss the local aspects of topological theories. It seems indeed to provide a construction of both the corresponding tautological actions and of a remarkable class of observables. It is definitely different from the cohomology associated with the gauge fixing of conventional gauge theories. In the latter case, the observables are functions on orbit space. These are not cohomology classes. One could alternatively interpret them as the cohomology of orbit space with maximum dimension (∞ !) and prescribed decrease properties, which does not help very much.

The choice of this topic, however underdeveloped has been sporadically justified during this conference: the observables of 2-d topological gravity (cf. C. Becchi's talk) can be constructed by the general technique alluded to in section 3; the similarity transformations described in M. Kato's talk, which pair apparently different topological conformal models are nothing else than the Kalkman automorphism (Eq. 2.8) suitably interpreted.

One final conclusion which applies to this lecture, the first one, and a few others in this conference concerns the necessity to reconcile -if possible- geometry and locality by resolving the Gribov horizon problem, either by patching as done in C. Becchi's talk or by direct use of a connection with non vanishing curvature.

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