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of a multiterminal network**

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## PROBABILITY DISTRIBUTION FUNCTION FOR THE CAPACITY OF A MULTITERMINAL NETWORK <sup>(1)</sup>

by Pierre DOULLIEZ <sup>(2)</sup>

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*Summary. — This paper presents a method for finding the probability distribution function for the capacity of a multiterminal network. The capacities of the arcs of the network are independant discrete random variables and the demands at the different « demand » nodes are given non-decreasing functions of time. The method has been applied to electrical power networks for which the reliability level must be known with accuracy. A numerical example is given and computational experience is discussed.*

### I. INTRODUCTION

In a multiterminal network, 'supply' nodes are connected to several 'demand' nodes through some intermediate nodes. The nodes of the network are joined by arcs and the capacity of an arc is an upper bound to the flow that may pass over it. The amount of flow that can be provided at each supply node has also an upper bound which is considered as the capacity of a fictitious arc joining each supply node to a common supply node. The demand required at each demand node is a given non-decreasing function of time.

When the arc capacities are known, the largest value of time up to which all demands are satisfied can be determined by a parametric network flow approach [1]. This time is called the 'critical time' and is denoted by ' $t^*$ '. The capacity of the network is the sum of demands at the critical time. At the critical time, there exists in the network a set of critical arcs which separates the set of nodes  $N$  into two subsets  $N_1$  and  $N_2$  ( $N_1 \cup N_2 = N$ ) with the common source node in  $N_1$  and at least one demand node in  $N_2$  and such that no flow augmenting path exists from  $N_1$  to  $N_2$  <sup>(3)</sup>.

In this article, it is assumed that the capacities of the arcs of the network are independant discrete random variables. Thus, the critical time  $t^*$  is also a

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(1) The work underlying this paper was performed while the author was at the Center for Operations Research and Econometrics (C.O.R.E.) University of Louvain, Belgium.

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(3) When there is only one demand node in the network, the critical set of arc is a minimal cut as defined in [3].

discrete random variable. The problem is to determine the probability distribution function  $F_{t^*}(t)$  of the critical time  $t^*$ . Then, the largest value of time  $T$  up to which the probability of satisfying all demands (or equivalently, up to which the probability that no set of critical arcs appears in the network) is above a given confidence level  $\beta$  can be found from the equation (1) :

$$1 - F_{t^*}(T) = \Pr [t^* \geq T] \geq \beta$$

In reference [1], the risk of non satisfying the demands due to unplanned arc capacity reductions has been taken into account in an approximate way. In that reference the critical time is defined as the largest value of time up to which all demands can be satisfied even if  $k$  arc capacity reductions occur on any set of  $k$  arcs. Therefore, an excess of network capacity is available at the critical time and this obviously reduces the risk of non satisfying the demands. However, in many situations, the risk of non satisfying the demands must be known with accuracy and the probability distribution function for the network capacity must be computed.

In section 2, the probability associated with a set of network states having an identical set of critical arcs is defined. In section 3 a method for finding the probability distribution function of the critical time is presented (2). Section 4 deals with some probability computation problems. A numerical example is given in section 5 and some conclusions are given in section 6.

## 2. NETWORK STATES WITH IDENTICAL SET OF CRITICAL ARCS

Let us assume that the capacities of the arcs of the network are independent discrete random variables. The critical time  $t^*$  is also a discrete random variable.

If  $H_j$  is a discrete random variable for arc  $j$ , and if the network has  $m$  arcs, then a state  $X$  of the network can be represented by an  $m$ -tuple of arc capacity values  $X = (H_{1v_1}, H_{2v_2}, \dots, H_{mv_m})$  where  $H_j$  assumes positive real values  $H_{j1}, H_{j2}, \dots, H_{jk}$ , with probabilities  $p_{j1}, p_{j2}, \dots, p_{jk}$ . The network state probability, denoted as  $\Pr [X]$ , equals the product  $p_{1v_1} \cdot p_{2v_2} \dots p_{mv_m}$ .

The total number of network states is  $\prod_{j=1}^m k_j$ . The enumeration of the network states would involve formidable calculations, even if the number of arcs is moderate. However, a great deal of network states will never be considered when the probability distribution of the network capacity is computed, if the property which follows is taken into account.

(1) An appropriate network confidence level  $\beta$  is chosen by considering the expenses needed for a given diminution of the risk of unsupplied demands. The economical aspects of this problem can be found in reference [5] and will not be discussed here.

(2) It is worth quoting the work done by Frank and Hakimi [4] for expressing mathematically the distribution function of the capacity of a network between two of its nodes. This function was obtained from the joint probability density function of the capacities assumed by the elementary cuts of the network. According to their statements, computing this function would represent, to say the least, a formidable task even for a digital computer.

For notational convenience, we suppose that  $H_{j_1} \leq H_{j_2} \dots \leq H_{j_{k_j}}$  for all  $j$ . Let us denote the critical time and the set of critical arcs of a given network state  $X$  respectively by  $t(X)$  and  $A(X)$ . These are easily found by using a method presented in reference [1].

Let  $J$  be the set of arcs in  $A(X)$ .

If  $A(X)$  is a set of critical arcs for a network state  $X = (H_{1v_1}, H_{2v_2}, \dots, H_{mv_m})$ , then  $A(X)$  is also a set of critical arcs for a network state  $X' = (H_{1v'_1}, H_{2v'_2}, \dots)$ , where  $v'_j \geq v_j$  for each  $j \notin J$  and  $v'_j = v_j$  for each  $j \in J$ .

The number of network states for which  $A(X)$  is a set of critical arcs is at least  $\prod_{j \notin J} (k_j - v_j + 1)$ . These network states constitute a set denoted by  $S(X)$  (1).

The probability that  $A(X)$  is a set of critical arcs is at least :

$$\begin{aligned} \Pr [S(X)] &= \prod_{j \notin J} \Pr [H_j \geq H_{jv_j}] \cdot \prod_{j \in J} \Pr [H_j = H_{jv_j}] \\ &= \prod_{j \notin J} \left( \sum_{h=v_j}^{k_j} p_{jh} \right) \cdot \prod_{j \in J} p_{jv_j}. \end{aligned} \tag{1}$$

For any network state  $X'$  defined above, we necessarily have  $S(X') \subseteq S(X)$  and the set  $S(X')$  need not to be retained for probability computations.

The probability  $\Pr [S(X)]$ , computed in (1), is not larger than the probability for the critical time  $t^*$  to equal  $t(X)$ . Indeed, the time  $t(X)$  may be also the critical time for some network states which are not in  $S(X)$ .

### 3. THE ALGORITHM (see flow chart)

The algorithm consists of finding successively each value of time  $t^*$  which is the critical time for at least one network state, so that the associated probability for  $t^*$  to be a critical time exists on the time interval  $(0, \infty)$ . For each such value  $t^*$ , the algorithm selects some network states  $X$  for which the critical time  $t(X) = t^*$  and such that the probability  $\Pr [t^* = t(X)]$  can be computed.

#### Step 1

Start with the network state  $X$  for which the capacity of each arc is the smallest that could be assumed. Then,  $X = (H_{11}, H_{21}, \dots, H_{m1})$ . Find the set of

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(1) The set  $S(X)$  may be enlarged when a maximal flow through  $X$  is available. Then, the flows in arcs  $j \notin J$  are not necessarily at their upper bounds  $H_{jv_j}$ . Let  $H_{j\bar{v}_j}$  be the capacity of an arc  $j \notin J$  immediately greater than the flow in arc  $j$ . We have  $\bar{v}_j \leq v_j$  and the number of network states for which  $A(X)$  is a set of critical arcs is at least  $\prod_{j \notin J} (k_j - \bar{v}_j + 1)$ .

critical arcs  $A(X)$  and the critical time  $t(X)$ . Let  $t^* = t(X)$  and let  $K$  be a vector in which some network states with critical time  $t^*$  will be stored. The network state  $X$  is a first component for  $K$ .

### Step 2

In vector  $K$ , take a network state  $X$  which has not yet been considered. The set of critical arcs  $A(X)$  and the critical time  $t(X)$  are known. Increase successively the capacity of each arc in  $A(X)$ . A set of network states  $X'$  is created.

### Step 3

For a network state  $X'$  not yet examined, compute  $t(X')$  and  $A(X')$ . If the time value  $t(X')$  is met for the first time, a new vector  $K'$  is associated with the value  $t(X')$  and let  $X'$  be the first component of  $K'$ . Go to step 5.

If the value  $t(X')$  has been already met, there exists a vector  $K'$  in which each component  $X''$  has a critical time  $t(X'') = t(X')$  (1). If in  $K'$ , there are no network states  $X''$  for which  $A(X'') \equiv A(X')$ , store  $X'$  in vector  $K'$  and go to step 5. Otherwise, consider the set  $V$  of network states  $X''$  for which  $A(X'') \equiv A(X')$ .

### Step 4

If there exists at least one  $X''$  in  $V$ , such that for each arc  $j$  not in the set of critical arcs, the capacity of arc  $j$  in  $X'$  is not less than the capacity of arc  $j$  in  $X''$ , then the network state  $X'$  must not be retained for probability computations. Go to step 5. If not, store  $X'$  in vector  $K'$ . If in  $K'$ , there exist some network states  $X''$  in  $V$  such that for each arc  $j$  not in the set of critical arcs, the capacity of arc  $j$  in  $X''$  is not less than the capacity of arc  $j$  in  $X'$ , these network states are cancelled from  $K'$ .

### Step 5

If all  $X'$  have been examined, go to step 6. Otherwise, go to step 3.

### Step 6

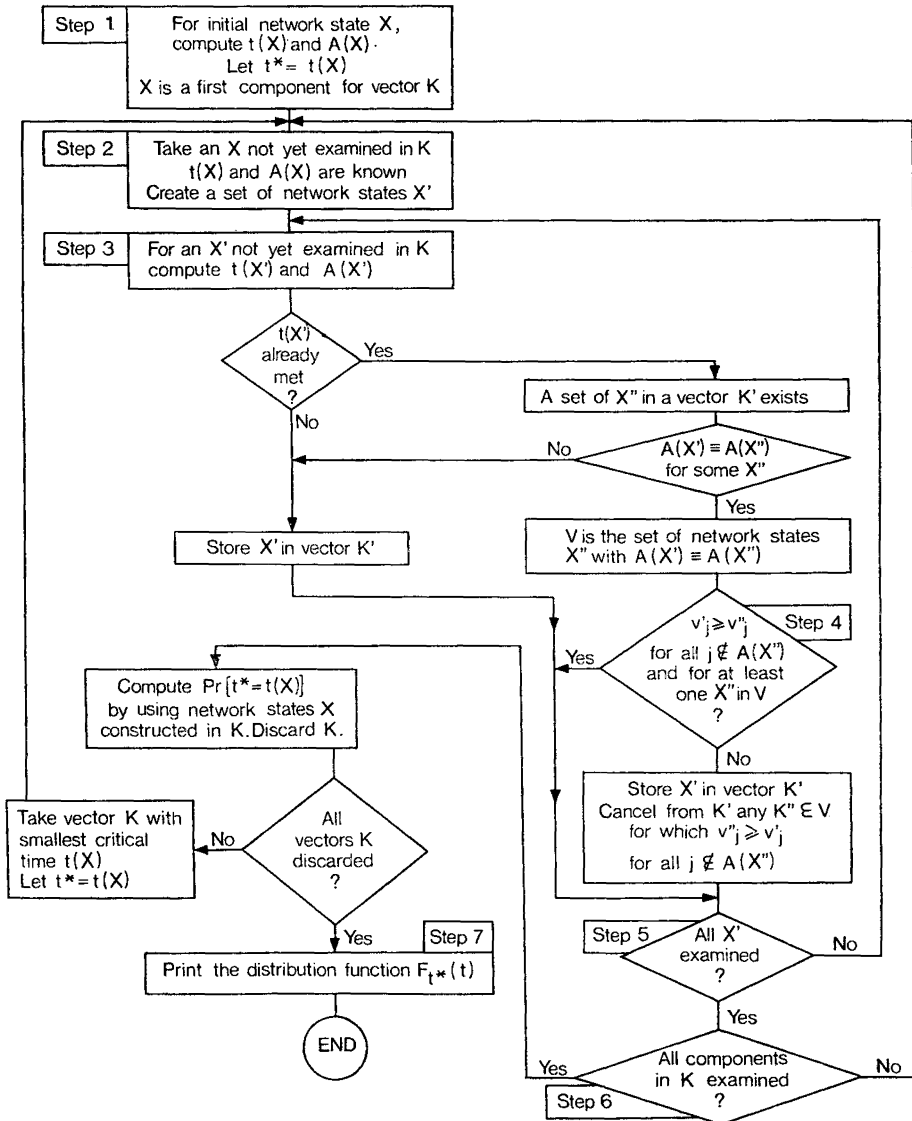
If all components in  $K$  have not been examined, go to step 2. Otherwise, compute the probability  $\Pr [t^* = t(X)]$  by using all components  $X$  constructed in vector  $K$ , as explained in section 4. Discard vector  $K$ . If all network state vectors have been exhausted, go to step 7. Otherwise, some network states with critical times greater than  $t^*$  have been constructed by the algorithm. Let the vector  $K$  be the vector in which the network states  $X$  have the smallest critical time  $t(X)$ . Let  $t^* = t(X)$  and go to step 2.

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(1)  $K' \equiv K$  if  $t(X') = t(X)$ .

Step 7

For any value  $t^*$  greater than  $t(X)$ , the probability for  $t^*$  to be a critical time does not exist. All values of time for which such a probability exists have been found and the probability distribution of the critical time is obtained END.



#### 4. PROBABILITY COMPUTATIONS

The network states  $X$  stored by the algorithm in a vector  $K$  are the only network states needed to compute the probability  $\Pr [t^* = t(X)]$ .

If there are no network states in  $K$  with identical sets of critical arcs we have

$$\Pr [t^* = t(X)] = \sum_{X \in K} \Pr [S(X)].$$

and the probability can be computed as in (1), section 2.

If, for only two network states  $X_q$  and  $X_r$  in  $K$ , we have  $A(X_q) \equiv A(X_r)$ , let  $J$  be the set of arcs which are in  $A(X_q)$  or in  $A(X_r)$ . Then, from the algorithm, we have  $v_j^q < v_j^r$  for at least one  $j \notin J$  and  $v_j^q > v_j^r$  for at least one  $j \notin J$  and therefore  $S(X_q) \not\subseteq S(X_r)$  and  $S(X_r) \not\subseteq S(X_q)$ .

Therefore,

$$S(X_{qr}) = S(X_q) \cap S(X_r) \neq \emptyset$$

$$\text{where } X_{qr} = (\max [H_{1v_1^q}, H_{1v_1^r}], \dots, \max [H_{mvm^q}, H_{mvm^r}]).$$

Consequently, the probability associated to network states  $X_q$  and  $X_r$  is : [2], pp. 89.

$$\Pr [S(X_q) \cup S(X_r)] = \Pr [S(X_q)] + \Pr [S(X_r)] - \Pr [S(X_{qr})]$$

and

$$\Pr [t^* = t(X)] = \sum_{X \neq X_q, X_r} \Pr [S(X)] + \Pr [S(X_q) \cup S(X_r)]$$

Each term of the right-hand side can be computed as in (1), section 2. More generally, if several network states  $X_1, X_2 \dots X_w$  in  $K$  have an identical set of critical arcs, let  $J$  be the set of critical arcs. Then, from the algorithm, we have  $v_j^q < v_j^r$  for at least one  $j \notin J$  and  $v_j^q > v_j^r$  for at least one  $j \notin J$  ( $q = 1 \dots w - 1; r = q + 1 \dots w$ ).

Therefore,  $S(X_{qr}) = S(X_q) \cap S(X_r) \neq \emptyset$  ( $q = 1 \dots w - 1; r = q + 1 \dots w$ ), but any set  $S(X_{qrs} \dots)$  may be empty.

Consequently, the probability associated to network states  $X_1, X_2 \dots X_w$  is :

$$\begin{aligned} \Pr [S(X_1) \cup S(X_2) \dots \cup S(X_w)] &= \sum_{q=1}^w \Pr [S(X_q)] - \sum_{\substack{q, r=1 \\ q \neq r}}^w \Pr [S(X_{qr})] \\ &\quad + \dots + (-1)^{w-1} \Pr [S(X_{12\dots w})]. \end{aligned}$$

Each term of the right-hand side, if non empty, is computed as in (1), section 2.

5. NUMERICAL EXAMPLE

The network with random arc capacities is represented in figure 2. The demands required at the demand nodes of the network are piecewise linear functions of time and are drawn in figure 3.

The data for the problem are given in figure 1. The capacity of each arc  $j$  is a discrete random variable which can assume only two values  $H_{j1}$  and  $H_{j2}$ . For each  $j$ , we have  $H_{j1} \leq H_{j2}$  and the associated probabilities are respectively  $p_{j1}$  and  $p_{j2}$ .

$j$	$H_{j1}$	$H_{j2}$	$p_{j1}$	$p_{j2}$
$s-1$	175	350	.03	.97
$s-2$	175	350	.03	.97
$s-3$	7.5	15	.03	.97
1-2	175	350	.03	.97
1-3	0	30	.03	.97
2-4	0	30	.03	.97
2-5	30	60	.03	.97
3-4	7.5	15	.03	.97
4-5	7.5	15	.03	.97

Figure 1  
The network data

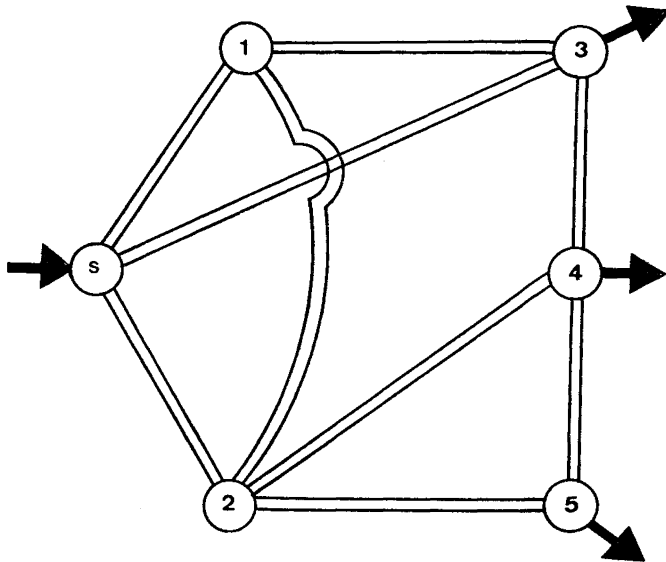


Figure 2  
The network with random arc capacities



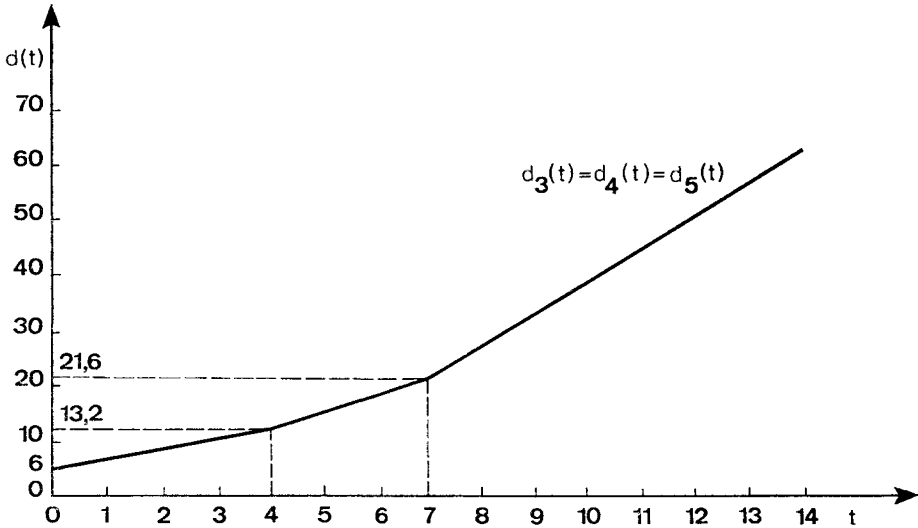


Figure 3  
The demand functions

The computed probability distribution function  $F_{t^*}(t)$  is given in figure 4, where the time parameter  $t$  is considered as a time interval (say, a year). Then, the probability  $\Pr [t^* = t]$  is the probability for critical time  $t^*$  to occur at year  $t$  (or equivalently, the probability for the network to have a set of critical arcs at year  $t$ ).

$t$	$\Pr [t^* = t]$	$F_{t^*}(t)$
1	.0	.0
2	.0	.0
3	.00000081	.00000081
4	.0	.00000081
5	.00005238	.00005319
6	.0	.00005319
7	.00089919	.00095238
8	.00002540	.00097778
9	.0	.00097778
10	.00659816	.00757594
11	.05237082	.05994676
12	.02820158	.08814834
13	.91185165	1.00000000

Figure 4  
The probability distribution function

From figure 4, it can be concluded that no set of critical arcs could appear in the network before  $t = 3$ . No set of critical arcs could occur at  $t = 4$ ,  $t = 6$

and  $t = 9$ . Beyond  $t = 13$ , no time value could satisfy the definition of critical time (unfeasibility region).

The largest value of time  $T$  up to which the probability of satisfying all demands is above a given confidence level can be determined.

If  $\beta = .99$ , the value  $T$  lies at the beginning of year  $t = 11$  since  $1 - F_{t^*}(10) = .993$  and  $1 - F_{t^*}(11) = .941$ .

The function  $F_{t^*}(t)$  gives the probability for the network to have a set of critical arcs in the time interval  $(0, t)$ . If we define  $\Pr [A(t)]$  as the probability for a set of arcs  $A$  to be critical in the time interval  $(0, t)$ , we have :

$$F_{t^*}(t) = \sum_A \Pr [A(t)] \text{ for all } A.$$

In the above example, the probability for a set of arcs  $A$  to be critical in the feasible time interval  $(0, 13)$  is different from zero for the only five sets of arcs shown in figure 5. The probability function  $\Pr [A(t)]$  for each of these sets of arcs is given in figure 6.

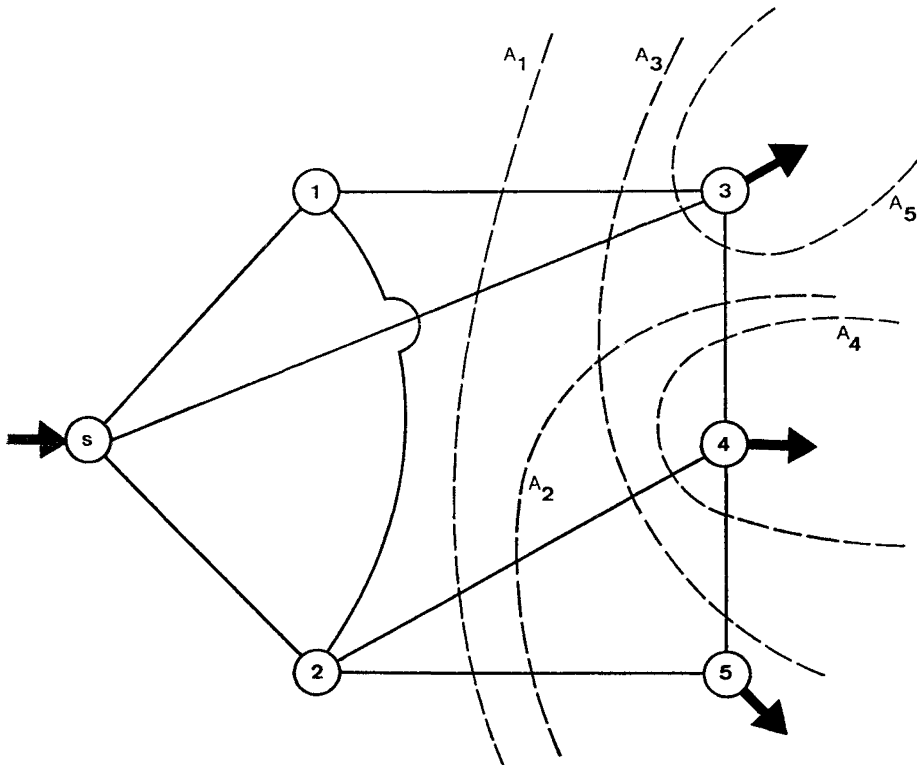


Figure 5  
The sets of critical arcs

$t$	$\text{Pr} [A_1(t)]$	$\text{Pr} [A_2(t)]$	$\text{Pr} [A_3(t)]$	$\text{Pr} [A_4(t)]$	$\text{Pr} [A_5(t)]$
1	.0	.0	.0	.0	.0
2	.0	.0	.0	.0	.0
3	.0	.0	.00000081	.0	.0
4	.0	.0	.00000081	.0	.0
5	.0	.0	.000053	.0	.0
6	.0	.0	.000053	.0	.0
7	.000025	.0	.000881	.000026	.000026
8	.000025	.000025	.000881	.000026	.000026
9	.000025	.000025	.000881	.000026	.000026
10	.000895	.000846	.002497	.001647	.001694
11	.001735	.000846	.028257	.001647	.027454
12	.028293	.001667	0.29078	.001647	.027454
13	.887023	.001667	.082252	.001647	.027454

Figure 6  
Probability functions for sets of arcs to be critical

From figure 6, it can be known, at any time  $t$ , which set of arcs  $A$  has the largest probability to be critical in the time interval  $(0, t)$ . Since a set of critical arcs is isolating a set of demand nodes from the common source node, the probability for this set of demand nodes to be isolated by a set of critical arcs in time interval  $(0, t)$  is also known from figure 6.

The results presented in this section were obtained after 20 seconds of computer time. The computer program was written in FORTRAN IV and was executed on a 360-40 IBM computer.

## 6. CONCLUDING REMARKS

The method presented in this paper finds the capacity distribution function of a network with random arc capacities and can be used to evaluate the network reliability. Usually, when a network has been constructed with least cost equipment, it must be verified whether a product can be shipped through the network with high enough reliability.

Increasing the reliability of service is one of the great, continuing challenges faced by industry. This is especially true for power supply industry to which the method presented in this paper has been applied. Because in an age of total or near-total electric homes and industries, power failure is more than a mere inconvenience. It can be both costly and serious. If the demands for electrical power are rigorous from residential customers, they are doubly so from industrial customers. Power failure can take out a production line; to the degree a plant is automated, it can cripple an entire plant. Therefore, ensuring

a high reliability of service at a least cost is the primary problem for the power supply industry.

Although the algorithm presented in this paper was efficient for finding the capacity distribution function of small networks, it is expected that the computer requirements will increase fairly rapidly with the size of the network and the number of levels each arc capacity can assume (1).

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(1) Since these lines were written, a different approach which is computationally more advantageous for finding the capacity distribution function of a large network has been established by the author.