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RAIRO. Recherche opérationnelle, tome 11, nº 4 (1977), p. 355-361

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SOME APPROXIMATE EQUILIBRIUM RESULTS FOR THE $E_k/M/r$ NO-QUEUE SYSTEM (*)

par George P. COSMETATOS (1)

Abstract. — Approximate formulae for the utilization of a group of servers in the $E_k/M/r$ no-queue system are derived. The degree of approximation achieved depends on the particular formula used and on the values of the various parameters involved; the relative percentage errors incurred seem, however, to be well below 1 per cent in absolute value. The derivation of the formulae is based on the underlying principle that two systems of the same general type $E_k/M/r$ with equal parameter k and equal traffic intensities but with unequal number of servers. have a "similar" behaviour.

Service systems where no queue is formed are quite common in practice, the original physical situation motivating their study, just after the turn of this century, being the simple telephone network [2]. Over the years many models of such systems have been devised, solved and applied in the design of telephone exchanges [3, 5]; and fairly recently these models have been used in the design of file organization systems and in the study of situations where either a queue is not allowed to form because of space limitations (limiting case of balking) or customers are unwilling to join a queue because of long expected queueing time (limiting case of impatience).

The system considered in this paper can be described as follows : there are r servers in parallel each having an exponential service time distribution with mean μ^{-1} . Intervals between successive arrivals have an Erlang – k distribution; the average rate of arrival of customers is λ , so that the traffic intensity $\rho = \lambda/r\mu$. No queue is formed : all customers that arrive and find every server busy are turned away.

When dealing with such a system one is usually faced with the problem of balancing the efficient use of the group of servers against the provision of acceptable service for the customers. One is therefore primarily interested in calculating the utilization factor of the servers (also known as server occupancy) which will be denoted U(r, k) and the probability of a customer being turned away (also known as probability of loss or blocking probability) which will be denoted B(r, k).

^(*) Reçu septembre 1976.

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R.A.I.R.O. Recherche opérationnelle/Operations Research, vol. 11, nº 4, nov. 1977

According to Takács [6], who discusses the GI/M/r no-queue system in detail,

$$U(r, k) = \frac{\rho \sum_{j=1}^{r} {r \choose j} D_j}{1 + \sum_{j=1}^{r} {r \choose j} D_j}$$
(1)
$$D_j = \sum_{i=1}^{j} \left[\frac{1 - \varphi(i\mu)}{\varphi(i\mu)} \right],$$

. .

with

where $\varphi(s)$ is the Laplace-Stieltjes transform of the inter-arrival times distribution given, in the case of Erlang inter-arrival times, by

$$\varphi(s) = \left[\frac{1}{1 + \frac{s}{k\rho r\mu}}\right]^{k}$$

The blocking probability can then be calculated as

$$B(r, k) = [\rho - U(r, k)]/\rho \qquad (2)$$

It is interesting to note that in a single-server no-queue system the complicated-looking formula (1) is greatly simplified :

$$U(1, k) = \rho \left[1 - \frac{(k \rho)^{k}}{(1 + k \rho)^{k}} \right]$$
(3)

DERIVATION OF APPROXIMATE FORMULAE FOR U(r, k)

With the assumption that inter-arrival times have an Erlang distribution, U(r, k) depends on ρ , r and k only; and, according to Benès [1], for ρ and r fixed, U(r, k) is an increasing function of k with limiting values U(r, 1), corresponding to the system M/M/r, and $U(r, \infty)$ if $k \to \infty$.

We now consider two such systems, $E_k/M/r_j$ and $E_k/M/r_i$, both with equal traffic intensities but with $r_j > r_i$ and proceed in writing a relationship of the general form :

$$\frac{U(r_j, k)}{U(r_i, k)} = S \frac{U(r_j, 1)}{U(r_i, 1)} + (1 - S) \frac{U(r_j, \infty)}{U(r_i, \infty)}$$
(4)

where S is a factor, yet to be specified, with limiting values 1 and 0 for k = 1 and ∞ respectively, regardless of r_i , r_i or ρ .

(4) is an expression giving $U(r_i, k)$ in terms of $U(r_i, k)$; consequently, if the relative percentage errors incurred are small,

R.A.I.R.O. Recherche opérationnelle/Operations Research

356

a) the hypothesis that the $E_k/M/r_j$ and $E_k/M/r_i$ no-queue systems with equal traffic intensities have a similar behaviour would be established, the quantity on the right hand side being regarded as the similarity ratio; and

b) (4) could be used as an approximate formula for the evaluation of $U(r_j, k)$ provided that $U(r_i, k)$ with $r_i = r_j - 1, r_j - 2, \ldots$ or even 1 and the similarity ratio were both known.

A method for the determination of S has now to be devised. It was felt initially that solving (4) for S could provide some indication on how sensitive S is to the values of r_i , r_j , k and ρ . A computer programme developed by Sellmeyer [4] was used for the calculation of the six utilization factors in (4) and the computed values of S were then tabulated for selected values of the various parameters involved. Quite unexpectedly, a promising result emerged, namely that S seemed to be rather insensitive to the value of ρ . On the basis of this information, the determination of S can be achieved by expressing (4) in the limiting case of heavy traffic conditions ($\rho \rightarrow \infty$). By expanding $[1 - \varphi(i\mu)]/\varphi(i\mu)$ into a power series of ρ , it can be easily verified that :

$$\lim_{\rho \to \infty} U(r, k) = \frac{2\rho r + 2(r-1) - (1+k)/k}{2\rho r + 2r}$$

in which case simple algebra yields S = 1/k so that :

$$U(r_j, k) \simeq \left[\frac{1}{k} \cdot \frac{U(r_j, 1)}{U(r_i, 1)} + \frac{k-1}{k} \cdot \frac{U(r_j, \infty)}{U(r_i, \infty)}\right] U(r_i, k)$$
(5)

We evaluated formula (5) for $2 \le r_j \le 100$, $1 \le r_i \le r_{j-1}$, k > 1 and $0.4 \le \rho \le 3.0$ and calculated the corresponding theoretical values for $U(r_j, k)$ using the programme in [4]. The results obtained are briefly outlined in the next section.

EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

Evaluation of formula (5) with $r_i = 1$

If we set $r_i = 1$, formula (5) is greatly simplified because the expression in (3) for U(1, k) can be worked out manually; some results for $U(r_j, 1)$ and $U(r_j, \infty)$ are given in Appendices 1 and 2 for easy reference.

a) For a given value of k, any value of $U(r_j, k)$ can be approximated with a relative percentage error :

e = 100 (approximate value - theoretical value)/(theoretical value)

which (i) is negative and tends to $0 \text{ as } \rho \to \infty$; (ii) is not sensitive to the value of r_j ; and (iii) is very small in absolute value, not exceeding 1 per cent over the whole range of values for ρ , k and r_j considered.

vol. 11, nº 4, nov. 1977

G. P. COSMETATOS

b) For given values of r_j and ρ , $U(r_j, k)$ can be approximated with a relative percentage error which varies with k as originally expected : it tends to 0 as k approaches 0 or 1 and attains a maximum absolute value somewhere in-between, namely for $k \simeq 2$. Table 1, for example, gives the relative percentage errors incurred if formula (5) with $r_i = 1$ is used for the evaluation of U(4, k) for $\rho = 0.5, 1.0, 1.5, 2.0, 3.0$ and k = 1, 2, 3, 4, 9, 25, 100 and ∞ .

	TABLE 1	
Relative	e percentage errors incurred if formula (5)
with r _i	= 1 is used for the evaluation of $U(4, k)$	k) [

k	. 1	2	3	4	9	25	100	. ∞
ρ								
0.5 1.0 1.5 2.0 3.0	0 0 0 0	$\begin{array}{r} - 0.97 \\ - 0.50 \\ - 0.39 \\ - 0.31 \\ - 0.19 \end{array}$	- 0.93 - 0.48 - 0.36 - 0.29 - 0.18	$\begin{array}{r} - 0.82 \\ - 0.41 \\ - 0.32 \\ - 0.25 \\ - 0.15 \end{array}$	$ \begin{array}{r} - 0.46 \\ - 0.23 \\ - 0.17 \\ - 0.14 \\ - 0.08 \\ \end{array} $	$\begin{array}{r} - \ 0.18 \\ - \ 0.09 \\ - \ 0.07 \\ - \ 0.05 \\ - \ 0.03 \end{array}$	0.05 0.02 0.02 0.01 0.01	0 0 0 0

Evaluation of formula (5) with $r_i > 1$

If a value of r_i exceeding unity is selected, formula (5) becomes difficult to apply; however, the relative percentage errors incurred are, generally, smaller in absolute value. In particular it was found that for given values of ρ and k, the degree of similarity in the behaviour of the no-queue systems $E_k/M/r_j$ and $E_k/M/r_i$, as expressed in (5), tends to increase when $(r_j - r_i)/r_j$ decreases. Table 2 gives the relative percentage errors incurred in evaluating U (10, 2) for $\rho = 0.5, 1.0, 1.5, 2.0, 3.0$ and $r_i = 1, 2, 3, 4, 6$ and 8.

ρ	l	2	3	4	6	8
0.5 1.0 1.5 2.0 3.0	$\begin{array}{r} - 0.74 \\ - 0.55 \\ - 0.43 \\ - 0.32 \\ - 0.19 \end{array}$	- 0.35 - 0.33 - 0.28 - 0.20 - 0.11	$\begin{array}{r} - 0.16 \\ - 0.23 \\ - 0.20 \\ - 0.14 \\ - 0.07 \end{array}$	$\begin{array}{r} - 0.05 \\ - 0.17 \\ - 0.15 \\ - 0.10 \\ - 0.04 \end{array}$	$\begin{array}{r} 0.04 \\ - 0.09 \\ - 0.08 \\ - 0.05 \\ - 0.02 \end{array}$	$\begin{array}{r} 0.04 \\ - 0.04 \\ - 0.03 \\ - 0.02 \\ - 0.01 \end{array}$

TABLE 2 Relative percentage errors incurred if formula (5) is used for the evaluation of U(10, 2)

R.A.I.R.O. Recherche opérationnelle/Operations Research

CONCLUSIONS

From a practical viewpoint formulae (5) with $r_i = 1$ and (2) appear to provide a reliable, simple and economical tool for the analysis of $E_k/M/r$ no-queue systems. If interested in redesigning an already existing system, where the arrival pattern may not lend itself to direct measurement, the values of ρ and k are unknown; provided that an Erlang distribution of inter-arrival times seems a reasonable distribution for one to adopt, it might be possible to estimate the values of U(r, v) and B(r, v) by other means and thus obtain a value for ρ by solving (2). Repetitive application of (5) could then provide an estimate for k.

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vol. 11, nº 4, nov. 1977

SERVER UTILIZATION	U(r,1)	IN	M/M/r	NO -	QUEUE	SYSTEMS

2	1	2	3	4	6	8
0.4	0.285714	0.339623	0.364090	0.377413	0.390256	0.395528
0.6	0.375000	0.452055	0.491840	0.516776	0.546516	0.563450
0.8	0.444444	0.536082	0.585275	0.617484	0.658576	0.684485
1.0	0.500000	0.600000	0.653846	0.689320	0.735078	0.764430
1.2	0.545455	0.649682	0.705134	0.741353	0.787611	0.816937
1.4	0.583333	0.689119	0.744387	0.779984	0.824695	0.852477
1.6	0.615385	0.721030	0.775117	0.809418	0.851739	0.877493
1.8	0.642857	0.747292	0.799678	0.832398	0.872085	0.895779
2.0	0.666667	0.769231	0.819672	0.850730	0.887822	0.909598
2.2	0.687500	0.787798	0.836214	0.865635	0.900291	0.920342
2.4	0.705882	0.803695	0.850096	0.877956	0.910377	0.928900
2.6	0.722222	0.817444	0.861891	0.888291	0.918682	0.935858
2.8	0.736842	0.829443	0.872024	0.897069	0.925626	0.941614
3.0	0.750000	0.840000	0.880814	0.904608	0.931510	0.946448
P	10	15	20	30	50	100
ρ <u>r</u> 0.4	10 0.397877	15 0.399643	20 0.399936	30 0.399998	50 0.400000	100 0.400000
0.4 0.6	10 0.397877 0.574115	15 0.399643 0.588079	20 0.399936 0.594123	30 0.399998 0.598427	50 0.400000 0.599867	100 0.400000 0.600000
p 0.4 0.6 0.8	10 0.397877 0.574115 0.702671	15 0.399643 0.588079 0.731417	20 0.399936 0.594123 0.748471	30 0.399998 0.598427 0.767903	50 0.400000 0.599867 0.785047	100 0.400000 0.600000 0.796806
r 0.4 0.6 0.8 1.0	10 0.397877 0.574115 0.702671 0.785418	15 0.399643 0.588079 0.731417 0.819684	20 0.399936 0.594123 0.748471 0.841108	30 0.3999998 0.598427 0.767903 0.867540	50 0.400000 0.599867 0.785047 0.895213	100 0.400000 0.600000 0.796806 0.924300
p 0.4 0.6 0.8 1.0 1.2	10 0.397877 0.574115 0.702671 0.785418 0.837690	15 0.399643 0.588079 0.731417 0.819684 0.871063	20 0.399936 0.594123 0.748471 0.841108 0.891501	30 0.399998 0.598427 0.767903 0.867540 0.916067	50 0.400000 0.599867 0.785047 0.895213 0.940658	100 0.400000 0.600000 0.796806 0.924300 0.964476
p 0.4 0.6 0.8 1.0 1.2 1.4	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259
r 0.4 0.6 0.8 1.0 1.2 1.4	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.922106	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105	100 0.400000 0.796806 0.924300 0.964476 0.978259 0.984593
p r 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.902126 0.922106 0.935738	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133
p 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735 0.924074	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.922106 0.935738 0.945512	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269 0.957386	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156 0.970241	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319 0.981391	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133 0.988133
p 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735 0.924074 0.933528	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.922106 0.935738 0.945512 0.952809	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269 0.957386 0.963343	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156 0.970241 0.974609	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319 0.981391 0.984248	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133 0.988133 0.990371 0.991907
p 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735 0.924074 0.933528 0.940971	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.902126 0.922106 0.935738 0.945512 0.952809 0.958441	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269 0.957386 0.963343 0.967881	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156 0.970241 0.974609 0.977884	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319 0.981391 0.984248 0.986356	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133 0.998133 0.990371 0.991907 0.993024
p 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735 0.924074 0.933528 0.940971 0.946965	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.922106 0.935738 0.945512 0.952809 0.958441 0.962905	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269 0.957386 0.963343 0.967881 0.967881	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156 0.970241 0.974609 0.977884 0.980424	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319 0.981391 0.981391 0.984248 0.986356 0.987973	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133 0.998373 0.990371 0.991907 0.993024 0.993872
p 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8	10 0.397877 0.574115 0.702671 0.785418 0.837690 0.871801 0.895102 0.911735 0.924074 0.933528 0.940971 0.946965 0.951886	15 0.399643 0.588079 0.731417 0.819684 0.871063 0.902126 0.922106 0.935738 0.945512 0.952809 0.958441 0.962905 0.966523	20 0.399936 0.594123 0.748471 0.841108 0.891501 0.920113 0.937676 0.949269 0.957386 0.963343 0.967881 0.967881 0.971442 0.974307	30 0.399998 0.598427 0.767903 0.867540 0.916067 0.940953 0.955189 0.964156 0.970241 0.974609 0.977884 0.980424 0.982449	50 0.400000 0.599867 0.785047 0.895213 0.940658 0.960676 0.971105 0.977319 0.981391 0.984248 0.986356 0.987973 0.989250	100 0.400000 0.600000 0.796806 0.924300 0.964476 0.978259 0.984593 0.988133 0.990371 0.991907 0.993024 0.993872 0.994538

SERVER UTILIZATION $U(\mathbf{r},\infty)$ IN $D/M/\mathbf{r}$ NO - QUEUE SYSTEMS

						and the second
Pr	1	2	.3	4	6	8
0.4	0.367166	0.388176	0.395246	0.397987	0.399610	0.399920
0.6	0.486675	0.534704	0.557958	0.571419	0.585605	0.592281
0.8	0.570796	0.636692	0.671852	0.694598	0.723069	0.740529
1.0	0.632121	0.706928	0.747212	0.773679	0.807689	0.829411
1.2	0.678482	0.756584	0.797811	0.824518	0.858284	0.879437
1.4	0.714642	0.792904	0.833053	0.858508	0.889898	0.908991
1.6	0.743582	0.820342	0.858572	0.882291	0.910841	0.927744
1.8	0.767244	0.841666	0.877711	0.899636	0.925480	0.940441
2.0	0.786939	0.858646	0.892501	0.912741	0.936181	0.949507
2.2	0.803580	0.872447	0.904226	0.922940	0.944297	0.956260
2.4	0.817822	0.883865	0.913722	0.931077	0.950638	0.961465
2.6	0.830148	0.893454	0.921554	0.937704	0.955717	0.965588
2.8	0.840917	0.901613	0.928115	0.943197	0.95 9869	0.968930
3.0	0.850406	0.908634	0.933685	0.947818	0.963323	0.971691
<u> </u>		······	· · · · · · · · · · · · · · · · · · ·			
P	10	15	20	30	50	100
ρ 0.4	10 0.399983	15 0.400000	20 0.400000	30 0.400000	50 0.400000	100 0.400000
0.4 0.6	10 0.399983 0.595713	15 0.400000 0.598931	20 0.400000 0.599716	30 0.400000 0.599978	50 0.400000 0.600000	100 0.400000 0.600000
ρ 0.4 0.6 0.8	10 0.399983 0.595713 0.752444	15 0.400000 0.598931 0.770395	20 0.400000 0.599716 0.780254	30 0.400000 0.599978 0.790259	50 0.400000 0.600000 0.797151	100 0.400000 0.600000 0.799815
ρ 0.4 0.6 0.8 1.0	10 0.399983 0.595713 0.752444 0.844891	15 0.400000 0.598931 0.770395 0.870063	20 0.400000 0.599716 0.780254 0.885733	30 0.400000 0.599978 0.790259 0.904987	50 0.400000 0.600000 0.797151 0.925046	100 0.400000 0.600000 0.799815 0.946017
ρ 0.4 0.6 0.8 1.0 1.2	10 0.399983 0.595713 0.752444 0.844891 0.894258	15 0.400000 0.598931 0.770395 0.870063 0.917770	20 0.400000 0.599716 0.780254 0.885733 0.931923	30 0.400000 0.599978 0.790259 0.904987 0.948608	50 0.400000 0.600000 0.797151 0.925046 0.964827	100 0.400000 0.600000 0.799815 0.946017 0.979862
ρ 0.4 0.6 0.8 1.0 1.2 1.4	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479
ρ r 0.4 0.6 0.8 1.0 1.2 1.4 1.6	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034
ρ r 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034 0.993934
ρ 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284 0.958160	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686 0.970633	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558 0.977345	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984 0.984434	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197 0.990416	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034 0.993934 0.995108
ρ 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284 0.958160 0.963949	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686 0.970633 0.974910	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558 0.977345 0.980741	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984 0.984434 0.986842	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197 0.990416 0.991940	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.99892034 0.993934 0.995108 0.995108
$p = \frac{r}{\rho}$ 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284 0.958160 0.963949 0.968367	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686 0.970633 0.974910 0.978122	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558 0.977345 0.980741 0.983267	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984 0.984434 0.986842 0.986842 0.988613	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197 0.990416 0.991940 0.993049	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034 0.992034 0.995108 0.995108 0.995904 0.996478
ρ 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284 0.958160 0.963949 0.968367 0.971841	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686 0.970633 0.974910 0.978122 0.980617	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558 0.977345 0.980741 0.983267 0.985215	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984 0.984434 0.986842 0.988613 0.989968	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197 0.990416 0.991940 0.993049 0.993049	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034 0.993934 0.995108 0.995904 0.995904 0.996478 0.996911
ρ 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8	10 0.399983 0.595713 0.752444 0.844891 0.894258 0.922042 0.939053 0.950284 0.958160 0.963949 0.968367 0.971841 0.974639	15 0.400000 0.598931 0.770395 0.870063 0.917770 0.942058 0.955935 0.964686 0.970633 0.974910 0.978122 0.980617 0.982608	20 0.400000 0.599716 0.780254 0.885733 0.931923 0.953604 0.965370 0.972558 0.977345 0.980741 0.983267 0.985215 0.986761	30 0.400000 0.599978 0.790259 0.904987 0.948608 0.966601 0.975670 0.980984 0.984434 0.986842 0.988613 0.989968 0.991037	50 0.400000 0.600000 0.797151 0.925046 0.964827 0.978431 0.984692 0.988197 0.990416 0.991940 0.993049 0.993892 0.994554	100 0.400000 0.600000 0.799815 0.946017 0.979862 0.988479 0.992034 0.993934 0.995108 0.995108 0.995904 0.995904 0.996911 0.997250