USING ANTICOMMUTING VARIABLES

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During the last twenty years or so a new kind of mathematics, whose objects are indentified by the prefix 'super' has been studied. This mathematics involves anticommuting variables in addition to commuting variables.

The motivation for such mathematics comes almost entirely from physics, particularly from the quantization of systems with Fermionic degrees of freedom and of systems with gauge symmetry. The prefix 'super' is derived from the idea of supersymmetry in physics. The term supersymmetry is used to denote a symmetry which mixes Bosonic and Fermionic degrees of freedom, that is to say, particles of integral spin and particles with half integral spin. Bosonic fields obey canonical commutation relationships, so that Bosonic theories can be formulated in terms of functions of conventional variable, while Fermionic fields obey canonical anticommutation relations so that an analogous formulation of Fermionic fields involves functions of anticommuting variables. Such a treatment of Fermions is particularly desirable when considering supersymmetry, because the two types of field related by the symmetry are then treated in a analogous fashion.

Anticommuting variables occur in the quantization of theories with symmetry as a means of handling the constrained phase spaces which arise in such theories. The constraints lead to a reduction in the dimension of the phase space, which can effectively be obtained by including extra anticommuting dimensions. This is because Gaussian integrals of anticommuting variables introduce a determinant (the Fadeev-Popov determinant) rather than an inverse determinant. Of course the extra dimensions are to be interpreted formally rather than as describing any physical reality.

These notes briefly summarise the basic analysis of functions of anticommuting variables. A more detailed account may be found (for instance) in [1], which also describes some applications to geometry.

There are two philosophies in supermathematics ; one is the concrete approach, in which actual sets parametrised by commuting and anticommuting variables are considered, and then functions between such sets analysed ; the other is more formal, with the algebra of function being extended, rather than the spaces themselves. The approach taken is partly a question of taste and partly a question of the particular problem being considered - the two approaches being largely interchangeable [2,3]. In these notes the concrete approach will be taken.

The starting point is the infinite-dimensional Grassmann algebra $\mathbb{R}_S$ over the reals with generators $1, \beta 1, \beta 2 \ldots$, and relations

$$1\beta_i = \beta_i 1 = \beta_i$$

$$\beta_i\beta_j = -\beta_j\beta_i$$

This algebra $\mathbb{R}_S$ is split into two parts $\mathbb{R}_{S_0}$ and $\mathbb{R}_{S_1}$, with $\mathbb{R}_{S_0}$ containing terms built from an even number of genrators and $\mathbb{R}_{S_1}$ those terms built from products of an odd number of generators. Even elements of the algebra commute with all elements, while odd elements anticommute with one another. The standard (m,n)— dimensional *superspace* (on which geometric objects such as supermanifolds may be locally modelled) is the space $\mathbb{R}_S^{m,n}$ product of $(\mathbb{R}_{S_0})^m$ and $(\mathbb{R}_{S_1})^n$ with typical element $(x^1,\dots,x^m;\xi^1,\dots,\xi^m)$

The analysis of functions on the purely odd superspace $\mathbb{R}_S^{0,n}$ is very simple ; the most general function considered takes the form

$$f(\xi^1,\dots,\xi^m) = \sum_{\mu\in M_n} f_\mu \xi^\mu$$

where $\mu = \mu_1 \dots \mu_{|\mu|}$ is a multi-index with $1 \vee \mu_1 < \dots < \mu_{|\mu|} \leq n$, M_n denotes the set of all such multi-indices (including the empty index) and $\xi^\mu = \xi^{\mu_1}\dots\xi^{\mu_{|\mu|}}$. The coefficients f_μ are constants - the space in which these are allowed to take values (which might be the reals, the complex numbers or $\mathbb{R}_S$) determining the particular function space under consideration. Differentiation of such functions is defined by the simple rule

$$\frac{\partial\xi^\mu}{\partial\xi^j} = (-1)^{l-1}\xi^{\mu_1}\dots\xi^{\mu_{l-1}}\xi^{\mu_{l+1}}\xi^{\mu_{|\mu|}} \text{ if } \mu_l = j$$

$$= 0 \text{ otherwise}$$

Integration is more difficult ; neither measure theory nor anti-differentiation leads to any useful definition ; the standard definition (due originally to Berezin [4]) is formal with no notion of limit. The integral is simply defined by the rule

$$\int d\xi^1 \dots d\xi^n f(\xi) = f_{1\dots n}$$

where $f_{1\dots n}$ denotes the coefficient of the highest degree term in the expansion (*) of f. This integral allows many standard techniques in functional analysis to be extended to the anticommuting case ; for instance the kernel of a differential operator H is the function $H(\xi^1,\dots,\xi^n,\eta^1,\dots,\eta^n)$ such that

$$H(\xi^1,\dots,\xi^n) = \int d\eta^1\dots\eta^n H(\xi^1,\dots,\xi^n,\eta^1,\dots,\eta^n) f(\eta^1,\dots,\eta^n)$$

The trace can be obtained from the kernel by the formula

$$\mathrm{trace}(H) = \int d\eta^1\dots\eta^n H(-\eta,\eta),$$

Since the delta function (or kernel of the identity operator) satisfies

$$\delta(\eta - \xi) = \int d^n \rho \exp(i \sum_{j=1}^{n} \rho^j (\eta^j - \xi^j))$$

Fourier transforms can defined which satisfy a simple Fourier inversion theorem.

On its own, analysis of functions of an anticommuting variable is a somewhat trivial subject ; bolted on to conventional analysis and geometry, some surprisingly userful and powerful techniques can be developed ; an example is the study of the Dirac operator (making use of the fact that the operators $\psi^i = \xi^i + \partial/\partial\xi^i$ satisfy the Clifford algebra relations $\psi^i\psi^j + \psi^j\psi^i = 2\delta^{ij}$ which is described in [1].

The literature on supermathematics and supersynunetry is vast, and no attempt will be made here to provide a complete bibliography. An account of supersymmetry and the use of superspace techniques may be found in [5].

Références

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