TOSHIO NAKAGAWA

SHUNJI OSAKI

The busy period of a repairman for redundant repairable systems

<http://www.numdam.org/item?id=RO_1975__9_3_69_0>

THE BUSY PERIOD OF A REPAIRMAN
FOR REDUNDANT REPAIRABLE SYSTEMS (*)

by Toshio NAKAGAWA [1] and Shunji OSAKI [2]

Abstract. — *The busy period of a repairman attending a repairable system composed of unreliable units has been discussed respectively for a parallel redundant system and a standby redundant system using the results of reliability theory and queueing theory. The results and models presented in this note might be of particular interest for reliability engineers and system designers.*

1. INTRODUCTION

In a recent contribution to this journal, Vanderperre [7] considered an $(n + 1)$ unit parallel redundant system with a single repairman, and derived directly the distribution of the busy period of the repairman using renewal theoretic arguments.

In this note, we first show that the $(n + 1)$ unit parallel redundant system of Vanderperre is formally coincident with a standby redundant system with many repairmen, and, from this fact, the result of Vanderperre is easily derived from the first-passage time distributions to system failure of such a standby redundant system, which have been discussed by Srinivasan [6] and Nakagawa [4]. In a similar fashion, we further derive the distribution of the busy period of the standby redundant system using the well-known results of the $G/M/1$ queues ([2], [3]). We finally discuss the busy period of a two-unit standby redundant system under more generalized assumptions.

2. PARALLEL REDUNDANT SYSTEM

Consider a parallel redundant system of $(n + 1)$ identical units with a single repairman in the context of Vanderperre (see *fig.* 1). We use his notation and assumptions unless otherwise mentioned.

We next consider a standby redundant system with many repairmen (*see* Case D in Barlow and Proschan [1], pp. 139-151, Srinivasan [6], and
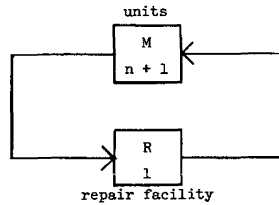
Figure 1

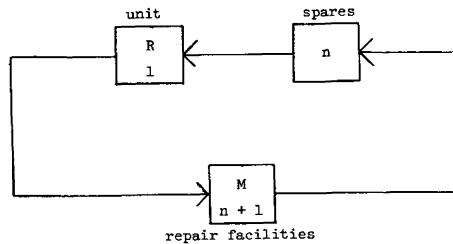An $(n + 1)$ unit parallel redundant system with a single repairman.

Figure 2

A standby redundant system with many repairmen.

Nakagawa [4]). That is, consider a standby redundant system with $(n + 1)$ identical units wherein one unit is operating and other units are in standby or under repair. Of course, a failed unit is repaired immediately and then put back into standby immediately. In particular, Barlow and Proschan [1], p. 150, have sketched figure 2 for illustration.

Comparing figure 1 with figure 2, we conclude that the above two models are equivalent one another wherein the notion of failure and repair changes one another. Thus, the results of Vanderperre will be easily obtained from those of the standby redundant system.

Let $F_{ik}(t)$ be the first-passage time distribution from state i to state k for the process generated by the system in figure 2, where state i ($i = 0, 1, 2, \ldots, n + 1$)

denote the numbers of units under repair. Then, it is evident that the distribution $B_n(t)$ of the busy period of the repairman for the process generated by the system in figure 1 (*see* Vanderperre [7]) corresponds to the first-passage time distribution $F_{n,n+1}(t)$ from state n to state $n+1$. Thus, from (22) of Nakagawa [4], we have the Laplace-Stieltjes (LS) transform of the distribution of the busy period:

$$b_n(s) = f_{n, n+1}(s)$$

$$= \frac{\sum_{j=0}^{n} \binom{n}{j} \Big/ C_{j-1}(s)}{\sum_{j=0}^{n+1} \binom{n+1}{j} \Big/ C_{j-1}(s)}, \tag{1}$$

where

$$C_j(s) = \prod_{i=0}^{j} \frac{r(s+i\lambda)}{1-r(s+i\lambda)} \qquad (j = 0, 1, 2, \ldots), \qquad C_{-1}(s) \equiv 1,$$

and, in general, the small letter functions denote the LS transforms of the corresponding capital ones throughout this note. This result in (1) agrees with that of Vanderperre [7].

We further have the relation :

$$b_n(s) = f_{n, n+1}(s) = f_{0, n+1}(s)/f_{0, n(s)}, \tag{2}$$

and hence, from (12) in Srinivasan [6], we can also obtain $b_n(s)$ in (1).

The mean busy period of the repairman is

$$l_n = \frac{1}{\mu} \sum_{j=0}^{n} \left[\binom{n}{j} \Big/ C_j \right] \qquad (n = 1, 2, \ldots), \tag{3}$$

where

$$C_j = \prod_{i=1}^{j} \frac{r(i\lambda)}{1-r(i\lambda)} \qquad (j = 1, 2, \ldots), \quad C_0 = 1.$$

3. STANDBY REDUNDANT SYSTEM

Consider a standby redundant system with a single repairman wherein one unit is operating and other n units are in standby or under repair. The behavior of the system is shown in figure 3. Moreover, we can see that the first-passage time from state i to state k $(i < k)$ for the system in figure 3 corresponds to that of the $G/M/1$ queue, where the state of the system denotes the number of units under repair or waiting for repair, and the state of the

queue denotes the queue size, respectively. That is, the first-passage time from state i to state k $(i < k \leq n + 1)$ of the system is identified with that of the system with infinite spares, which has the same stochastic structures for the $G/M/1$ queue
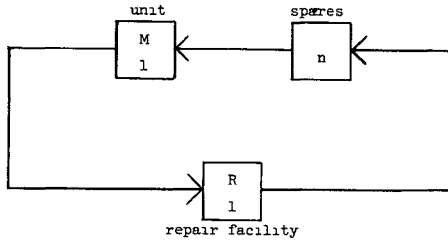

Figure 3

A standby redundant system with a single repairman.

Therefore, the distribution of the busy period of the system in figure 3 is equal to the first-passage time distribution $F_{n,n+1}(t)$ from state n to state $n + 1$ of the $G/M/1$ queue. Thus, from Cohen ([2], [3]), the LS transform of the busy period is

$$b_n(s) = \frac{1 + [1 - r(s)] A_n(s)}{1 + [1 - r(s)] A_{n+1}(s)} \qquad (n = 1, 2, \ldots), \qquad (4)$$

where $A_n(s)$ is given by

$$\sum_{j=0}^{\infty} A_j(s) z^j = \frac{z^2}{(1-z)\{r[s + \lambda(1-z)] - z\}} \qquad \text{for } |z| < 1.$$

Further, the mean busy period is

$$l_n = \frac{1}{\mu} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial z^{n-1}} \frac{1}{r[\lambda(1-z)] - z}\bigg|_{z=0} \qquad (n = 1, 2, \ldots). \qquad (5)$$

4. TWO-UNIT STANDBY REDUNDANT SYSTEM

If $n = 1$, i. e., the system is composed of two identical units, the busy periods of the repairman both in figure 1 and figure 3 are identical and

$$b_1(s) = \frac{r(s+\lambda)}{1 - r(s) + r(s+\lambda)}, \qquad (6)$$

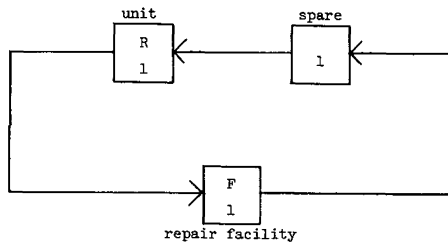$$l_1 = 1/[\mu r(\lambda)]. \qquad (7)$$

Figure 4

A two-unit standby redundant system with a single repairman.

Furthermore, when the failure time of each unit has a general distribution $F(t)$ in figure 4, we can obtain the following $b_1(s)$ and l_1 of the system in figure 4 :

$$b_1(s) = \frac{\int_0^\infty e^{-st} [1 - F(t)] \, dR(t)}{1 - \int_0^\infty e^{-st} F(t) \, dR(t)}, \qquad (8)$$

$$l_1 = 1 \Big/ \left\{ \mu \int_0^\infty [1 - F(t)] \, dR(t) \right\}, \qquad (9)$$

which were derived by Nakagawa and Osaki [5].

REFERENCES

[1] R. E. BARLOW and F. PROSCHAN, *Mathematical theory of reliability*, Wiley, New York, 1965.

[2] J. W. COHEN, *The distribution of the maximum number of customers present simulataneously during a busy period for the queueing system M/G/1 and G/M/1*, J. Appl. Prob., 4, 1967, p. 162-179.

[3] J. W. COHEN, *The single server queue*, North-Holland, Amsterdam-London, 1969.

[4] T. NAKAGAWA, *The expected number of visits to state k before a total system failure of a complex system with repair maintenance*, Oper. Res., 22, 1974, p. 108-116.

[5] T. NAKAGAWA and S. OSAKI, *Stochastic behaviour of a two-unit standby redundant system*, INFOR, 2, 1974, p. 66-70.

[6] V. S. SRINIVASAN, *First emptiness in the spare parts problem for repairable components*, Oper. Res., 16, 1968, p. 407-415.

[7] E. J. VANDERPERRE, *The busy period of a repairman attending a (n + 1) unit parallel system*, R.A.I.R.O., 7e année, V-1, 1973, p. 124-126.