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A RENEWAL THEOREM FOR STATIONARY POPULATIONS

par M. Krakowski

Résumé. — *Add a newborn baby to a random sample of k individuals out of a stationary population. Then, (a) at the instant of the first death the k survivors are again a randomly and independently distributed sample of the stationary population; (b) the probability density of âge at death of the first individual (among the $k + 1$) to depart is the probability density of lifespans for the stationary population; and (c) the expected time till the first death (among the $k + 1$) is an average lifespan divided by $k + 1$.*

In demographic and epidemiological applications a stationary population is one whose constant birth rate equals its death rate, and whose age distribution (for very large populations) and mortality function do not depend on calendar time. (It is a stable population of zero growth.) We refer to individuals, births, and deaths, but the population may consist of any objects, births and deaths meaning entries and departures.

Let :

$s(t)$ = probability that a newborn will survive (at least) to age t; $s(0) = 1$, of course; $s(x) \to 0$ as $x \to \infty$.

$m(t)$ = mortality function (sometimes called hazard function), i.e. $m(t)dt$ = probability that an individual of age t will die within dt.

$f(t)$ = lifespan density, i.e. probability density that a newborn will die at age t.

$p(t)$ = age density of the stationary population, i.e. $p(t)\mathrm{d}t$ = fraction of individuals of ages between t and $t + \mathrm{d}t$.

M = expected lifespan of a newborn.

It is shown in the Appendix that :

1) $p(t) = p(0)s(t)$; this relation is used repeatedly in the Proofs.

2) $f(t) = -s'(t) = -p'(t)/p(0)$.

3) $m(t) = -s'(t)/s(t) = -p'(t)/p(t)$.

4) $M = 1/p(0)$; $p(0)$ = birth rate = death rate.

5) probability that an individual of age x will survive (at least) t more years is $s(x + t)/s(x)$.

Theorem

Let the age density of the stationary population be $p(t)$, its life-span distribution $f(t)$, and the average life-span be M.

Grab a « random sample » of k individuals and add to it a new-born baby. Then :

a) At the instant of the first death the k survivors are a random sample (1) of the original population;

b) The probability density that the first death among the $k + 1$ individuals occurs at an age t is $f(t)$;

c) The expected time till the first death among the $k + 1$ individuals is $M/(k + 1)$.

In order to simplify the typography the proof will be carried out for the case $k = 1$ only, i. e. when a newborn is paired with one random individual (more precisely, a randomly selected individual from the population, the probability of drawing an individual aged t being $\mathrm{p}(t)$). The extension to arbitrary k will be apparent.

Proof of a).

1. The probability that the newborn will still be alive at age t, (i.e. t time units after the pairing) is $s(t)$. The probability density that the random member will die t time units after the pairing is

$$\int_0^\infty p(x)\frac{p(x+t)}{p(x)}m(x+t)\,\mathrm{d}x = -\int_0^\infty p(x+t)\frac{p'(x+t)}{p(x+t)}\,\mathrm{d}x = p(t).$$

Therefore, conditional upon the newborn being the survivor, his age at the time of his partner's death has the probability density $\mathrm{s}(t)\mathrm{p}(t)$.

2. The probability density of the joint event « newborn will die exactly when random member reaches age t » is

$$\int_0^t p(x)\frac{p(t)}{p(x)}f(t-x)\,\mathrm{d}x = -p(t)\int_0^\infty s'(t-x)\,\mathrm{d}x = p(t) - p(t)s(t).$$

Adding the probability densities of 1. and 2. we get $p(t)$, the age density of a random selection.

(1) That is, the probability density that the ages of the k survivors are $t_1, t_2, \ldots t_k$ is $p(t_1), p(t_2) \ldots p(t_k)$ when these survivors are ordered, say, alphabetically; or the joint probability density is $k!p(t_1)p(t_2) \ldots p(t_k)$ when their order is immaterial.

Proof of b)

1. The probability density that the newborn will die at the age t and that the random member will then be still alive, is

$$-s'(t)\int_0^\infty p(x)\frac{p(x+t)}{p(x)}\,\mathrm{d}x = -s'(t)\int_t^\infty p(x)\,\mathrm{d}x.$$

2. The probability density that the random member will die at age t while the newborn still survives, is :

$$\int_0^\infty p(x)\frac{p(t)}{p(x)}\frac{-s'(t)}{s(t)}s(t-x)\,\mathrm{d}x = -s'(t)\int_0^t p(t-x)\,\mathrm{d}x = -s'(t)\int_0^t p(x)\,\mathrm{d}x.$$

Adding the probability densities 1. and 2. we find that the density of the random variable « age at which first death occurs », is $-s'(t) = f(t)$; cf. 1). This is the life-span distribution, as stated in the part *b)* of the Theorem.

Proof of c)

The joint survival function of the newborn and the random member, i.e. the probability that both will be alive at time t after the pairing, is

$$S(t) = s(t)\int_0^\infty p(x)\frac{p(x+t)}{p(x)}\,\mathrm{d}x = s(t)[1-P(t)];\; P(t) = \int_0^t p(x)\,\mathrm{d}x.$$

The corresponding probability density of joint life spans is, cf. 1),

$$-S'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\{\,s(t)[1-P(t)]\,\}.$$

Therefore, the expected time till the next death is

$$-\int_{x=0}^\infty x d\{\,s(x)[1-P(x)]\,\} =$$

$$-xs(x)[1-P(x)]\Big|_{x=0}^{\infty} + \int_0^\infty s(x)[1-P(x)]\,\mathrm{d}x$$

$$= -\frac{1}{p(0)}\int_{x=0}^\infty [1-P(x)]d[1-P(x)]$$

$$= \frac{-1}{2p(0)}[1-P(x)]^2\Big|_{x=0}^{\infty} = \frac{1}{2p(0)} = M/2, \text{ qed.}$$

APPENDIX

Derivation of Relations 1) through 5)

These relations are classical and can be found in Lotka and Keyfitz (cf. Bibliography), among others. Brief proof outlines are given here for completeness of presentation. p.d. = probàbility density.

Proof of 5)

The probability that a newborn will survive to age $t + x$ (at least) = probability that ehe will survive till age t, multiplied by the conditional probability that an individùal aged t will survive to $t + x$; thus $s(t + x) = [s(t)]$ [probability of survival till age $t + x$, conditional upon being of age t]. This is equivalent to 5).

Proof of 1)

The p.d. that an individual will be t gears old $t - x$ gears from now = p.d. that he is x years old now, multiplied by the probability that he will survive from age x till age t. Because of stationarity the first p.d., referring to the time $t - x$ hence, is the same as the p.d. now, i.e. $p(t)$. The second p.d. is $p(x)$. The probability of survival from given age x to age $x + t$ is, using the already proven 5), $s(t)/s(x)$.

Therefore, $p(t) = p(\mathrm{x})s(t)/s(x)$; when $x \to 0$, $s(x) \to 1$, and $p(t) = p(0)s(t)$, q.e.d.

Proof of 2)

The probability that a newborn dies before age t is $F(t) = 1 - s(t)$; therefore the p.d. of lifespan distributions is $F't) = f(t) = - s'(t)$, and in view of 1), equals also $-p'(t)/p(0)$.

Proof of 3)

The p.d. of dying exactly at age t equals the survival probability till i, multiplied by the mortality at t. Thus, $f(t) = s(t)m(t)$, or in view of 2) $-s'(t) = s(t)m(t)$. Using 1) it follows that $m(t) = - s'(t)/s(t) = -p'(t)/p(t)$.

$$M = \int_0^\infty xf(x)\,\mathrm{d}x = - \int_0^\infty x\,\mathrm{d}s(x)$$

$$= - xs(x) \int_0^\infty + \int_0^\infty s(x)\,\mathrm{d}x = \int_0^\infty p(x)\,\mathrm{d}x/p(0) = 1/p(0).$$

We assume that $s(x) \to 0$ rapidly enough so that $xs(x) \to 0$.

The death rate

$$= \int_0^\infty p(x)m(x)\,\mathrm{d}x = -\int_0^\infty p(x)\frac{p'(x)}{p(x)}\,\mathrm{d}x = -\int_0^\infty \mathrm{d}p(x) = p(0) - p(\infty) = p(0).$$

Finally, stationarity implies that birth rate = death rate.

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